On the role of the gain collision term in the theory of the anomalous skin effect

M. I. Kaganov, G. Ya. Lyubarskii, and J. Czerwonko

Kapitsa Institute for Physical Problems, Russian Academy of Sciences, Moscow; Physicotechnical Institute, Ukrainian Academy of Sciences, Khar'kov; and Instytut Fizyki, Politechnika Wrocławska, Poland

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By accounting for the gain term in the collision integral, the relaxation time (or τ) approximation is avoided in calculating the surface impedance of a metallic half-space. The probability density per unit length that an electron will be scattered through an angle θ is taken to be $W(\theta) = (4\pi l)^{-1}(1+\alpha\cos\theta)$, where $|\alpha|<1$ and l is the loss-related mean free path of the electron. The dependence on α is derived for various values of the ratio l/δ (δ being the usual skin depth) for electrons reflected either specularly or diffusively from the boundary.

I. The kinetic theory of the skin effect for electrons with the quadratic isotropic dispersion law[1)]

$$\varepsilon=p^2/2m \qquad (1)$$

dates back as far as 1948 to Reuter and Sondheimer.[1] They based their analysis on the τ approximation, in which the collision integral of the problem is described solely in terms of its loss term $-f_1/\tau$, where f_1 is linear in the wave electric field and represents the nonequilibrium correction to the equilibrium electron distribution, the Fermi function f_F. The electron gas was assumed to be degenerate, i.e.,

$$-\partial f_F/\partial\varepsilon=\delta(\varepsilon-\varepsilon_F).$$

To justify the use of the τ approximation the following arguments may be advanced.

1. Under normal skin-effect conditions ($l\ll\delta$), both the surface impedance ζ of the metal and the electromagnetic field penetration depth are expressible in terms of the static conductivity

$$\sigma_{tr}=ne^2l_{tr}/p_F, \qquad (2)$$

dependent on the momentum-transfer mean free path l_{tr} given by

$$\frac{1}{l_{tr}}=\int_0^{\pi}\int_0^{2\pi} W(\theta)\,(1-\cos\theta)\,dO', \quad dO'=\sin\vartheta'\,d\vartheta'\,d\varphi', \qquad (2')$$

where $W(\theta)$ represents the probability density per unit length that an electron will be scattered through an angle θ,

$$\cos\theta=\cos\vartheta\cos\vartheta'+\sin\vartheta\sin\vartheta'\cos(\varphi-\varphi'). \qquad (3)$$

The spherical angles ϑ, φ and ϑ', φ' in this equation specify the propagation direction of the electron prior to and after the scattering event, respectively.

Equation (2) is in fact equivalent to the τ approximation, that is, to replacing the true collision integral by the quantity $-f_1/\tau_{tr}$, with $\tau_{tr}=l_{tr}/v_F$. This replacement, however, is only valid under normal skin effect conditions where the angular electron distribution depends solely on the direction of the electric field **E**, i.e., $f_1\propto(\mathbf{E}v)=Ev\sin\vartheta\cos\varphi$, see the discussion below.

2. In the extreme anomalous skin effect limit ($l\gg\delta$ or, strictly speaking $l\to\infty$) the surface impedance expression does not contain the carrier lifetime at all and the integral (or gain) term in the collision integral is of no importance. This last circumstance arises from the fact that, as opposed to the static conductivity, the impedance ζ in this limit does not depend on the totality of the Fermi electrons but rather on those of them moving parallel to the sample surface $z=0$ [on which the plane electromagnetic wave $\mathbf{E}=\mathbf{E}(z)e^{-i\omega t}$ is incident]. Formally, this is because the function f_1 has a singularity at $v_z=0$ in the $l\to\infty$ limit. In the gain collision term this function enters through the integrand and hence its singularity—even if it survives—is of less importance than in the loss term.

We note also that the theory of the anomalous skin effect depends on the solution of the set of Maxwell's equations, and hence requires a knowledge of a material equation of the form

$$j_i=\int_0^{\infty}K_{ik}(z,z')E_k(z')\,dz'$$

relating the current density **j** to the electric field **E**. The conductivity operator $K_{ik}(z,z')$ in this equation is constructed from the solution of the kinetic equation for the electron distribution function. If we allow the electric field to be nonuniform, $\mathbf{E}=\mathbf{E}(z)$, and take into account the electron/boundary interaction, an effective solution to the relevant integro-differential kinetic equation is in fact impossible to obtain unless some simplifying assumptions concerning the collision integral and the manner in which electrons are reflected by the surface are made. Accordingly, Reuter and Sondheimer consider only loss terms in the collision integral and restrict their analysis to the limiting cases of electrons suffering either specular or diffusive reflection. The point to bear in mind, however, is that for arbitrary l/δ values, the results of Ref. 1 hold strictly for $W(\theta)=\text{const}$ only.

Because the gain term is unimportant in the extreme anomalous limit ($l\gg\delta$), it has proven possible to generalize the theory in some respects so as to consider, for example, a complex dispersion law for the conduction electrons; to elucidate the role of the surface as a scatterer of electrons; or to incorporate the effects of a steady magnetic field (refer to the books by Lifshitz, Azbel', and Kaganov[2] and by Abrikosov,[3] where a bibliography of original work is also given; for a review, see Ref. 4).

In the literature, quite a number of formulas have been developed for the impedance ζ and other metal properties, supposedly valid for arbitrary l/δ values (see Ref. 5, for ex-

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ample). These formulas, however, have been mostly derived within the τ approximation and it has already been mentioned—and will be made clear below—that their accuracy is therefore questionable.

To our knowledge, so far three attempts[6-8] have been made to calculate the temperature-dependent correction to the surface impedance near the extreme anomalous limit. All three closely parallel each other in using τ-type approximations when treating the collision integral and a specular reflection model for electrons incident on the boundary. As for the bulk electrons, these are either assumed to interact with each other,[6] or with phonons,[7] or else with local vibration modes.[8]

The skin-effect in metals is one of the fundamental problems in physical kinetics, and its rigorous and self-consistent solution for even a simple model would be highly desirable. For arbitrary values of the l/δ ratio, it has been remarked earlier that the τ approximation may be justified by assuming the probability density $W(\theta)$ to be independent of the scattering angle. As a consequence, only the s-wave electron scattering from isotropic impurities is eligible for consideration.

II. The present study is motivated by the desire to avoid the τ approximation and to be able, at the same time, to solve the surface impedance ζ problem exactly within the framework of the model used. Accordingly, the theory of the skin effect is constructed under the assumptions that electrons are scattered elastically and that the probability density W is fully determined by the angle between the electron momenta prior to and after the scattering event [cf. Eqs. (2′) and (3)] so that the collision integral takes the form

$$I\{f_1\}=-\int_0^\pi\int_0^{2\pi} W(\theta)\{f_1(\mathbf{n})-f_1(\mathbf{n}')\}dO', \quad \mathbf{n}\mathbf{n}'=\cos\theta,$$
$$\mathbf{n}=\mathbf{p}/p, \quad \mathbf{n}'=\mathbf{p}'/p. \tag{4}$$

Furthermore, the requirement for an exact solution to exist restricts the functions $W(\theta)$ to those for which

$$W(\theta)=W_0(1+\alpha\cos\theta), \tag{5}$$

which is equivalent to assuming that only the s-scattering and p-scattering (described by the respective constants W_0 and α) are of importance. Replacing the arbitrary function $W(\tau)$ by (5)—which excludes small-angle scattering—is of course the single most restrictive assumption we make in this study.

With the collision integral as given by (4), the Boltzmann equation may be conveniently written as

$$\cos\vartheta\frac{\partial\chi(z,\mathbf{n})}{\partial z}+\frac{1}{l}\chi(z,\mathbf{n})-\int_0^\pi\int_0^{2\pi} W(\theta)\chi(z,\mathbf{n}')\sin\vartheta'\,d\vartheta'\,d\varphi'=E(z)\cos\varphi\sin\vartheta, \tag{6}$$

where the quantities $\chi(z,\mathbf{n})$ and l are defined by

$$f_1=-\frac{\partial f_F}{\partial\varepsilon}e\chi(z,\mathbf{n}), \quad n_x=\sin\vartheta\cos\varphi,$$
$$n_y=\sin\vartheta\sin\varphi, \quad n_z=\cos\vartheta, \tag{7}$$

and

$$\frac{1}{l}=W_0\int_0^\pi\int_0^{2\pi}(1+\alpha\cos\theta)\,dO'=4\pi W_0, \tag{8}$$

with f_1 denoting the nonequilibrium correction to the Fermi function f_F. The quantity $E(z)$ in (6) represents the x component of the electric field $\mathbf{E}$ in the metal, and it is assumed that an x-polarized electromagnetic wave of frequency ω is incident on the surface of the metal.[2)] The field component $E(z)$ is given by

$$\frac{d^2E(z)}{dz^2}+\frac{4\pi i\omega}{c^2}j(z)=0, \tag{9}$$

and the current density $j(z)$, remembering the assumed degeneracy of the electron gas, is

$$j(z)=\frac{2e^2p_F^2}{(2\pi\hbar)^3}\int_0^\pi\int_0^{2\pi} n_x\chi(z,n)\,dO=\frac{3\sigma}{4\pi l}\int_0^\pi\int_0^{2\pi} n_x\chi(z,\mathbf{n})\,dO. \tag{10}$$

