Critical state of a nonuniform Josephson junction

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The critical-state problem is solved exactly for a Josephson junction with a periodic array of closely spaced defects under the assumption that the interaction between vortices is exponential. A surface barrier for the vortices is taken into account. The complete profile of the magnetic field penetrating into the junction is found. The field dependence of the critical current and the dependence of the penetrating flux on the external field are also found. Possible generalizations to a multidimensional Josephson structure and a granular superconductor are discussed.

1. INTRODUCTION

Most papers reporting experimental studies of the critical state in hard type-II superconductors make use of the well-known phenomenological approaches of Bean[1,2] and Kim and Anderson.[3,4] Those approaches start from a functional dependence of the critical current on the magnetic induction, $J_c(B)$, which is specified on the basis of general considerations. This dependence, which is usually the end result of the corresponding experiments, can be derived only on the basis of a microscopic theory. The results found experimentally on the behavior of the critical current as a function of the magnetic induction for many granular superconducting systems differ sharply from the standard Bean dependence ($J_c = \text{const}$) or the standard Kim-Anderson dependence $[J_c \propto B^{-1}$ or $J_c \propto (B_0 + B)^{-1}]$. For high T_c superconducting ceramics, the experimental dependence found in Refs. 5–7 is described by $J_c \propto \exp[(B_0/B)^\alpha]$, where $\alpha \approx 0.5$. Other functional dependences have also been used.[8]

In the present paper we determine the conditions under which the various functional dependences apply, using as an example the very simple model of a Josephson junction with a periodic array of closely spaced defects. We take a microscopic approach.

We consider a "1*D* model": a long Josephson junction on which pinning centers are arranged periodically at a small period. This model was discussed in Refs. 9–13. Results for corresponding sparse structures, in which the lattice constant is much greater than the dimensions of a vortex, were found in Refs. 11 and 13.

The critical-state problem is generally classical and can be outlined as follows: An external magnetic field is turned on. We watch how the magnetic flux begins to penetrate into the hard type II superconductor. Because of a pinning of vortices by defects, the rapid initial relaxation is followed by the establishment of an extremely nonuniform spatial distribution of the magnetic induction, $B(x)$, in the sample. This "critical" distribution undergoes an extremely slow (logarithmic) relaxation to a uniform distribution which minimizes the free energy. At each point on the critical profile, the total force exerted on a vortex by the other vortices and by the external field is equal to the maximum pinning force at which a defect still confines a vortex.

To find the critical profile, we also need to solve the problem of the penetration of Josephson vortices (fluxons) through the surface barrier into the structure.[14–17] This part of the problem corresponds to finding a boundary condition for one of Maxwell's equations: determining the abrupt change near the surface due to the onset of surface currents. One can work from this abrupt change in the field and from the observed $J_c(B)$ dependence to reconstruct the critical profile $B(x)$ with the help of Maxwell's equations.

In the present paper we will therefore do the following: (a) Solve the surface-barrier problem for the type of problem under consideration here and determine the relationship between the external field and the flux which penetrates into the system. (b) Find the overall shape of the critical profiles $B(x)$ and the field dependence of the critical current. In particular, we show that in strong fields this behavior is of the Bean type.

We take two approaches below. We first carry out an analysis by a discrete approach, in which the critical profile is found from a system of equations for the coordinates of individual vortices. The critical state is then analyzed by means of an integral equation for the density of vortices.

We conclude with some estimates which generalize our results to the case of a multidimensional Josephson structure and a granular superconductor.

2. DISCRETE APPROACH

We consider a plane Josephson junction which lies in the *xz* plane, with pinning centers arranged periodically along the x axis at intervals of L. Each of these centers is parallel to the z axis. The external magnetic field H_0 and the fluxon field are both directed along the z axis. The junction occupies the half-plane $x > 0$.

As the defects (the pinning centers) we could use, for example, cavities of area s which are parallel to the z axis, or we could use Josephson junctions with a finite length $2l$ along the y axis, which intersect the main junction in the planes $x = mL, y, z$ (Refs. 11–13; see Fig. 1 of the present paper). The relationship between these two models is specified by $s = 2ld$, where $d = d' + 2\lambda_L$ is the effective thickness of the junction, λ_L is the London depth, and d' is the thickness of the insulating layer of the junction. In other words, the relationship between the two models is established by equating the areas through which the flux penetrates. Here we are assuming $l \ll \delta$, where δ is the Josephson length. (There are other pinning-center models which we

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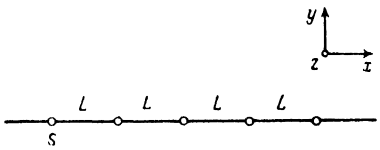

FIG. 1. Model of a linear (planar) Josephson junction with periodic defects. The line is the projection of the plane of the junction onto the xy plane. Here $x_k = kL$ are the coordinates of the pinning centers, which are cylindrical cavities with an area $s = 2ld$ ($d = d' + 2\lambda_L$), which are oriented parallel to the z axis. The magnetic field is directed along the z axis.

could use.[18]) In all cases we assume that the conditions $\lambda_L \ll L$ and $l \ll \delta$ hold.

This problem can be solved explicitly in the two limiting cases (1) $l \ll L$, which is the case of "weak" pinning, and (2) $l \gg L$, which is the case of "strong" pinning.