From (6) and (3) it is easy to show that

$$\chi(z,\mathbf{n})=\chi(z,\vartheta)\cos\varphi, \tag{11}$$

where the function $\chi(z,\vartheta)$ satisfies

$$\cos\vartheta\frac{\partial\chi(z,\vartheta)}{\partial z}+\frac{1}{l}\chi(z,\vartheta)-\int_0^\pi\sin\vartheta'\overline{W}(\vartheta,\vartheta')\chi(z,\vartheta')\,d\vartheta'=E(z)\sin\vartheta, \tag{12}$$

which is simpler than (6) and in which

$$\overline{W}(\vartheta,\vartheta')=\int_0^{2\pi}W(\cos\vartheta\cos\vartheta'+\sin\vartheta\sin\vartheta'\cos\varphi')\cos\varphi'\,d\varphi'. \tag{13}$$

From (10) and (11), the current density becomes

$$j(z)=\frac{3\sigma}{4l}\int_0^\pi\chi(z,\vartheta)\sin^2\vartheta\,d\vartheta. \tag{14}$$

In view of the assumption (5) the kernel (13) of the integro-differential Eq. (12) is degenerate,

$$\overline{W}(\vartheta,\vartheta')=\frac{1}{4}\frac{\alpha}{l}\sin\vartheta\sin\vartheta' \tag{15}$$

and using this together with (14) reduces (12) to

$$\cos\vartheta\frac{\partial\chi(z,\vartheta)}{\partial z}+\frac{1}{l}\chi(z,\vartheta)=S(z)\sin\vartheta, \tag{16}$$

with

$$S(z)=E(z)+\frac{\alpha}{3\sigma}j(z). \tag{17}$$

The transition from the integro-differential kinetic equation (12) to the differential equation (16) is, of course, the most important consequence of the assumption (5). From this point on the problem will require no further simplifications.

It is now necessary to supplement (16) with appropriate boundary conditions. In the limit $z\to\infty$, the boundary conditions to apply in the skin-effect context are naturally

$$E(z)|_{z\to\infty}=0,\ \chi(z,\vartheta)|_{z\to\infty,\ \pi/2\leqslant\vartheta\leqslant\pi}=0. \tag{18}$$

The second of these ensures that the electrons in the bulk of the metal are in equilibrium, the restriction on the angles reflecting the fact that, for $z\to\infty$ and $0\leqslant\vartheta\leqslant\pi/2$, $\chi(z,\vartheta)=0$ in view of (16). On the metal surface ($z=0$), the boundary conditions are of course surface dependent and there exist a vast literature on their derivation for surface types of practical interest (see, for example, Ref. 9, where references to the original work may also be found). Following previous work,[1] and concentrating as we are on the role of the gain term of the collision integral, the boundary conditions will be restricted to those proposed by Fuchs,[10]

$$\chi(0,\vartheta)=Q\chi(0,\pi-\vartheta),\quad 0\leqslant\vartheta\leqslant\pi/2, \tag{19}$$

where the parameter Q measures the proportion of electrons reflected specularly from the boundary ($0\leqslant Q\leqslant 1$). Following the same practice, we will consider only the two extremes, $Q=1$ (pure specular reflection) and $Q=0$ (pure diffusive reflection).

Using (14) together with (16)–(19), we show in the usual way that the current density $j(z)$ and the function $S(z)$, Eq. (17), are related by

$$j(z)=\sigma\int_0^\infty\left\{K\left(\frac{z-z'}{l}\right)+QK\left(\frac{z+z'}{l}\right)\right\}S(z')\frac{dz'}{l}, \tag{20}$$

where

$$K(u)=\tfrac{3}{4}\int_0^{\pi/2}\frac{\sin^3\vartheta}{\cos\vartheta}\exp\left(-\frac{|u|}{\cos\vartheta}\right)d\vartheta. \tag{21}$$

Eliminating $S(z)$ and $j(z)$ from (9), (17), and (20) and changing to the dimensionless length $\xi=z/l$ now results in the following integro-differential equation:

$$\frac{d^2E}{d\xi^2}=\int_0^\infty[K(\xi-\xi')+QK(\xi+\xi')]\times\left[\frac{\alpha}{3}\frac{d^2E(\xi')}{d\xi'^2}-2i\frac{l^2}{\delta^2}E(\xi')\right]d\xi'. \tag{22}$$

III. We shall first consider the specular reflection limit, $Q=1$. In this case Eq. (22) may be given a much simpler form if we continue the function $E(\xi)$ ($\xi>0$) evenly onto the negative half-axis by demanding that

$$E(-\xi)=E(\xi).$$

For $\xi>0$ and $Q=1$ Eq. (22) becomes

$$\frac{d^2E(\xi)}{d\xi^2}=\int_{-\infty}^\infty K(\xi-\xi')\left[\frac{\alpha}{3}\frac{d^2E(\xi')}{d\xi'}-2i\frac{l^2}{\delta^2}E(\xi')\right]d\xi', \tag{23}$$

which, by the evenness of $K(\xi)$ and $E(\xi)$, is valid for all negative ξ's as well. Introducing the notation

$$\bar E(\varkappa)=\int_{-\infty}^\infty E(\xi)e^{i\varkappa\xi}d\xi \tag{24}$$

for the Fourier transform of a function and applying this transform to (23) we obtain

$$\bar E(\varkappa)=-\frac{2E'(0)}{\varkappa^2-(2il^2/\delta^2)\mu(\varkappa)}, \tag{25}$$

where

$$E'(0)=\left.\frac{dE}{d\xi}\right|_{\xi=0},\quad \mu(\varkappa)=\frac{\bar K(\varkappa)}{1-(\alpha/3)\bar K(\varkappa)} \tag{25'}$$

and

$$\bar K(\varkappa)=\int_{-\infty}^\infty K(\xi)e^{i\varkappa\xi}d\xi=\frac{3}{2}\int_0^1\frac{(1-y^2)dy}{1+\varkappa^2y^2}. \tag{26}$$

Note that the function $\bar K(\varkappa)$ is expressible in terms of elementary functions,

$$\bar K(\varkappa)=\frac{3}{4}(1+\varkappa^{-2})(i\varkappa)^{-1}\ln\frac{1+i\varkappa}{1-i\varkappa}-\frac{3}{2\varkappa^2}, \tag{27}$$

with the logarithm of the fraction uniquely determined by the condition

$$\left|\operatorname{Im}\ln\frac{1+i\varkappa}{1-i\varkappa}\right|<\pi.$$

Because of this condition, the function $\bar K(\varkappa)$ undergoes discontinuities on the half-axes $(-i\infty,-i)$ and $(i,i\infty)$.

Application of the inversion procedure

$$E(\xi)=\frac{1}{2\pi}\int_{-\infty}^\infty\bar E(\varkappa)e^{-i\varkappa\xi}d\varkappa$$

now yields an explicit expression for the field $E(\xi)$—and for $E(0)$ as a special case—and noting that

$$H(0)=(c/i\omega l)E'(0),$$

we find that the impedance $\zeta=E(0)/H(0)$ is given by

$$\zeta_{Q=1}=\frac{\omega l}{i\pi c}\int_{-\infty}^\infty\frac{d\varkappa}{\varkappa^2-(2il^2/\delta^2)\mu(\varkappa)}. \tag{28}$$

In this expression the parameter α enters only through $\mu(\varkappa)$ and since $|\alpha|<1$ holds and the function $\bar K(\varkappa)$ monotonically decreases with its argument, it follows that, while the magnitude of the impedance is of course α-dependent, qualitatively neither the impedance $\zeta_{Q=1}$ nor the electric field distribution in the metal is affected by the introduction of the gain term into the collision integral.

Applying Fourier transforms to equations (9) and (17), we find that the Fourier component $\bar j$ of the current and that of the field, $\bar E$, are related by

$$\bar j(\varkappa)=\sigma\mu(\varkappa)\bar E(\varkappa).$$

Setting $\varkappa=0$ and noting that $\bar K(0)=1$ from (26), we find, in agreement with Eqs. (2) and (5), that the static conductivity is

$$\sigma_{tr}=\frac{\sigma}{1-\alpha/3}, \tag{29}$$

or (referring to footnote 1)

$$\sigma_{tr}=\frac{ne^2l_{tr}}{p_F},\quad l_{tr}=\frac{l}{1-\alpha/3}. \tag{30}$$

In the extreme cases $l/\delta\ll1$ or $l/\delta\gg1$, the impedance is calculated by noting that

$$\mu(\varkappa)\approx\begin{cases}[1-\varkappa^2/5(1-\alpha/3)](1-\alpha/3)^{-1}, & \varkappa\ll1,\\ (3\pi/4\varkappa)[1+(\alpha\pi-4/\pi)/\varkappa], & \varkappa\gg1.\end{cases} \tag{31}$$

For $l \ll \delta$, the integral (28) is dominated by the residue at the point

$$\varkappa = \varkappa_0 \sim l/\delta,$$

where the denominator of the integrand vanishes. Using the first of equations (31) we find that

$$\zeta_{Q=1} = \zeta_{\rm norm}\,(1 - l_{tr}^2/5\delta^2),\quad l \ll \delta, \tag{32}$$

where

$$\zeta_{\rm norm} = (\omega/4\pi i\sigma_{tr})^{1/2} \tag{32'}$$

represents the impedance as calculated for the normal skin-effect conditions, with the quantities $\sigma_{\rm tr}$ and $l_{\rm tr}$ defined by (29) and (30). It should be emphasized that the penetration depth δ in the correction term contains the conductivity σ rather than $\sigma_{\rm tr}$!