Case 1 is analyzed in detail in the Appendix. In this case, the pinning centers cause a slight renormalization of the solitons (magnetic vortices) in comparison with isolated solitons in a uniform junction. The solitons are characterized in this case by an energy $E_J = 4\hbar j_c \delta/e$, by a length $\delta = (\hbar c^2/8\pi j_c e d)^{1/2}$, and by a lower critical field $H_{c1} = 4\pi E_J/\Phi_0 = 2\Phi_0/\pi^2 \delta d = 16 j_c \delta/c$, where j_c is the critical current density for the uniform junction, e is the charge of an electron, and $\Phi_0 = \pi\hbar c/e$ is the flux quantum.

Case 2 is characterized by a pronounced renormalization of the solitons. As was shown in Refs. 12 and 15, this case can be described by the Frenkel'-Kontorova model:

$$E=E_J\frac{\delta}{16l}\sum_k\left[\frac{1}{2}(\theta_{k+1}-\theta_k)^2+\lambda(1-\cos\theta_k)\right]. \tag{1}$$

Here θ_k is the phase difference between the superconducting wave functions at the banks of the junction, averaged over the intervals between pinning centers, and λ is given by $\lambda = 2Ll/\delta^2$. As was shown in Refs. 12 and 13, under the condition $\lambda \ll 1$, in the lowest-order approximation, Hamiltonian (1) corresponds to ordinary Josephson vortices with a renormalized length $\delta^* = \delta(L/2l)^{1/2} = L/\lambda^{1/2}$, an energy E_J^*, and a lower critical field H_{c1}^*, found from

$$\frac{E_J^*}{E_J}=\frac{H_{c1}^*}{H_{c1}}=\frac{\delta^*}{\delta}=\left(\frac{L}{2l}\right)^{1/2}. \tag{2}$$

In the Frenkel'-Kontorova model, the vortex energy can be written[19]

$$\Omega_p=E_J^*\left[1-\frac{2\pi^2}{\lambda^{1/2}}\exp\left(-\frac{\pi^2}{\lambda}\right)\cos\frac{2\pi x}{L}\right], \tag{3}$$

where x is the distance from the center of the vortex to the pinning center. The corresponding maximum pinning force [cf. the corresponding expression (A13) for model 1] is

$$F_p=\frac{4\pi^3\delta^*}{L^2}E_J^*\exp\left(-\frac{\pi^2}{\lambda^{1/2}}\right). \tag{4}$$

The interaction energy of two vortices separated by a distance $\mathscr{L}$ is[20] [cf. the corresponding expression (A8) for model 1]

$$\Omega_{vv}=4\zeta^2E_J^*\exp(-\mathscr{L}/\delta^*). \tag{5}$$

This expression is valid, strictly speaking, under the condition $\mathscr{L}/\delta^* > 1$. The introduction of the additional numerical factor of ζ^2, where $\pi/4 < \zeta < 1$, is discussed in the Appendix. The vortex-antivortex interaction energy is the same as (5), aside from a change in sign [see (A8)].

The force exerted on vortex i by all the other vortices is [cf. (A10)]

$$F_{i\,vv}=4E_J^*\zeta^2\delta^{*-1}\sum_{j\neq i}\mathrm{sign}(x_i-x_j)\,(\exp(-|x_i-x_j|/\delta^*), \tag{6}$$

$$F_{i\,av}=-4E_J^*\zeta^2\delta^{*-1}\sum_j\exp[-(x_i+x_j)/\delta^*]. \tag{6a}$$

Since the lattice of pinning centers is assumed to be dense in both models, the size of a vortex is greater than the distance between defects: $L \ll \delta$ (for model 1) and $L \ll \delta^*$ (for model 2). Below, where we deal with multivortex configurations, we also require that the field H, in the junction not be too strong, so there is less than one flux quantum between pinning centers; i.e., $H \ll \Phi_0/Ld$ or $H \ll H_{c1}\delta/L$ (for model 1) and $H \ll H_{c1}^*\delta^*/L$ (for model 2).

The case of a sparse lattice, with $L \gg \delta, \delta^*$, was studied in Refs. 11 and 13. The critical state in the Frenkel'-Kontorova model with large values of λ was studied by a dynamic-modeling method in Ref. 21. In each of the last two situations, the critical field profile has turned out to be approximately linear (approximately a Bean profile).[1,2]

Finally, a force F_{iH} is exerted by the external field H_0 on a fluxon near the surface. For model 1, this force is given by expression (A11). To go over to model 2 all we have to do is make the substitutions $E_J \to E_J^*$, $\delta \to \delta^*$, $H_{c1} \to H_{c1}^*$.

We turn now to the construction of a balance equation for the forces exerted on vortex i by the other vortices, by the image antivortices, and by the external field and also for the maximum pinning forces. As was mentioned back in the Introduction, this balance is realized in the critical state. (The reasons why it is legitimate to introduce image antivortices in this problem are discussed in the Appendix.) Using (A10), (A10a), (A11), and (A13), for model 1 or (4), (6), and (6a), for model 2, we find the equilibrium condition in the critical state for vortex i (Fig. 2):

$$\begin{aligned}p=\{&-a_1a_2\ldots a_i-a_2\ldots a_i-\ldots-a_i\\&+a_{i+1}+a_{i+1}a_{i+2}+\ldots+a_{i+1}+\ldots+a_{N-2}+a_{i+1}\ldots a_{N-1}\}\\&+a_{i+1}a_{i+2}\ldots a_N\{h-a_N-a_Na_{N-1}-\ldots-a_Na_{N-1}\ldots a_1\}.\end{aligned} \tag{7}$$

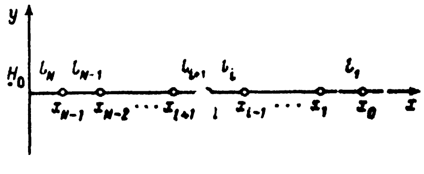

FIG. 2. Configuration of the vortex structure penetrating into the junction. The field $\mathbf{H}_0$, which is parallel to z, is applied from the left. $x = 0$—Coordinate of the surface; x_i—coordinates of the vortices; x_0—coordinate of the rightmost vortex (the front of the critical distribution); l_i—distance between vortices.