For $l \gg \delta$, the integral (28) is dominated by the large values of $\varkappa$, and using the second of equations (31) yields

$$\zeta_{Q=1} = \zeta_{\rm anom}\,[1 + (2/3\pi)^{1/3}(i + 3^{1/2})(4/\pi - \alpha\pi/4)(\delta^2/3l^2)],\quad l \gg \delta, \tag{33}$$

where

$$\zeta_{\rm anom} = \frac{2}{9}\left(\frac{3^{1/2}\omega^2 l}{\pi^2 c\sigma}\right)^{1/3}(1 - i3^{1/2})$$

is the surface impedance expression as calculated in the extreme anomalous limit ($l/\delta \to \infty$) for electrons reflected specularly from the sample boundary.

IV. We now turn to consider the case of diffusive reflection, $Q = 0$. The formulas we present below are derived in the Appendix to this paper, which may be omitted by those taking no interest in the details.

Equation (22) now takes the form

$$\frac{d^2E(\xi)}{d\xi^2} = \int_0^\infty K(\xi - \xi')\left[\frac{\alpha}{3}\frac{d^2E(\xi')}{d^2\xi'^2} - \left(\frac{2il^2}{\delta^2}\right)E(\xi')\right]d\xi' \tag{34}$$

and is amenable to solution by the Wiener–Hopf method.[11] Writing

$$\bar{E}(\varkappa) = \int_0^\infty E(\xi)e^{i\varkappa\xi}d\xi \tag{35}$$

to define the Fourier transform of a function, we find explicitly that

$$\bar{E}(\varkappa) = \frac{iE(0) - (\alpha/3)H_+(\varkappa)}{\varkappa + \varkappa_0}\exp[-G_+(\varkappa)]. \tag{36}$$

The functions $H_+(\varkappa)$ and $G_+(\varkappa)$ in (36) may be calculated by the following algorithm: Let

$$F(\varkappa) = \varkappa^2 - [(\alpha/3)\varkappa^2 + 2il^2/\delta^2]\bar{K}(\varkappa), \tag{37}$$

and let $\varkappa_0$ be the (only) zero the function $F(\varkappa)$ has in the first quadrant. Further,

$$G_\pm(\varkappa) = \frac{1}{2\pi i}\int_{\mp i\varepsilon - \infty}^{\mp i\varepsilon + \infty}\frac{d\xi}{\xi - \varkappa}\ln\frac{F(\xi)}{\xi^2 - \varkappa_0^2} \tag{38}$$

and

$$H_+(\varkappa) = \frac{1}{2\pi i}\int_{-i\varepsilon_1 - \infty}^{-i\varepsilon_1 + \infty}\frac{d\xi}{\xi - \varkappa}\,\frac{iE(0)\xi - E'(0)}{\xi - \varkappa_0}\bar{K}(\xi)\exp[G_-(\xi)], \tag{39}$$

where the (positive) numbers $\varepsilon > 0$ and $\varepsilon_1 > \varepsilon$ are sufficiently small that their choice does not affect the integrals above (which means that integrands in question have no singularities in the strip $|{\rm Im}\ \xi| < \varepsilon_1$).

Now if we let $\xi = 0$ in $E(\xi)$ as calculated from (36), a trivial identity will result. We may proceed, however, by employing the device used by Reuter and Sondheimer,[1] to obtain

$$E'(0) = \lim_{\varkappa \to i\infty}[i\varkappa E(0) - \varkappa^2\bar{E}(\varkappa)], \tag{40}$$

which leads to the impedance expression of the form

$$\zeta_{Q=0} = \frac{\omega l}{c}i\xi_0\frac{1 - S^2}{1 + S^2}, \tag{41}$$

where

$$\xi_0 = \frac{l}{\delta}\left(-\frac{6i}{\alpha}\right)^{1/2},\quad {\rm Im}\,\xi_0 > 0,\quad S = \frac{\xi_0\exp[-G_+(\xi_0)]}{\xi_0 + \varkappa_0}. \tag{42}$$

We thus see that to calculate the impedance $\zeta_{Q=0}$ requires the evaluation of one of the integrals (38) for $\varkappa = \xi_0$.

In the $\alpha \to 0$ limit, it is shown in the Appendix that Eq. (41) reduces to

$$\zeta_{Q=0} = \frac{\omega l}{c}\left(\varkappa_0 - \frac{1}{2\pi i}\int_{-\infty}^{\infty}d\varkappa\ln\frac{F(\varkappa)}{\varkappa^2 - \varkappa_0^2}\right)^{-1}, \tag{43}$$

which retrieves the Reuter–Sondheimer result.[1]

In what follows, approximate impedance expressions for the limiting cases $l \ll \delta$ and $l \gg \delta$ are given.

For $l \ll \delta$ and α of order unity (or, more precisely, for $l/\delta\alpha \ll 1$), we have

$$\zeta_{Q=0} = \zeta_{\rm norm}\left[1 + \frac{l}{\delta}\Phi(\alpha)\right]. \tag{44}$$

The function Φ in the last equation is defined by

$$\Phi(\alpha) = \frac{1+i}{(\alpha/3)(1-\alpha/3)^{1/2}}\times\frac{1}{(2\pi i)^2}\int_{C'}\frac{d\varkappa}{\varkappa^2}\ln\frac{1-(\alpha/3)K_-(\varkappa)}{1-(\alpha/3)K_+(\varkappa)}\ln(\varkappa+i), \tag{45}$$

where the integration contour C' starts and terminates at the point $\varkappa = -i$ and encloses the segment $(-i, i)$ and all the zeros of the functions

$$1 - (\alpha/3)K_\pm(\varkappa).$$

While the functions $K_\pm(\varkappa)$ are both defined by the same equation (27) defining $\bar{K}(\varkappa)$, they differ from one another in that in calculating

$$\ln\frac{1+i\varkappa}{1-i\varkappa}$$

different branches of the logarithmic function are taken, so that the branch with the imaginary part varying from 0 to 2π [from -2π to 0] defines the function $K_+(\varkappa)$ [$K_-(\varkappa)$].

In the limit $\alpha \to 0$ Eq. (45) exhibits an indeterminacy which is easily evaluated by noting that

$$\ln\frac{1-(\alpha/3)K_-(\varkappa)}{1-(\alpha/3)K_+(\varkappa)}=\frac{\alpha}{3}[K_+(\varkappa)-K_-(\varkappa)]+O(\alpha^2)$$

and using the exact result

$$K_+(\varkappa)-K_-(\varkappa)=\frac{3\pi}{2}\left(\frac{1}{\varkappa}+\frac{1}{\varkappa^3}\right)$$

to give

$$\Phi(0)=\frac{3}{16}(1-i).$$

We thus see that the right-hand side of (44) reproduces the corresponding result of Dingle,[5] and since this latter was derived for $\alpha = 0$, $l/\delta \ll 1$, it follows that (44) holds under these assumptions as well rather than being restricted to the condition $l/\delta\alpha \ll 1$.

The values of the function $\Phi(\alpha)$ are listed in Table I. For $l \gg \delta$,

$$\zeta_{Q=0}=\tfrac{9}{8}\zeta_{\text{anom}}\left[1+\frac{2i3^{1/3}}{\pi^2\lambda}\exp\left(-\frac{i\pi}{3}\right)(\ln\lambda+a+\alpha b)\right],\tag{46}$$

where

$$\lambda=\left(\frac{3\pi l^2}{2\delta^2}\right)^{1/3}\gg 1,\quad a=\frac{\pi^2}{4i}\left(\frac{1}{2}+\frac{2}{3\pi}+i\sigma_+\right),$$

$$\sigma_+=\frac{1}{2}\int_1^\infty\left[\operatorname{arctg}\frac{1}{\pi}\left(\ln\frac{s+1}{s-1}+\frac{2s}{s^2-1}\right)-\frac{4}{\pi s}\right]ds,$$

and

$$b=-\frac{\pi^3}{48}\left(1+\frac{2}{3^{1/2}}-2i\right).$$

For $\alpha = 0$, Eq. (46) is consistent with the corresponding result of Dingle.[5]

V. Equations (28) and (41), together with (32), (44) for $l \ll \delta$ or (33), (46) for $l \gg \delta$, constitute the solution to the problem. As argued earlier, the theory of the anomalous skin effect remains qualitatively unchanged by the inclusion of the gain term in the collision integral. In particular, in the limit $l/\delta \to \infty$ surface impedance measurements are again independent of bulk dissipation processes and as such may be usefully employed in spectroscopic applications. On the other hand, the magnitude of the impedance depends strongly on the value of α (i.e., the gain term)—so much so that the correction terms in Eqs. (32), (33), and (44), (46) may vary severalfold as α is varied between -1 and $+1$.

As we see it, the most important result of our study is that, for the Fuchs parameter Q fixed, the surface impedance of a metal is by no means a unique function of the mean free path l (nor of the transport-related mean free path l_{tr}) but rather is controlled by the value of the parameter α, the specific form of its α-dependence being different for $l/\delta < 1$ and $l/\delta > 1$.

Attention should also be paid to the far-from-trivial role of the reflection conditions in determining the relevant dependence on α. In the specular reflection limit, both the surface impedance and the bulk field distribution are determined by the conductivity

$$\sigma(\varkappa)=\sigma\frac{\bar K(\varkappa)}{1-(\alpha/3)\bar K(\varkappa)},$$

which although wavevector-dependent, represents a very natural generalization of the collisional conductivity formula (29). In the diffusive reflection case, it is seen from equations (41) through (46) that the parameter α is in fact totally unpredictable as to its precise location in the impedance expression.

Finally, the discussion above seems to have demonstrated that, for arbitrary l/δ values, augmenting Eq. (5) to obtain a more realistic form of the collision integral makes it virtually impossible to derive an analytical expression for the surface impedance of the metal.

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APPENDIX: SOLUTION OF THE INTEGRO-DIFFERENTIAL EQUATION (34)

1. Let us consider the function

$$f_-(\xi)=E''(\xi)-\int_{-\infty}^{\infty}K(\xi-\xi')\times[(\alpha/3)E''(\xi')-(2il^2/\delta^2)E(\xi')]d\xi',\tag{A1}$$

so defined after the unknown function $E(\xi)$ has been continued onto the entire real axis by setting $E(\xi) = 0$ for $\xi < 0$. From the basic equation (34) it follows that the (unknown) function f_- vanishes for all $\xi > 0$. The Fourier transform of (A1) gives the equation

$$\bar E(\varkappa)F(\varkappa)-e(\varkappa)\left[1-\frac{\alpha}{3}\bar K(\varkappa)\right]=\bar f_-(\varkappa),\tag{A2}$$

where

$$e(\varkappa)=iE(0)\varkappa-E'(0),\quad F(\varkappa)=\varkappa^2-(\alpha\varkappa^2/3+2il^2/\delta^2)\bar K(\varkappa)\tag{A3}$$

and

$$\bar K(\varkappa)=\frac{3}{2}\int_0^1\frac{1-t^2}{1+\varkappa^2t^2}dt=\frac{3}{4i\varkappa^3}(1+\varkappa^2)\ln\frac{1+i\varkappa}{1-i\varkappa}-\frac{3}{2\varkappa^2},\tag{A4}$$

where the relevant branch of the logarithm is the one which is zero for $\varkappa = 0$.

Following the Wiener–Hopf method as we do here, our first objective is to represent the prefactor of $\bar E(\varkappa)$ in (A2) as a ratio of two functions, one analytic in the upper and the

TABLE I.

α	−0,9	−0,7	−0,5	−0,3	−0,1	0	0,1	0,3	0,5	0,7	0,9
$\mathrm{Re}\Phi(\alpha)$	0,169	0,173	0,177	0,182	0,186	0,1875	0,189	0,193	0,197	0,200	0,202
$-\mathrm{Im}\Phi(\alpha)$	0,128	0,139	0,151	0,165	0,174	0,1875	0,195	0,214	0,234	0,266	0,282

other in the lower half-plane. The specific form of such factorization depends strongly on the behavior of $F(\varkappa)$ on the real axis, and so the next step in the argument is to examine more closely some of the properties of the functions $\overline{K}(\varkappa)$ and $F(\varkappa)$.

From its two representations afforded by (A4), it is seen that the function $\overline{K}(\varkappa)$ is analytic everywhere in the complex $\varkappa$ plane with cuts along $(-i\infty, -i)$ and $(i, i\infty)$ and that everywhere in this region the estimate

$$|\overline{K}(\varkappa)|=\frac{3\pi}{4|\varkappa|}+O(|\varkappa|^{-2}), \quad |\varkappa|\gg 1 \tag{A5}$$

holds. Also, for all real $\varkappa$ the function $\overline{K}(\varkappa)$ is strictly positive, with the implication that the function $F(\varkappa)$ is analytic and has its imaginary part negative on this axis, and that its argument may be regarded as varying the range from $-\pi$ to 0. It will be shown in Section 7 of this Appendix that $F(\varkappa)$ has precisely two zeros, $\pm\varkappa_0$, in the plane cut along $(-i\infty, -i)$ and $(i, i\infty)$, the point $\varkappa_0$ lying in the first quadrant of the plane. The function

$$\frac{F(\varkappa)}{\varkappa^2-\varkappa_0^2}$$

has neither zeros nor singular points in the plane so cut. The definition (A3) implies the following representation:

$$\frac{F(\varkappa)}{\varkappa^2-\varkappa_0^2}-1=-\frac{\alpha}{3}\overline{K}(\varkappa)+O(|\varkappa|^{-2})=O(|\varkappa|^{-1}), \quad |\varkappa|\gg 1. \tag{A6}$$

Now let us introduce

$$G(\varkappa)=\ln\frac{F(\varkappa)}{\varkappa^2-\varkappa_0^2}, \tag{A7}$$

by choosing the logarithm branch which vanishes when $\varkappa\to+\infty$. As $|\varkappa|\to\infty$, it follows from (A6) that over the entire right half-plane $G(\varkappa)$ approaches zero as $(|\varkappa|^{-1})$, whereas on the left half-plane its limit is a multiple of $2\pi i$. The following simple argument shows that this limit is in fact zero: Since there are no branch points in $G(\varkappa)$, this is obviously a single-valued function in the cut $\varkappa$ plane. Now as $\varkappa$ is varied from $+\infty$ to $-\infty$ along the real axis, the argument of the quantity $(\varkappa^2-\varkappa_0^2)$ remains unchanged, whereas that of $F(\varkappa)$ (as discussed earlier) changes by π at most. But then the imaginary part of $G(-\infty)$ does not exceed π in its absolute value and, since it is a multiple of 2π, has to be zero. From the above it follows that in the cut complex plane

$$G(\varkappa)=-\frac{\alpha}{3}\overline{K}(\varkappa)+O(|\varkappa|^{-2}), \quad |\varkappa|\gg 1. \tag{A8}$$

We next turn to the factorization of $F(\varkappa)$. By the Cauchy integral theorem,

$$G(\varkappa)=\frac{1}{2\pi i}\oint_L\frac{d\xi}{\xi-\varkappa}G(\xi), \quad |\operatorname{Im}\varkappa|<1,$$

where the (arbitrary) contour L lies in the cut complex plane and encloses the point $\varkappa$. We may now employ the estimates (A8) and (A5) to deform L into a pair of straight lines,

$$\operatorname{Im}\xi=\pm\varepsilon,$$

with the number ε chosen at will from the interval

$$|\operatorname{Im}\varkappa|<\varepsilon<1.$$

The function $G(\varkappa)$ may thus be represented as

$$G(\varkappa)=G_+(\varkappa)-G_-(\varkappa) \quad \left(=\ln\frac{F(\varkappa)}{\varkappa^2-\varkappa_0^2}\right), \tag{A9}$$

where we have defined

$$G_+(\varkappa)=\frac{1}{2\pi i}\int_{-i\varepsilon-\infty}^{-i\varepsilon+\infty}\frac{d\xi}{\xi-\varkappa}\ln\frac{F(\xi)}{\xi^2-\varkappa_0^2}, \quad \operatorname{Im}\varkappa>-\varepsilon,$$

$$G_-(\varkappa)=\frac{1}{2\pi i}\int_{i\varepsilon-\infty}^{i\varepsilon+\infty}\frac{d\xi}{\xi-\varkappa}\ln\frac{F(\xi)}{\xi^2-\varkappa_0^2}, \quad \operatorname{Im}\varkappa<\varepsilon. \tag{A10}$$

The function $G_+(\varkappa)[G_-(\varkappa)]$ is analytical in the uncut half-plane $\operatorname{Im}\varkappa>-\varepsilon$ $[\operatorname{Im}\varkappa<\varepsilon]$. Thus

$$F(\varkappa)=(\varkappa^2-\varkappa_0^2)\frac{\exp[G_+(\varkappa)]}{\exp[G_-(\varkappa)]},$$

which completes the factorization procedure $F(\varkappa)$. Equation (A2) may now be given the form

$$\overline{E}(\varkappa)(\varkappa+\varkappa_0)\exp[G_+(\varkappa)]-\frac{e(\varkappa)}{\varkappa-\varkappa_0}\exp[G_-(\varkappa)]$$
$$+\frac{\alpha}{3}\overline{K}(\varkappa)e(\varkappa)\frac{\exp[G_-(\varkappa)]}{\varkappa-\varkappa_0}=\overline{j}_-(\varkappa)\frac{\exp[G_-(\varkappa)]}{\varkappa-\varkappa_0} \tag{A11}$$

in which (almost) all terms are analytic either in the upper or in the lower (cutless) half-plane. The only exception is the term

$$H(\varkappa)\equiv\frac{\alpha}{3}\overline{K}(\varkappa)\frac{\exp[G_-(\varkappa)]}{\varkappa-\varkappa_0}e(\varkappa),$$

but this too can be represented as a difference of two functions of this kind,

$$H(\varkappa)=H_+(\varkappa)-H_-(\varkappa), \tag{A12}$$

where

$$H_+(\varkappa)=\frac{1}{2\pi i}\int_{-i\varepsilon_1-\infty}^{-i\varepsilon_1+\infty}\frac{d\xi}{\xi-\varkappa}H(\xi), \quad \operatorname{Im}\varkappa>-\varepsilon_1, \quad 0<\varepsilon_1<\varepsilon,$$

$$H_-(\varkappa)=\frac{1}{2\pi i}\int_{i\varepsilon_1-\infty}^{i\varepsilon_1+\infty}\frac{d\xi}{\xi-\varkappa}H(\xi), \quad \operatorname{Im}\varkappa<\varepsilon_1 \tag{A13}$$

and the estimate (A5), together with the boundedness of $\overline{K}(\varkappa)$ and $G_-(\varkappa)$, ensures the convergence of the integrals (A13) and the truth of equation (A12). As shown in Section 4 of this Appendix, the functions $G_-(\varkappa)$ and $G_+(\varkappa)$ are not only bounded in their respective half-planes, but also tend to zero as $|\varkappa|\to\infty$, with their absolute values obeying the inequality

$$|G_\pm(\varkappa)|<\frac{\text{const}}{|\varkappa|}|\ln\varkappa|. \tag{A14}$$

Making use of (A12), equation (A11) may now be rewritten as

$$\overline{E}(\varkappa)(\varkappa+\varkappa_0)\exp[G_+(\varkappa)]+H_+(\varkappa)=\overline{j}_-(\varkappa)\frac{\exp[G_-(\varkappa)]}{\varkappa-\varkappa_0}$$
$$+\frac{e(\varkappa)}{\varkappa-\varkappa_0}\exp[G_-(\varkappa)]+H_-(\varkappa).$$