The dimensionless maximum pinning force,

$$p=\frac{2\pi^3 l\delta^2}{L^3\zeta^2}\exp\left(-\frac{\pi^2\delta}{L}\right)\ll 1 \tag{8a}$$

for model 1 or

$$p=\frac{\pi^3}{\lambda\zeta^2}\exp\left(-\frac{\pi^2}{\lambda^{1/2}}\right)\ll 1 \tag{8b}$$

for model 2, stands on the left side of this equation. We see that p is an exponentially small quantity in both models. In (7) we have used the notation $a_i=\exp(-l_i/\delta)$ [or $\exp(-l_i/\delta^*)$, for model 2], where l_i is the distance between vortices i and $i+1$ (Fig. 2). We assume that there are N vortices in the system and that the coordinate (x_0) of vortex $i=0$ determines the front of the field distribution. The distance l_N describes the depth to which the first vortex penetrates into the interior of the junction.

The expression in the first set of braces on the right side of (7) corresponds to an interaction with vortices (the minus sign corresponds to vortices to the right of the i th vortex, and the plus sign corresponds to vortices to its left). The term proportional to h in the second set of braces corresponds to the interaction with the external field:

$$h=H_0/\pi H_{c1}\zeta \tag{9a}$$

for model 1 or

$$h=H_0/\pi H_{c1}^*\zeta \tag{9b}$$

for model 2. The other terms in the second set of braces describe the interaction with antivortices.

Although the system of N equations in (7) is highly nonlinear, an exact solution can be found. The decisive circumstance which makes this possible is the exponential nature of the interaction, which allows us write the quantity $\exp(-|x_i-x_j|/\delta)$ as the product $a_i a_{i-1}\ldots a_{j-1}$ $(j<i)$.

We introduce the two quantities G_{i+1} and K_i $(i=0,1,\ldots,N-1)$, which are determined from the recurrence relations

$$G_{i+1}=a_{i+1}(1+G_i),\quad K_i=a_i(1+K_{i+1}), \tag{10}$$

supplemented by the conditions $G_0=K_N=0$. Equations (7) can then be rewritten in the compact form

$$p=K_{i+1}-G_i+a_{i+1}\ldots a_N(h-G_N). \tag{11}$$

We now sum both parts of expression (11) over i from 0 to $m-1$. On the left we find the pinning force acting on m vortices adjacent to the front, and on the right we have the forces exerted on this group of vortices by all other vortices, by all antivortices, and by the field. In the course of this summation, the forces acting between the vortices of the group from 0 to $(m-1)$ of course drop out. As a result, using (10), we find $(m=1,2,\ldots,N)$

$$pm=(1+G_{m-1})[K_m+a_m\ldots a_N(h-G_N)]. \tag{12}$$

Now setting $i=m-1$ in (11), we find the quantity K_{m-1} from this relation for $m=1,2,\ldots,N-1$, and we substitute the result into (12). We find a closed equation for G_{m-1}:

$$pm=(1+G_{m-1})(p+G_{m-1}). \tag{13}$$

Solving this quadratic equation, we find

$$G_{m-1}=-(p+1)/2\pm[(p+1)^2/4+p(m-1)]^{1/2}. \tag{14}$$

Since $G_m>0$, we should use only the plus sign. Since we have $a_m=G_m/(1+G_{m-1})$ according to (10), we find

$$a_m=\frac{[(1+p)^2+4mp]^{1/2}-1-p}{[(1-p)^2+4mp]^{1/2}+1-p} \tag{15}$$

for $m=1,2,\ldots,N-1$.

We are left with finding the quantity a_N, which determines the distance from the leftmost vortex to the surface. For this purpose we set $m=N$ in (12), and we use $K_N=0$ and $a_N(1+G_{N-1})=G_N$. As a result we find

$$pN=G_N(h-G_N),$$

and thus

$$G_N=h/2\pm(h^2/4-pN)^{1/2}.$$

Correspondingly, we find

$$a_N=\frac{h\pm(h^2-4pN)^{1/2}}{[(1-p)^2+pN]^{1/2}+1-p}. \tag{16}$$

From (16) we find the equilibrium value of the distance l_N from the leftmost vortex to the surface $[a_N=\exp(-l_N/\delta^*)^N]$ for the given dimensionless external field h. We now see that there is yet another special value of l_N. To choose the correct solution, we draw an (extremely schematic) plot of the vortex energy U versus the distance l_N for our model (Fig. 3). There is a maximum here; it approaches the point $l_N=0$ as h is increased. The position of this maximum is given by (16) when we choose the plus sign. The minimum on the curve corresponds to (16) with the minus sign. The increase in the energy at large l_N is a consequence of pinning $(p\neq 0)$. As $p\to 0$, the curve of $U(l_N)$ reaches saturation at large l_N, and the minimum goes off to infinity $(a_N\to 0)$. The second solution (with the minus sign), like the first (with the plus sign), is thus not suitable for our purposes. However, there is yet another equilibrium point: $l_N=0$ $(a_N=1)$. This point does not correspond to a minimum on the curve of $U(l_N)$ $[(\partial U/\partial l_N)_{l_N=0}\neq 0]$, since in our approximation there is a break in the $U(l_N)$ curve at this point. This break results from the nonanalytic nature of the function describing the interaction of the vortices with the external field and with each other [see (6), (A10), and (A11)] in the exponential interaction model