The left-hand and right-hand sides of this equation are analytic in the half-planes $\operatorname{Im} \varkappa > -\varepsilon$ and $\operatorname{Im} \varkappa < \varepsilon$, respectively, and are seen to be equal on the real axis. As a consequence, either side represents an analytical continuation of the other into its respective half-plane or, equivalently, both of them are analytic everywhere in the uncut $\varkappa$ plane. As $|\varkappa| \to \infty$, either side increases no faster than the first power of $|\varkappa|$ in its respective half-plane and hence is described by some polynomial in $(a\varkappa + b)$, of degree at most unity. It is also important to note that as $\varkappa \to -\infty$, the right-hand side of the last equation is asymptotic to $iE(0)$ and indeed is equal to $iE(0)$ because of its being a polynomial of the kind just mentioned. With this in mind, the Fourier transform $\bar{E}(\varkappa)$ is found to be given by

$$\bar{E}(\varkappa)=\frac{\exp[-G_+(\varkappa)]}{\varkappa-\varkappa_0}\{iE(0)-H_+(\varkappa)\}. \tag{A15}$$

2. We next proceed to calculate the $E(0)/E'(0)$ ratio. Following Reuter and Sondheimer, we begin by writing the identity

$$\int_0^\infty E''(\xi)e^{i\xi\varkappa}d\xi=-\varkappa^2\bar{E}(\varkappa)+i\varkappa E(0)-E'(0)$$

and taking the limit $\varkappa \to i\infty$ to obtain

$$E'(0)=\lim_{\varkappa\to i\infty}\{-\varkappa^2\bar{E}(\varkappa)+i\varkappa E(0)\}.$$

Substituting (A15) for $\bar{E}(\varkappa)$ and noting that

$$G_+(\varkappa)=O\left(\frac{1}{\varkappa}\right), \quad \varkappa\to i\infty,$$

from the definition of the function $G_+(\varkappa)$, we find that

$$E'(0)=i\varkappa_0E(0)+\lim_{\varkappa\to i\infty}\{\varkappa[H(\varkappa)+iE(0)G(\varkappa)]_+\}, \tag{A16}$$

where, as before,

$$[f]_+=\frac{1}{2\pi i}\int_{-i\varepsilon-\infty}^{-i\varepsilon+\infty}\frac{d\xi}{\xi-\varkappa}f(\xi).$$

Now if

$$f(\xi)=O\left(\frac{1}{\xi^2}\right), \quad \xi\to-i\varepsilon\pm\infty, \tag{A17}$$

then

$$\lim_{\varkappa\to i\infty}\{\varkappa[f]_+\}=-\frac{1}{2\pi i}\int_{-i\varepsilon-\infty}^{-i\varepsilon+\infty}f(\xi)d\xi \tag{A18}$$

because

$$\lim_{\varkappa\to i\infty}\{\varkappa[f]_+\}=\lim_{\varkappa\to i\infty}\frac{1}{2\pi i}\int_{-i\varepsilon-\infty}^{-i\varepsilon+\infty}\frac{\varkappa f(\xi)}{\xi-\varkappa}d\xi$$

$$=-\frac{1}{2\pi i}\int_{-i\varepsilon-\infty}^{-i\varepsilon+\infty}f(\xi)d\xi+\lim_{\varkappa\to i\infty}\int_{-i\varepsilon-\infty}^{-i\varepsilon+\infty}\frac{\xi f(\xi)}{\xi-\varkappa}d\xi$$

and, from (A17),

$$\left|\int_{-i\varepsilon-\infty}^{-i\varepsilon+\infty}\frac{\xi f(\xi)d\xi}{\xi-\varkappa}\right|<\text{const}\int_{-i\varepsilon-\infty}^{-i\varepsilon+\infty}\frac{|d\xi|}{|\xi-\varkappa|\,|\xi|},$$

where the integral on the right vanishes as $\varkappa\to i\infty$.

If the function $f(\varkappa)$ is taken to be

$$H(\varkappa)+iE(0)G(\varkappa),$$

the condition (A17) is easily shown to hold giving

$$E'(0)=i\varkappa_0E(0)-\frac{1}{2\pi i}\int_{-i\varepsilon-\infty}^{-i\varepsilon+\infty}[H(\xi)+iE(0)G(\xi)]d\xi,$$

or, more explicitly,

$$E'(0)=i\varkappa_0E(0)-\frac{1}{2\pi i}\int_{-i\varepsilon-\infty}^{-i\varepsilon+\infty}\left\{\frac{\alpha}{3}\bar{K}(\varkappa)e(\varkappa)\frac{\exp[G_-(\varkappa)]}{\varkappa-\varkappa_0}+iE(0)\ln\frac{F(\varkappa)}{\varkappa^2-\varkappa_0^2}\right\}d\varkappa.$$

By the definition Eq. (A3) of the function $e(\varkappa)$,

$$\frac{E(0)}{E'(0)}=\frac{1}{i}\frac{1-B}{\varkappa_0-A}, \tag{A19}$$

where

$$A=\frac{1}{2\pi i}\int_{-i\varepsilon-\infty}^{-i\varepsilon+\infty}\left[\frac{\alpha}{3}\frac{\varkappa}{\varkappa-\varkappa_0}\bar{K}(\varkappa)\exp[G_-(\varkappa)]+\ln\frac{F(\varkappa)}{\varkappa^2-\varkappa_0^2}\right]d\varkappa,$$

$$B=\frac{\alpha}{6\pi i}\int_{-i\varepsilon-\infty}^{-i\varepsilon+\infty}\bar{K}(\varkappa)\exp[G_-(\varkappa)]\frac{d\varkappa}{\varkappa-\varkappa_0}. \tag{A20}$$

3. Because of the function $G_-(\varkappa)$ involved in the integrands in (A20), and because this function itself is of integral form, the integrals A and B are rather difficult to evaluate and their behavior under various extreme conditions is difficult to analyze. For $\alpha=0$ the problem is much simplified and Eq. (A19) reduces to

$$\frac{E(0)}{E'(0)}=\frac{1}{i}\left(\varkappa_0-\frac{1}{2\pi i}\int_{-i\varepsilon-\infty}^{-i\varepsilon+\infty}d\varkappa\ln\frac{F(\varkappa)}{\varkappa^2-\varkappa_0^2}\right)^{-1}, \tag{A21}$$

but even for $\alpha\neq0$ it turns out that Eqs. (A20) may be rendered no less tractable.

A few preliminary remarks should be made first. As already pointed out, the functions $\bar{K}(\varkappa)$ and $F(\varkappa)$ are both analytic in the complex $\varkappa$ plane containing two cuts.

With the second of the forms (A4), the function $\bar{K}(\varkappa)$ admits of two different Laurent expansions in the ring $|\varkappa|>1$,

$$K_\pm(\varkappa)=\pm\frac{3\pi}{4\varkappa}-\frac{3}{\varkappa^2}\pm\frac{3\pi}{4\varkappa^3}-\frac{4}{\pi\varkappa^4}\psi(\varkappa),$$

$$\psi(\varkappa)=\sum_{m=0}^\infty\left(-\frac{1}{\varkappa^2}\right)^m\frac{1}{(2m+1)(2m+3)},$$

where $K_+(\varkappa)$ and $K_-(\varkappa)$ are identical to $\bar{K}(\varkappa)$ in the right and left half-planes, respectively. We also note that

$$K_+(\varkappa)-K_-(\varkappa)=\frac{3\pi}{2}\left(\frac{1}{\varkappa}+\frac{1}{\varkappa^3}\right), \quad K_+(-\varkappa)=K_-(\varkappa) \tag{A21'}$$

and that the functions $K_\pm(\varkappa)$ continue analytically on the entire complex $\varkappa$ plane with the cut along $(-i,i)$.