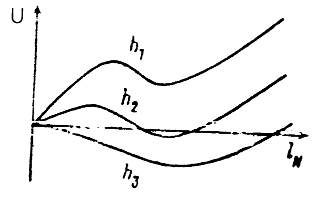

FIG. 3. Energy of the system of vortices as a function of the distance from the leftmost vortex to the surface for various values of the external field.

which we are using. The nonanalytic nature of the interaction, on the other hand, arises because of a central "core" of the model fluxon. In other words, the phase is discontinuous at this point. Because of this circumstance, the equilibrium value of a_N in our model is always equal to one ($l_N = 0$). This equilibrium is disrupted as the field is increased, at the point at which the maximum disappears from the $U(l_N)$ curve, after the point $l_N = 0$ is reached. At this time, the system of fluxons "breaks through" into the junction, and the total number of fluxons increases by one.

We can thus work from (16) to determine the relationship between the external field h and the number N of vortices in the critical state. For this purpose we need to set $a_N = 1$ in (16) and take the plus sign (i.e., we need to find the time at which the maximum disappears). As a result we have

$$N=[h^2-(1-p)^2]/4p. \tag{17}$$

Relation (17) characterizes the height of the surface barrier for the fluxons, since they begin to penetrate into the structure at $h>(1+p)\approx 1$, i.e., $H_0 > \pi\zeta H^*_{c1}$ (for model 1 $H^*_{c1} \to H_{c1}$).

To determine the spatial position of vortex m, we need to sum $\ln a_m$ [see (15)] over m from m to $N-1$. At $p \ll 1$, and in the continuum approximation, however, the problem simplifies dramatically. At $p \ll 1$, but for arbitrary pm, expression (15) becomes

$$m=\{4p\,\mathrm{sh}^2(1/2\delta^* n_m)\}^{-1}, \tag{18}$$

where $n_m = l_m^{-1}$ is the number density of vortices. In the continuum approximation, this number density is related to m by

$$n_m=-dm/dx. \tag{19}$$

Differentiating (18) with respect to x, and using (19), we find

$$\frac{dn}{dx}=-pn^3\delta^*\,\mathrm{sh}^3(1/2\delta^* n)\,\mathrm{ch}^{-1}(1/2\delta^* n), \tag{20}$$

where $n = n(x) \equiv n_m$.

We now need to relate the vortex concentration $n(x)$ to the local field $\overline{H}(x)$ in the continuum approximation. If a vortex carries a flux Φ_0, then

$$\overline{H}(x)=\frac{\Phi_0}{d}n(x)=\frac{\pi^2}{2}H_{c1}^*\delta^* n(x). \tag{21}$$

Substituting (21) into (20), we find an equation for the field profile:

$$\frac{d\overline{H}}{dx}=-p\frac{\overline{H}}{\delta^*}\left(\frac{4\overline{H}}{\pi^2 H_{c1}^*}\right)^2\mathrm{sh}^3\left(\frac{\pi^2 H_{c1}^*}{4\overline{H}}\right)\mathrm{ch}^{-1}\left(\frac{\pi^2 H_{c1}^*}{4\overline{H}}\right). \tag{22}$$

Comparing (22) with Maxwell's equation for the critical state,

$$d\overline{H}/dx=-(4\pi/c)J_c(\overline{H}),$$

where $J_c(\overline{H})$ is the critical current, we find

$$J_c=\pi p j_c\left(\frac{4\overline{H}}{\pi^2 H_{c1}^*}\right)^3\mathrm{sh}^3\left(\frac{\pi^2 H_{c1}^*}{4\overline{H}}\right)\mathrm{ch}^{-1}\left(\frac{\pi^2 H_{c1}^*}{4\overline{H}}\right). \tag{23}$$

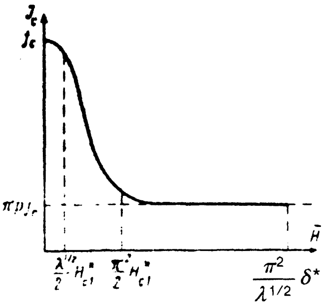

FIG. 4. Field dependence of the critical current. The values of $\overline{H}$ at which a transition from one regime to another occurs are shown for the particular case of model 2.

[Again, for model (1) we have $H^*_{c1} \to H_{c1}$.] Figure 4 shows a plot of $J_c(\overline{H})$.

We see from (23) that $J_c(\overline{H})$ has two asymptotes: the transition between the two occurs at $n \sim 1/\delta^* \times (\overline{H} \sim \pi^2 H^*_{c1}/2)$. In weak fields, $\overline{H} < \pi^2 H^*_{c1}/2$, i.e., when the overlap of vortices is only slight ($n < 1/\delta^*$), we have

$$J_c\approx 2\pi p j_c\left(\frac{2\overline{H}}{\pi^2 H_{c1}^*}\right)^3\exp\left(\frac{\pi^2 H_{c1}^*}{2\overline{H}}\right). \tag{24}$$

Since the overlap of vortices is weak in this region, we could in principle derive (24) while restricting force balance equation (7) to the interaction between nearest neighbors.

Expression (24) has an applicability limit which is a lower limit of $\overline{H}$, since the relation $J_c < j_c$ obviously holds. At logarithmic accuracy we can write

$$\overline{H}>\pi^2 H_{c1}^*/2\ln(p^{-1}). \tag{25}$$

Since the pinning force p in (8) is exponentially small, the lower limit of $\overline{H}$ is also small—much smaller than H^*_{c1}. The mathematical reason why expressions (23) and (24) are not valid at very small values of $\overline{H}$ is that the continuum approximation breaks down near the front of the critical profile (where $\overline{H}$ vanishes).

In strong fields, $\overline{H} > \pi^2 H^*_{c1}/2$, the vortices overlap to a great extent ($n > 1/\delta^*$), and the dependence becomes a Bean dependence:

$$J_c\approx\pi p j_c. \tag{26}$$

Since $p \ll 1$, we have $J_c < j_c$ in (26), as we should. In this region, we could not hope that the model of interacting isolated fluxons which we are using here would give us anything better than a qualitative description.

We thus find the Bean form of $J_c \propto \exp(\pi^2 H^*_{c1}/2\overline{H})$ in strong fields and the behavior $J_c(\overline{H})$ in weak fields. Similar results were found in Ref. 22 for the model of a nonuniform Josephson junction on the basis of a theory of collective pinning.[23] Strangely enough, a similar functional dependence $J_c(\overline{H})$ [the parameters were of course different from those in (24) and (26)] was found in Refs. 11–13 for a fundamentally different model: for a nonuniform Josephson junction with sparsely distributed pinning centers ($L \gg \delta$).

To find the critical profile $\overline{H}(x)$ from differential equation (22) we need a boundary condition: the value of the field $\overline{H}$ at the $x = 0$ boundary. Generally speaking, this field value is not the same as the external field H_0. It can be found from (17) and (18) by setting $m = N$ in the latter. Assuming $p \ll 1$, we find

$$h^2 - 1 = \mathrm{sh}^{-2}(1/2\delta^* n_N).$$

Using relation (21) at $x = 0$ along with definition (9), we find the boundary condition to be

$$H_i = \frac{\pi^2}{4} H_{c1}^* / \mathrm{arth}\, \frac{\pi\zeta H_{c1}^*}{H_0}, \tag{27}$$

where $H_i \equiv \overline{H}(x = +0)$ is the field at the boundary. Figure 5 shows a plot of $H_i(H_0)$.

This dependence has a threshold at $H_0 = \pi\zeta H_{c1}^*$ due to the appearance of a surface barrier to the penetration of vortices. In strong external fields, the plot of $H_i(H_0)$ becomes linear: $H_i \approx \pi H_0/4\zeta$. From this form of the asymptotic behavior we see the meaning of the factor ζ, $\pi/4 < \zeta < 1$, which is introduced in the Appendix. This factor depends weakly on the vortex concentration, i.e., on $\overline{H}(x)$. As is mentioned in the Appendix, at a high vortex concentration we have $\zeta \to \pi/4$, and in this region the asymptotic behavior assumes a reasonable form, $H_i \approx H_0$. In other words, the abrupt change in the field the surface due to the onset of surface currents becomes negligible.

We can now find the critical profile and its dependence on the external magnetic field, integrating Eq. (22) over x from 0 to x (or over $\overline{H}$ from H_i to $\overline{H}$):

$$x = \frac{\delta^*}{2p}\left(\frac{z}{\mathrm{sh}^2 z} + \mathrm{cth}\, z\right)\Bigg|_{z=\pi^2 H_{c1}^*/4\overline{H}}^{z=\pi^2 H_{c1}^*/4H_i}. \tag{28}$$

In particular, the field penetration depth x_0 is found by setting $\overline{H} = 0$ here:

$$x_0 = \frac{\delta^*}{2p}\left(\frac{\pi^2 H_{c1}^*}{4H_i}\,\mathrm{sh}^{-2}\frac{\pi^2 H_{c1}^*}{4H_i} + \mathrm{cth}\,\frac{\pi^2 H_{c1}^*}{4H_i} - 1\right). \tag{29}$$

In strong external fields, $H_0 \approx H_i \gg \pi^2 H_{c1}^*/2$, we have

$$x_0 \approx \frac{\delta^*}{p}\,\frac{4H_0}{\pi^2 H_{c1}^*}. \tag{30}$$

In other words, the penetration depth increases linearly with the field. If the field is just slightly above the threshold value, $(H_0 - H_1)/H_1 \ll 1$, where $H_1 \equiv \zeta\pi H_{c1}^*$, the dependence $x_0(H_0)$ becomes stronger:

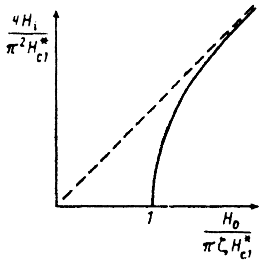

FIG. 5. The average internal field near the surface as a function of the external magnetic field H_0.

$$x_0 \approx \frac{\delta^*}{2p}\,\frac{H_0 - H_1}{H_1}\ln\frac{H_1}{H_0 - H_1}, \tag{31}$$

so we have $(dx_0/dH_0)_{H_0 \to H_1} \to \infty$.

The profile itself, $\overline{H}(x)$ [see (28)], turns out to be linear in strong fields:

$$\overline{H} \approx \pi^2 p H_{c1}^* (x_0 - x)/4\delta^*$$

(in accordance with the Bean model). Near the front, x_0, the dependence becomes logarithmic:

$$\overline{H} \propto (x_0 - x)\ln[(x_0 - x)^{-1}]$$

(Fig. 6).