Defining

$$F_\pm(\varkappa)=\varkappa^2-\left(\frac{\alpha\varkappa^2}{3}+2i\frac{l^2}{\delta^2}\right)K_\pm(\varkappa), \qquad \text{(A21'')}$$

we obtain two analytical continuations of $F(\varkappa)$ onto the same region. The function $G_+(\varkappa)$ may also be analytically continued on the entire plane with a cut $(-i\infty,-i)$, the edges of the cut including. To do this, it is sufficient to convert (A9) into

$$G_+(\varkappa)=G_-(\varkappa)+\ln\frac{F(\varkappa)}{\varkappa_2-\varkappa_0^2}, \qquad \text{(A22)}$$

which yields

$$G_+(\varkappa-0)=G_-(\varkappa)+\ln\frac{F_-(\varkappa)}{\varkappa^2-\varkappa_0^2},$$

$$G_+(\varkappa+0)=G_-(\varkappa)+\ln\frac{F_+(\varkappa)}{\varkappa^2-\varkappa_0^2} \qquad \text{(A23)}$$

if we denote by $G_+(\varkappa-0)$ and $G_+(\varkappa+0)$ the limiting values of the function $G_+\,[\varkappa\in(-i\infty,-i)]$. From (A22),

$$\exp[G_+(\varkappa-0)]-\exp[G_+(\varkappa+0)]$$
$$=\frac{\exp[G_+(\varkappa-0)]}{F_-(\varkappa)}[F_-(\varkappa)-F_+(\varkappa)]$$

and

$$\exp[G_+(\varkappa-0)]-\exp[G_+(\varkappa+0)]$$
$$=\frac{\exp[G_-(\varkappa)]}{\varkappa^2-\varkappa_0^2}\left(\frac{\alpha\varkappa^2}{3}+2i\frac{l^2}{\delta^2}\right)[\bar K(\varkappa+0)-\bar K(\varkappa-0)]. \qquad \text{(A24)}$$

We turn our attention next to the A and B integrals. To get started, we displace their integration contours to the edges of the cut $(-i\infty,-i)$ to obtain

$$A=\frac{1}{2\pi i}\int_{-i\infty}^{-i}\left\{\frac{\alpha}{3}\frac{\varkappa}{\varkappa-\varkappa_0}[\bar K(\varkappa-0)-\bar K(\varkappa+0)]\exp[G_-(\varkappa)]+\ln\frac{F_-(\varkappa)}{F_+(\varkappa)}\right\}d\varkappa,$$

$$B=\frac{\alpha}{6\pi i}\int_{-i\infty}^{-i}[\bar K(\varkappa-0)-\bar K(\varkappa+0)]\frac{\exp[G_-(\varkappa)]}{\varkappa-\varkappa_0}d\varkappa,$$

which is fully legitimate since the integrands in (A20) fall off faster than $|\varkappa|^{-1}$ as $|\varkappa|\to\infty$.

With the aid of the identities (A23) and (A24), the last two expressions become

$$A=\frac{1}{2\pi i}\int_{-i\infty}^{-i}\left\{\frac{\alpha}{3}\frac{\varkappa(\varkappa+\varkappa_0)}{\alpha\varkappa^2/3+2il^2/\delta^2}\times\{\exp[G_+(\varkappa+0)]-\exp[G_+(\varkappa-0)]\}+G_+(\varkappa-0)-G_+(\varkappa+0)\right\}d\varkappa. \qquad \text{(A25)}$$

$$B=\frac{\alpha}{2\pi i}\int_{-i\infty}^{-i}\frac{\varkappa+\varkappa_0}{\alpha\varkappa^2/3+2il^2/\delta^2}\times\{\exp[G_+(\varkappa+0)]-\exp[G_+(\varkappa-0)]\}d\varkappa$$

and it is now helpful to compare these with the respective integrals

$$\frac{1}{2\pi i}\oint_{C_R}\left\{G_+(\xi)-\frac{\alpha\xi(\xi+\varkappa_0)}{3(\alpha\xi^2/3+2il^2/\delta^2)}\{\exp[G_+(\xi)]-1\}\right\}d\xi \qquad \text{(A26)}$$

and

$$\frac{1}{2\pi i}\oint_{C_R}\frac{1}{3}\frac{\alpha(\xi+\varkappa_0)}{\alpha\xi^2/3+2il^2/\delta^2}\{\exp[G_+(\xi)]-1\}d\xi, \qquad \text{(A27)}$$

where the contour C_R includes the twice-traversed cut $(-i,-iR)$ and the circle $|\xi|=R$, as shown in Fig. 1. From the estimate (A14) it immediately follows that for $R\to\infty$ the circle contributes vanishingly little into the integrals. On the other hand, the integrals (A26) and (A27) both become independent of R for $R>|\xi_0|$, where

$$\xi_0=\frac{l}{\delta}\left(-\frac{6i}{\alpha}\right)^{1/2}, \qquad \operatorname{Im}\xi_0>0$$

and we may then write

$$A=-\frac{\alpha}{6\pi i}\oint_{C_R}\frac{\xi(\xi+\varkappa_0)}{\alpha\xi^2/3+2il^2/\delta^2}\{\exp[G_+(\xi)]-1\}d\xi,$$

$$B=-\frac{\alpha}{6\pi i}\oint_{C_R}\frac{\xi+\varkappa_0}{\alpha\xi^2/3+2il^2/\delta^2}\{\exp[G_+(\xi)]-1\}d\xi.$$

Since $G_+(\xi)$ is analytic and single-valued inside C_R, the integrals A and B both reduce to the sum of residues at the points $\varkappa=\pm\xi_0$, giving

$$A=-\frac{1}{2}(\xi_0+\varkappa_0)\exp[G_+(\xi_0)]+\frac{1}{2}(\xi_0-\varkappa_0)\exp[G_+(-\xi_0)]+\varkappa_0,$$

$$B=-\frac{1}{2\xi_0}(\xi_0+\varkappa_0)\exp[G_+(\xi_0)]-\frac{1}{2\xi_0}(\xi_0-\varkappa_0)\exp[G_+(-\xi_0)]+1,$$

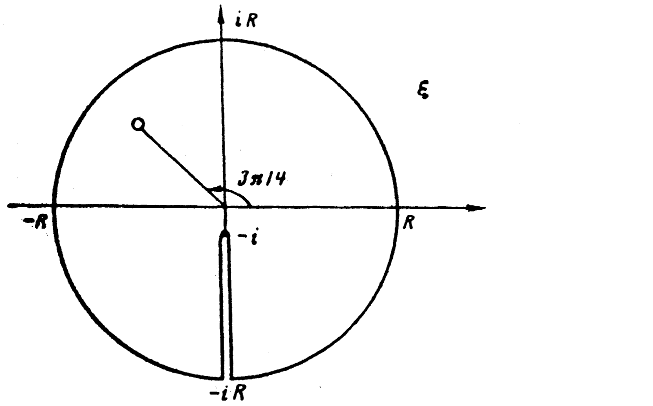

FIG. 1. C_R: integration contour for Eqs. (26) and (27).

which when substituted into (A19) leads to

$$\frac{E(0)}{E'(0)}$$
$$=\frac{1}{i\xi_0}\frac{(\xi_0+\varkappa_0)\exp[G_+(\xi_0)]+(\xi_0-\varkappa_0)\exp[G_+(-\xi_0)]}{(\xi_0+\varkappa_0)\exp[G_+(\xi_0)]-(\xi_0-\varkappa_0)\exp[G_+(-\xi_0)]}. \tag{A27'}$$

This may be given a somewhat simpler form by noting that

$$G_+(-\xi_0)=G_-(-\xi_0)+\ln\frac{F(-\xi_0)}{\xi_0^2-\varkappa_0^2},\quad F(\pm\xi_0)=\xi_0^2$$

and

$$G_-(\xi)=-G_+(\xi)$$

to give, finally,

$$\frac{E(0)}{E'(0)}=\frac{1}{i\xi_0}\frac{1+S^2}{1-S^2}, \tag{A28}$$

where

$$S=\frac{\xi_0}{\xi_0+\varkappa_0}\exp[-G_+(\xi_0)]. \tag{A29}$$

4. The calculation of the ratio $E(0)/E'(0)$ thus reduces to evaluating the integral

$$G_+(\xi_0)=\frac{1}{2\pi i}\int_{-i\varepsilon-\infty}^{-i\varepsilon+\infty}\frac{d\varkappa}{\varkappa-\xi_0}\ln\frac{F(\varkappa)}{\varkappa^2-\varkappa_0^2}. \tag{A30}$$

Let us transform the integral for $G_+(\xi)$ in (A10) into some closed-contour integral. This will be advantageous from a computational point of view and, most important, will simplify the analysis for the extreme cases $\alpha\ll 1$, $l\ll\delta$, or $l\gg\delta$.

As the first step, the contour of integration in (A10) [or (A30)] is displaced to the edges of the cut $(-i\infty,-i)$ to give

$$G_+(\xi_0)=\frac{1}{2\pi i}\int_{-i\infty}^{-i}\frac{d\varkappa}{\varkappa-\xi_0}\ln\frac{F(\varkappa-0)}{F(\varkappa+0)}, \tag{A31}$$

which may now be rewritten as

$$G_+(\xi_0)=\frac{1}{2\pi i}\int_{-i\infty}^{-i}\frac{d\varkappa}{\varkappa-\xi_0}\ln\frac{F_-(\varkappa)}{F_+(\varkappa)} \tag{A32}$$

or, referring to Fig. 1, as

$$G_+(\xi_0)=\frac{1}{(2\pi i)^2}\int_{C_R}\frac{d\varkappa}{\varkappa-\xi_0}\ln\frac{F_-(\varkappa)}{F_+(\varkappa)}\ln(\varkappa+i). \tag{A33}$$

The parameter R here is sufficiently large that the contour C_R encloses all the zeros of the functions $F_-(\varkappa)$ and $F_+(\varkappa)$ and we also note that the logarithms in the integrand are uniquely determined by the conditions

$$\ln\frac{F_-(\xi_0)}{F_+(\xi_0)}=0,\quad -\frac{\pi}{2}<\operatorname{Im}\ln(\varkappa+i)<\frac{3\pi}{2}.$$

The contour C_R in (A33) may be replaced by any other contour C which lies in the $\varkappa$ plane cut along $(-i,i)$; starts and terminates at the point $\varkappa=-i$ and encloses all the zeros of the functions $F_-(\varkappa)$ and $F_+(\varkappa)$. Thus

$$G_+(\xi)=\frac{1}{(2\pi i)^2}\oint_C\frac{d\varkappa}{\varkappa-\xi}\ln\frac{F_-(\varkappa)}{F_+(\varkappa)}\ln(\varkappa+i). \tag{A34}$$