3. INTEGRAL EQUATION FOR THE NUMBER DENSITY OF VORTICES

Far from the front, where the vortex concentration is fairly high, the results of the preceding section could also be derived with the help of an integral equation for the number density of vortices, $n(x)$. We take the standard approach (Ref. 24, for example), multiplying both sides of (7) by $\Sigma_m \delta(x - x_m)$ and using the relations

$$n(x) = \sum_m \delta(x - x_m), \tag{32}$$

$$\sum_m \delta(x - x_m)\delta(x' - x_m) = n(x)\delta(x - x'). \tag{33}$$

As a result we find the integral equation

$$pn(x) = \int dx' F(x - x') n(x) n(x') \,\mathrm{sign}(x') + h\exp(-x/\delta^*) n(x), \tag{34}$$

where the kernel is

$$F(x) = -\mathrm{sign}(x)\exp(-|x|/\delta^*). \tag{35}$$

We denote by $x = x_0$ the position of the front of the critical profile, while $x = 0$ is the surface of the sample. We then find

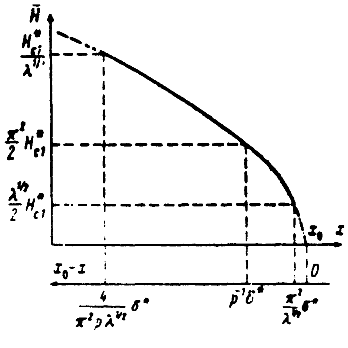

FIG. 6. The critical profile $\overline{H}(x)$. Here x_0 is the coordinate of the front. The values of the field and the coordinate at which the transition from one regime to the other occurs are shown for the particular case of model 2. The dot-dashed line shows the region of low values of $\overline{H}$ in which the large distances between vortices rule out the use of an equation like (22).

$$p=-\int_x^{x_0} dyn(y)\exp\left(\frac{x-y}{\delta^*}\right)+\int_0^x dyn(y)\exp\left(\frac{y-x}{\delta^*}\right)$$

$$+h\exp\left(-\frac{x}{\delta^*}\right)-\int_0^{x_0} dyn(y)\exp\left(-\frac{y+x}{\delta^*}\right). \quad (36)$$

A solution of Eq. (36) can be found directly; it is

$$n(x)\delta^*=p(x_0-x+\delta^*)/2\delta^*,\ \ 0<x<x_0, \quad (37)$$

where x_0 is related to the dimensionless external field:

$$x_0/\delta^*=h/p-1. \quad (38)$$

Relation (37) leads to the asymptotes $J_c(\overline{H})$ and $\overline{H}(x)$ found in the preceding section of this paper.

4. CRITICAL STATE OF GRANULAR SUPERCONDUCTORS

We conclude with a rough estimate of the dependence of the critical current through a granular superconductor on the magnetic induction, $J_c(B)$. This problem might be thought of as a direct generalization of the problem discussed above. We will not discuss the particular value which the dimensionless maximum pinning force reaches for one specific model or other (Ref. 25, for example); we simply treat it as a small parameter of the theory: $p \ll 1$.

To derive at least a very crude estimate of the critical profile, we make use of the circumstance that the vortices penetrate into the sample along the normal to its surface (along the x axis). In this sense the problem is "one-dimensional." Proceeding as in Sec. 2, we find an estimate for

$$x_{m-1}-x_m \sim 2^{-1/2}\delta\ln[(pm)^{-1}]$$

near the front (the number of the vortex, m, is again counted from the front). We now allow for the two-dimensional nature of the vortex structure. Assuming that this structure is locally isotropic, we find $n_m=(x_{m-1}-x_m)^2$, and the magnetic induction is $B=n_m\Phi_0$. As a result we find an estimate for the critical profile,

$$\frac{B\delta^2}{2\Phi_0}\sim \ln^{-2}\frac{\delta}{2^{1/2}xp}, \quad (39)$$

and for the critical current,

$$J_c \propto \exp\left[\left(\frac{2\Phi_0}{\delta^2 B}\right)^{1/2}\right]. \quad (40)$$

This behavior of the critical current has been seen[5-7] in several granular superconductors, over an even broader range of the applied field than was considered in the derivation of (40).

We wish to thank A. V. Goltsev and A. N. Samukhin for useful discussions.

APPENDIX. ENERGY OF A SYSTEM OF VORTICES IN A BOUNDED JOSEPHSON JUNCTION WITH PINNING CENTERS

The energy of a semi-infinite ($x>0$) nonuniform Josephson junction in an external field H_0 can be written

$$\Omega=\Omega_0+\Omega_H+\Omega_p, \quad (A1)$$

where

$$\Omega_0=\frac{1}{8}E_J\int_0^\infty dz\left(1-\cos\theta+\frac{1}{2}\theta'^2\right), \quad (A2)$$

$$\Omega_H=-\frac{H_0 d}{4\pi}\int_0^\infty dzH=-E_J\frac{H_0}{2\pi H_{c1}}\int_0^\infty dz\theta', \quad (A3)$$

$$\Omega_p=-E_J\frac{l}{8\delta}\int_0^\infty dz\theta'^2\sum_m\delta(z-z_m). \quad (A4)$$

Here Ω_0, Ω_H, and Ω_p are the energies of the Josephson junction, of the interaction with the field, and of the interaction with the pinning centers, respectively; $E_J=4\hbar j_c\delta/c$ is the fluxon energy; $z=x/\delta$ is a dimensionless coordinate; δ is the Josephson length; j_c is the critical current of the junction; $d=d'+2\lambda_L$ is the thickness of the junction; and $H_{c1}=16j_c\delta/c$ is the lower critical field. The pinning centers are at the points $z_i=x_i/\delta$. The parameter l, which has the dimensionality of a length, describes the pinning force (Sec. 2). The choice of the pinning energy in the form in (A4) leads to an abrupt change in the phase θ at a pinning center, $\Delta\theta=2l\theta'(z_n)$, and to a continuity of the fluxon magnetic field, $H=(\Phi_0/2\pi d)\theta'$ (Refs. 10 and 26). We will discuss below only model 1 (see Sec. 2).

The basic approximation of the present study is to choose the multifluxon phase $\theta(z)$ as a linear superposition of isolated solitons, each of which is centered at some point and is described by the phase

$$\theta_0(z-z_i)=4\,\mathrm{arctg}[\exp(z-z_i)]. \quad (A5)$$

This superposition of solitons does not satisfy the boundary conditions: the vanishing of the total fluxon magnetic field at $z=0$ and the vanishing of the normal component of the current at the same point. These conditions can be satisfied in the approximation which we are using here if we make use of a "mirror reflection," i.e., if we introduce a system of antisolitons at the points $-z_i$. As a result, the trial function which we are using becomes

$$\theta(z)=\sum_i[\theta_0(z-z_i)+\theta_0(-z-z_i)]=\theta(-z). \quad (A6)$$

The introduction of antisolitons is analogous to the use of vortex-antivortex pairs in a study of the penetration of Abrikosov vortices into a type-II superconductor. In that case, the introduction of the vortex-antivortex pairs leads to the appearance of a Bean-Livingston barrier. In contrast with the linear problem of the interaction of an Abrikosov vortex with a surface, this mapping of the phase onto the $z<0$ half-plane in the nonlinear problem for fluxons is only a very simple and convenient approximation (not an exact solution), even in the problem of the interaction of one fluxon with a surface.

Another necessary condition for the validity of approximation (A6) is that the deformation of the fluxon due to the interaction with pinning centers be weak (the pinning must be weak). As we will see below, this condition corresponds to the condition $l/L \ll 1$, where L is the distance between pinning centers. This condition corresponds to a situation in which the magnetic flux accumulated by a pinning center is small in comparison with the flux which is incident on the section of the junction between centers. In this Appendix, we restrict the discussion to the case of weak pinning.

After we use a trial function in the form in (A6), the energy Ω_0 in (A2) becomes

$$\Omega_0=\frac{E_J}{16}\int_{-\infty}^{\infty}dz\left\{1-\mathrm{Re}\prod_{i,s}\left(1-\frac{1}{2}\theta_{is}'^2+i\theta_{is}''\right)+\frac{1}{2}\sum_{i,i',s,s'}\theta_{is}'\theta_{i's'}'\right\},$$

where $s=1, 2$; $\theta_{i1}=\theta_{0i}(z-z_i)$; and $\theta_{i2}=\theta_{0i}(-z-z_i)$. Here we have used the relations $\sin\theta_0=\theta_0''$ and $\cos\theta_0=1-\theta_0'^2/2$ for isolated solitons. The energy Ω_0 can be written as a sum $\Omega_{01}+\Omega_{02}+\Omega_{03}$, where Ω_{0k} is the energy of the interaction of the "particles"—the fluxons ($s=1$) and the antifluxons ($s=2$). The energy of the noninteracting particles is

$$\Omega_{01}=\frac{1}{16}E_J\sum_{i,s}\int_{-\infty}^{\infty}dz\,\theta_{is}'^2=E_JN,$$

where N is the total number of fluxons. The quantity Ω_{01} merely shifts the origin of the energy scale and can be omitted. The two-particle interaction is

$$\Omega_{02}=\frac{E_J}{32}\sum_{ii',ss'}{}'\int_{-\infty}^{\infty}dz(\theta_{is}'\theta_{i's'}'+\theta_{is}''\theta_{i's'}''-\theta_{is}'^2\theta_{i's'}'^2/4),$$

where the prime on the summation sign means that the term with $i=i'$, $s=s'$ is excluded. Using (A5), we then find

$$\Omega_{02}=\frac{1}{2}E_J\left\{\sum_{i,i'}{}'\left[\mathrm{sh}^{-2}\left(\frac{z_i-z_{i'}}{2}\right)\left(1-\frac{z_i-z_{i'}}{\mathrm{sh}(z_i-z_{i'})}\right)\right]-\sum_{i,i'}\left[\mathrm{ch}^{-2}\left(\frac{z_i+z_{i'}}{2}\right)\left(1+\frac{z_i+z_{i'}}{\mathrm{sh}(z_i+z_{i'})}\right)\right]\right\}. \tag{A7}$$

The first term in braces is the contribution to the interaction between vortices, while the second is the contribution to the interaction between vortices and antivortices. This expression can be simplified greatly if the distance between vortices and the distance between vortices and antivortices are large in comparison with their size (i.e., if $|x_i-x_{i'}|>\delta$ and $x_i+x_{i'}>\delta$):

$$\Omega_{02}\approx 2E_J\left[\sum_{i,i'}{}'\exp(-|z_i-z_{i'}|)-\sum_{i,i'}\exp(-z_i-z_{i'})\right]. \tag{A8}$$