With this result, we are now able to derive the asymptotic behavior of $G_+(\xi)$ and in particular to obtain the bound (A14). Let us deform the contour C in such a way as to have the point $\varkappa=\xi$ outside of it while preserving all the other relevant properties of the contour. Adding the relevant residue then gives

$$G_+(\xi)=\frac{1}{2\pi i}\ln\frac{F_-(\xi)}{F_+(\xi)}\ln(\xi+i)+\frac{1}{(2\pi i)^2}\oint_{C_0}\frac{d\varkappa}{\varkappa-\xi}\ln\frac{F_-(\varkappa)}{F_+(\varkappa)}\ln(\varkappa+i), \tag{A34'}$$

which implies that

$$G_+(\xi)=\frac{1}{2\pi i}\ln\frac{1-[\alpha/3+(2il^2/\delta^2)(1/\xi^2)]K_-(\xi)}{1-[\alpha/3+(2il^2/\delta^2)(1/\xi^2)]K_+(\xi)}+O(\xi^{-1}).$$

For large $|\xi|$ we have

$$\ln\frac{1-[\alpha/3+(2il^2/\delta^2)(1/\xi^2)]K_-(\xi)}{1-[\alpha/3+(2il^2/\delta^2)(1/\xi^2)]K_+(\xi)}\approx\left(\frac{\alpha}{3}+2i\frac{l^2}{\delta^2}\frac{1}{\xi^2}\right)[K_+(\xi)-K_-(\xi)]+O(\xi^{-2}).$$

By the identity (A34),

$$G_+(\xi)=\frac{\alpha}{6i}\frac{\ln\xi}{\xi}+O(\xi^{-2})$$

and the representation (A34′) shows that, for $\xi=\xi_0$,

$$G(\xi_0)=\frac{1}{(2\pi i)^2}\oint_{C_0}\frac{d\varkappa}{\varkappa-\xi_0}\ln\frac{F_-(\varkappa)}{F_+(\varkappa)}\ln(\varkappa+i).$$

Note. If $\alpha\to 0$, then $\xi_0\to\infty$ and to the first nonvanishing approximation,

$$G_+(\xi_0)=\frac{I}{\xi_0},$$
$$I=\frac{1}{4\pi^2}\int_C\ln\frac{\varkappa^2-(2il^2/\delta^2)K_-(\varkappa)}{\varkappa^2-(2il^2/\delta^2)K_+(\varkappa)}\ln(\varkappa+i)\,d\varkappa. \tag{A35}$$

Now for S and $E(0)/E'(0)$, by using (A28) and (A29) we obtain:

$$S=1+\frac{1}{\xi_0}(I-\varkappa_0),\quad \frac{E(0)}{E'(0)}=\frac{1}{i(I-\varkappa_0)},$$

which is consistent with (A21) in view of the identity

$$I=\frac{1}{2\pi i}\int_{-i\varepsilon-\infty}^{-i\varepsilon+\infty}\ln\frac{\varkappa^2-(2il^2/\delta^2)K_-(\varkappa)}{\varkappa^2-(2il^2/\delta^2)K_+(\varkappa)}d\varkappa.$$

5. We consider the case $l/\delta\gg 1$ next. Depending on their behavior in the $l/\delta\to\infty$ limit, the zeros of $F_\pm(\varkappa)$ can be of either the first kind (that remain in some fixed bounded region) or the second kind (that go to infinity). If $\varkappa$ is a zero of the second kind, then

$$K_\pm(\varkappa)\approx\pm\frac{3\pi}{4\varkappa},$$

and the equation

$$F_\pm(\varkappa)=0$$

may be approximated by

$$\varkappa^2 \mp \left(\frac{\alpha\varkappa^2}{3} + 2i\frac{l^2}{\delta^2}\right)\frac{3\pi}{4\varkappa} = 0.$$

From this we see that the function $F_+(\varkappa)$ has precisely three zeros of the second kind, whose approximate values are

$$\varkappa_j^+ \approx \lambda\theta_j^+, \; j=1,\, 2,\, 3,$$

where

$$\lambda = \left(\frac{3l^2}{2\pi\delta^2}\right)^{1/3}, \quad \theta_1^+ = e^{\pi i/6}, \quad \theta_2^+ = e^{5\pi i/6}, \quad \theta_3^+ = e^{-\pi i/2}. \tag{A36}$$

A similar argument shows that $F_-(\varkappa)$ also has three zeros of the second kind,

$$\varkappa_j^- \approx \lambda\theta_j^-,$$

with

$$\theta_1^- = e^{-\pi i/6},\; \theta_2^- = e^{\pi i/2},\; \theta_3^- = e^{7\pi i/6}. \tag{A37}$$

More precisely

$$\varkappa_j^+ = \lambda\theta_j^+ + \beta, \qquad \varkappa_j^- = \lambda\theta_j^- - \beta, \qquad \beta = \frac{\alpha\pi}{12} - \frac{4}{3\pi}. \tag{A38}$$

To transform the integral (A34) for $G_+(\xi_0)$ it is helpful to note that its integrand is a single-valued function on any closed contour C_1 which encloses the points $\varkappa_1^-$ and $\varkappa_1^+$; does not enclose any other zeros of F_- and F_+; and lies as a whole in the $\varkappa$ plane with a cut $(-i,i)$. The contour C_1 can be split off from C in the manner shown in Fig. 2. By applying the same procedure to the remaining two pairs $\varkappa_j^-$, $\varkappa_j^+$ $(j=2,3)$, we find that

$$G_+(\xi_0) = \sum_{j=1}^{3} I_j + I', \tag{A39}$$

where

$$I_j = \frac{1}{(2\pi i)^2}\oint_{C_j} \ln\frac{F_-(\varkappa)}{F_+(\varkappa)}\,\frac{\ln(\varkappa+i)}{\varkappa-\xi_0}\,d\varkappa,$$

$$I' = \frac{1}{(2\pi i)^2}\oint_{C'} \ln\frac{F_-(\varkappa)}{F_+(\varkappa)}\,\frac{\ln(\varkappa+i)}{\varkappa-\xi_0}\,d\varkappa,$$

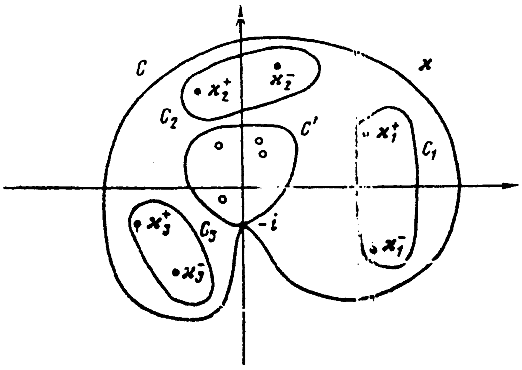

FIG. 2. Contours C_j $(j=1,2,3)$ are obtained by splitting off from the contour C which originates and terminates at the point $-i$ and encloses all the zeros of the functions $F_\pm(\varkappa)$. Contour C_j encloses only the zeros $\varkappa_j^\pm$, while contour C' encloses all the remaining zeros of $F_\pm(\varkappa)$ (marked by crosses).

and the contour C' only encloses the zeros of the first kind (see Fig. 2).

It is readily seen that

$$I_j = \frac{1}{2\pi i}\int_{\xi_j^-}^{\xi_j^+} \ln(\varkappa+i)\frac{d\varkappa}{\varkappa-\xi_0}$$

and with the accuracy up to values of order of ξ_j^{-1}, we obtain

$$I_j = -\frac{1}{2\pi i}\int_{\xi_j^-}^{\xi_j^+}\left(\frac{1}{\xi_0} + \frac{\varkappa}{\xi_0^2} + \frac{\varkappa^2}{\xi_0^3}\right)\ln(\varkappa+i)\,d\varkappa.$$

The above integrals are all expressible in terms of elementary functions giving

$$\sum_{j=1}^{3} I_j = \frac{\lambda}{\xi_0}\frac{e^{-\pi/3}}{3^{1/2}} + \frac{\lambda^2}{2\xi_0^2}e^{\pi i/3} + \frac{4i}{\pi^2}\frac{\ln\lambda}{\xi_0} + \frac{1}{\xi_0}\left(\frac{i}{2} - \frac{8}{3\pi} - \frac{\alpha\pi i}{12}\right) + o(\xi_0^{-1}).$$

We next turn to consider the integral I'. Since its integration variable is of order unity in absolute value, the integral is of the same order of magnitude as ξ_0^{-1}, and dropping the higher-order terms we may write

$$I' = -\frac{1}{\xi_0}\frac{1}{(2\pi i)^2}\oint_{C'} \ln\frac{K_-(\varkappa)}{K_+(\varkappa)}\ln(\varkappa+i)\,d\varkappa.$$

Precisely which branch of the logarithm

$$\ln\frac{K_-(\varkappa)}{K_+(\varkappa)}$$

should be taken here is totally immaterial because in view of the equation

$$\oint_{C'} \ln(\varkappa+i)\,d\varkappa = 0$$

their interchange has no effect on the value of the integral. By the same argument

$$I' = -\frac{1}{\xi_0}\frac{1}{(2\pi i)^2}\oint_{C'} \ln\left(-\frac{K_-(\varkappa)}{K_+(\varkappa)}\right)\ln(\varkappa+i)\,d\varkappa.$$