Strictly speaking, it is only in this case that the superposition of isolated vortices in (A6) is applicable. In the opposite case we should take account of three-body, etc., interactions; the problem becomes essentially unsolvable. In this paper, however, we use expressions like (A8) even in the case of a pronounced overlap of the solitons, in which case we could hope for no more than a qualitatively correct description by an expression in this form. This form is exact if the soliton phase in (A5) is written in the form $4\exp(-|z-z_i|)$. The phase difference, however, turns out to be 8, not 2π. In other words, the magnetic flux of the soliton, Φ_0, turns out to be equal to $4\hbar c/e$, not the flux quantum $\pi\hbar c/e$. In order to conserve the size of the flux quantum in the exponential approximation of the phase we thus need to write

$$d\theta_0/dz\approx\pi\exp(-|z-z_i|). \tag{A9}$$

The introduction of the factor of $\pi/4$ in the approximate expression for the phase leads to the appearance of an additional factor $(\pi/4)^2\equiv\zeta^2$ in (A8). The reasoning behind the introduction of a factor $\zeta\neq 1$ in the expression for the energy is discussed in the text proper. From (A.8) we can now finally find the force exerted on vortex i by the other vortices,

$$F_{i\,vv}=4E_J\zeta^2\delta^{-1}\sum_{i'\neq i}\mathrm{sign}(z_i-z_{i'})\exp(-|z_i-z_{i'}|), \tag{A10}$$

as well as the force exerted by the antivortices,

$$F_{i\,av}=-4E_J\zeta^2\delta^{-1}\sum_{i'}\exp(-z_i-z_{i'}). \tag{A10a}$$

We now consider the interaction with the external field H_0. Substituting (A6) into (A3), we find the following expression for the force exerted on vortex i by the field H_0:

$$F_{iH}=\frac{2}{\pi}E_J\delta^{-1}\frac{H_0}{H_{c1}}\mathrm{ch}^{-1}z_i\,.$$

Alternatively, switching to an approximation of the type in (A9), we find

$$F_{iH}=\frac{4}{\pi}\zeta E_J\delta^{-1}\frac{H_0}{H_{c1}}\exp(-z_i), \tag{A11}$$

where ζ is again a factor which depends weakly on the vortex concentration, having the behavior $\zeta\to 1$ at $z_i>1$ and the behavior $\zeta\to\pi/4$ at $z_i<1$.

Finally, the interaction with pinning centers, (A4), takes the following form when we use (A6):

$$\Omega_p=-\frac{l}{2\delta}E_J\sum_{n,i,i'}[\mathrm{ch}^{-1}(z_i-z_n)\mathrm{ch}^{-1}(z_{i'}-z_n)+\mathrm{ch}^{-1}(z_i+z_n)\mathrm{ch}^{-1}(z_{i'}+z_n)-2\mathrm{ch}^{-1}(z_i-z_n)\mathrm{ch}^{-1}(z_{i'}+z_n)].$$

We restrict the discussion to the case of periodically positioned defects: $z_n=Ln/\delta$, $n=0, 1,\ldots$, where L is the distance between defects. In the case of a dense lattice, $L/\delta\ll 1$, which is the case of interest in this paper, it is convenient to sum over n by the Poisson method:

$$\Omega_p=-\frac{2\pi l}{L}E_J\sum_{m=-\infty}^{\infty}\sum_{i,i'}\mathrm{sh}^{-1}\left(\frac{2\pi m\delta}{L}\right)\times\left\{\frac{\sin[\pi m\delta(z_i-z_{i'})/L]\cos[\pi m\delta(z_i+z_{i'})/L]}{\mathrm{sh}(z_i-z_{i'})}-\frac{\sin[\pi m\delta(z_i+z_{i'})/L]\cos[\pi m\delta(z_i-z_{i'})/L]}{\mathrm{sh}(z_i+z_{i'})}\right\}. \tag{A12}$$

The terms of the series in m in (A12) fall off rapidly at $\delta/L \gg 1$. The term with $m=0$,

$$\Omega_p^{(0)} = -\frac{l}{L} E_J \left\{ \sum_i \left(1 - \frac{2z_i}{\operatorname{sh} 2z_i}\right) + {\sum_{i,i'}}' \left[\frac{z_i - z_{i'}}{\operatorname{sh}(z_i - z_{i'})} - \frac{z_i + z_{i'}}{\operatorname{sh}(z_i + z_{i'})} \right] \right\},$$

renormalizes Ω_{01}, which is the energy of a noninteracting soliton (Σ_i), and also Ω_{02}, which is the energy of the soliton-soliton and soliton-antisoliton interaction ($\Sigma'_{i,i'}$) [see (A7)]. In the case of weak pinning, $l/L \ll 1$, this renormalization is of minor importance; we ignore it below. The quantity $\Omega_p^{(0)}$ also has a contribution which describes the repulsion of a vortex from a surface. This force acts at distances on the order of δ from the surface, and we ignore this effect.

The primary component of the actual pinning energy comes from terms with $m = \pm 1$, which are, for $L/\delta \ll 1$ (and also $|z_i - z_{i'}| > 1$),

$$\Omega_p \approx -\frac{4\pi^2 l\delta}{L^2} E_J \exp\left(-\frac{\pi^2\delta}{L}\right) \sum_i \cos\frac{2\pi\delta z_i}{L}.$$

Correspondingly, the maximum pinning force (which is directed toward the surface) is

$$F_{ip} = -\frac{1}{\delta}\frac{\partial\Omega}{\partial z_i}\left(z_i = \frac{L}{\delta}\left(k_i + \frac{1}{4}\right)\right) = -\frac{8\pi^3 l\delta}{L^3}\exp\left(-\frac{\pi^2\delta}{L}\right). \qquad \text{(A13)}$$

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