The quantity I' may now be transformed into an integral along the positive axis. Consider the integral

$$\mathcal{I} \equiv -\frac{1}{\xi_0}\frac{1}{2\pi i}\int_{-i\infty}^{-i}\left[\ln\left(-\frac{K_-(\varkappa)}{K_+(\varkappa)}\right) - \frac{8}{\pi\varkappa}\right]d\varkappa.$$

By deforming the integration contour we obtain

$$\mathcal{I} = -\frac{1}{\xi_0}\frac{1}{(2\pi i)^2}\oint_{C_R}\left[\ln\left(-\frac{K_-(\varkappa)}{K_+(\varkappa)}\right) - \frac{8}{\pi\varkappa}\right]\ln(\varkappa+i)\,d\varkappa. \tag{A40}$$

Noting that the contour C_R may be replaced by C' and that

$$\frac{1}{(2\pi i)^2}\oint_{C'}\frac{\ln(\varkappa+i)}{\varkappa}\,d\varkappa = \frac{1}{4i},$$

we find that

$$\mathcal{I} = I' + \frac{2}{i\pi\xi_0}. \tag{A41}$$

The integral $\tilde{I}$ may now be simplified by noting that the functions $-\varkappa K_-(\varkappa)$ and $\varkappa K_+(\varkappa)$ are complex conjugate so that

$$\ln\left(-\frac{K_-(\varkappa)}{K_+(\varkappa)}\right)=-2i\operatorname{arctg}\frac{\operatorname{Im}\varkappa K_+(\varkappa)}{\operatorname{Re}\varkappa K_+(\varkappa)},\qquad \varkappa\in(-i\infty,-i).$$

Noting, further, that

$$\ln\frac{1+i\varkappa}{1-i\varkappa}=\ln\left|\frac{1+i\varkappa}{1-i\varkappa}\right|+\pi i,$$

on the right edge of the $(-i\infty,-i)$ cut, and setting $\varkappa=-it$, we find that

$$\varkappa K_+(\varkappa)=\frac{3}{4}\left(1-\frac{1}{t^2}\right)(-i)\left[\ln\left|\frac{1+t}{1-t}\right|+i\pi\right]-\frac{3i}{2t},$$

giving

$$\frac{\operatorname{Im}\varkappa K_+(\varkappa)}{\operatorname{Re}\varkappa K_+(\varkappa)}=-\frac{1}{\pi}\left[\ln\frac{t+1}{t-1}+\frac{2t}{t^2-1}\right]$$

and

$$\tilde{I}=\frac{1}{i\pi\xi_0}\int_1^\infty\left\{\operatorname{arctg}\left[\frac{1}{\pi}\left(\ln\frac{t+1}{t-1}+\frac{2t}{t^2-1}\right)\right]-\frac{4}{\pi t}\right\}dt. \tag{A42}$$

Equations (A39) through (A42) yield the quantity $G_+(\xi_0)$ to $o(\xi_0^{-1})$, and $G_+(\xi_0)$ is:

$$G_+(\xi_0)=O\left(\frac{\lambda}{\xi_0}\right)=O\left(\frac{\delta^{1/2}}{l^{1/2}}\right).$$

Consequently,

$$\exp[-2G_+(\xi_0)]\approx1-2G_+(\xi_0)+2G_+^2(\xi_0)-\frac{4}{3}G_+^3(\xi_0)+o\left(\frac{\delta}{l}\right),$$

and we also note that

$$\left(\frac{\xi_0}{\xi_0+\varkappa_0}\right)^2=1-2\frac{\varkappa_0}{\xi_0}+3\frac{\varkappa_0^2}{\xi_0^2}-4\frac{\varkappa_0^3}{\xi_0^3}+o\left(\frac{\delta}{l}\right).$$

Equation (46) of the text is now obtained by merely substituting the above equations into (A28) and (A29) and dropping the terms $o(\delta/l)$.

6. Turning to the limit $l\ll\delta$ we confine our attention to the case where the stronger inequality

$$\frac{l}{\delta}\ll|\alpha|^{1/2}, \tag{A43}$$

also holds, thereby leaving out of account a small region of α values defined by

$$|\alpha|\lesssim l^2/\delta^2.$$

The inequality (A43) implies that $|\xi_0|\ll1$ and hence

$$G_+(\pm\xi_0)=G_+(0)\pm\xi_0G_+'(0)+O(\xi_0^2),$$

which on substituting into (A27′) yields

$$\frac{E(0)}{E'(0)}=\frac{1}{i\varkappa_0}\left[1+\frac{3\varkappa_0}{\alpha}G_+'(0)+O(\xi_0^2)\right].$$

For $l\ll\delta$, $\varkappa_0$ may be represented as

$$\frac{l}{\delta}\frac{1+i}{(1-\alpha/3)^{1/2}}+O\left(\frac{l^3}{\delta^3}\right)$$

and we are led to

$$\frac{E(0)}{E'(0)}=\frac{\delta}{l}\frac{(1-\alpha/3)^{1/2}}{-1+i}\times\left[1+\frac{l}{\delta}\frac{1+i}{(\alpha/3)(1-\alpha/3)^{1/2}}G_+'(0)\right]+O(\xi_0).$$

From (A10),

$$G_+'(0)=\frac{1}{2\pi i}\int_{-i\infty}^{-i}\frac{d\varkappa}{\varkappa^2}\ln\frac{F_-(\varkappa)}{F_+(\varkappa)}.$$

and so

$$G_+'(0)=\frac{1}{2\pi i}\int_{-i\infty}^{-i}\frac{d\varkappa}{\varkappa^2}\ln\frac{1-(\alpha/3)K_-(\varkappa)}{1-(\alpha/3)K_+(\varkappa)}+O\left(\frac{l^2}{\delta^2}\right).$$

The final result is conveniently written by defining

$$\Phi(\alpha)=\frac{1+i}{(\alpha/3)(1-\alpha/3)^{1/2}}\frac{1}{2\pi i}\int_{-i\infty}^{-i}\frac{d\varkappa}{\varkappa^2}\ln\frac{1-(\alpha/3)K_-(\varkappa)}{1-(\alpha/3)K_+(\varkappa)}$$

to give

$$\frac{E(0)}{E'(0)}=-\frac{\delta}{l}\left(\frac{1-\alpha/3}{2}\right)^{1/2}e^{-i\pi/4}\left[1+\frac{l}{\delta}\Phi(\alpha)\right]+O(\xi_0),\quad l\ll\delta|\alpha|^{1/2}. \tag{A44}$$

Note that as $\alpha\to0$, $\Phi(\alpha)$ tends to $(3/16)(1-i)$ and the right-hand side of the last equation becomes

$$-\frac{\delta}{l}\frac{e^{-i\pi/4}}{2}\left[1+\frac{l}{\delta}\frac{3}{16}(1-i)\right].$$

It is entirely straightforward to verify that in the small l/δ limit the exact formula (A35) leads to precisely the same result for $E(0)/E'(0)$. The implication of this is that apart from being true when (A43) holds, Eq. (A44) is also valid when $\alpha=0$ and (A43) accordingly breaks down.

7. In this section we show that the function $F(\varkappa)$ has precisely two zeros, $\pm\varkappa_0$, in the $\varkappa$ plane cut along $(-i\infty,-i)$ and $(i,i\infty)$. We denote by D the domain of definition of $F(\varkappa)$ and we include the edges of the cuts in this domain.

Referring to the analytical form of $F(\varkappa)$, Eqs. (A3) and (A4), it is seen that this function is bounded in D and tends to zero as $\varkappa\to\infty$. Let us show that the difference

$$1-\frac{\alpha}{3}\bar{K}(\varkappa) \tag{A45}$$

does not vanish in D. If $\varkappa$ is either real or belongs to the segment $(-i,i)$, it follows from (A4) that $\bar{K}(\varkappa)$ is positive and less than 3/2 in magnitude and hence the difference (A45) fails to vanish. Everywhere else in D, excluding for a moment the rims of the cuts (A45) is again nonzero because the imaginary part of $\bar{K}(\varkappa)$ is nonzero. And as for the rims, finally, here $\varkappa=it$ (for $t<-1$ or $t>1$), and $\bar{K}(\varkappa)$ is given by

$$\bar{K}(it)=-\frac{3}{4}\left(1-\frac{1}{t^2}\right)\frac{1}{t}\left[\ln\frac{t-1}{t+1}+i\pi\operatorname{sgn}t\right]+\frac{3}{2t^2}, \tag{A46}$$

so that the imaginary part of $\bar{K}(\varkappa)$ is again nonzero.

It thus follows that for any bounded and closed subdomain of D, a positive number must exist which bounds the absolute value of (A45) from below. Since (A45) tends to unity as $\varkappa \to \infty$, the lower bound is

$$\inf|1-(\alpha/3)\bar{K}(\varkappa)|>0,$$

showing that the function

$$\mu(\varkappa)=\frac{\bar{K}(\varkappa)}{1-(\alpha/3)\bar{K}(\varkappa)}$$

is bounded in D:

$$|\mu(\varkappa)|<M.$$

If $\varkappa$ is a zero of $F(\varkappa)$, then

$$\varkappa^2=2i\frac{l^2}{\delta^2}\mu(\varkappa). \tag{A47}$$

This implies that the zeros of $F(\varkappa)$ are all inside the circle

$$|\varkappa|\leqslant\frac{l}{\delta}(2M)^{1/2} \tag{A48}$$

and hence tend to zero as $l/\delta \to 0$, which enables them to be found (in this limit) by means of the iterative procedure defined by

$$\varkappa_{n+1}=\pm\frac{l}{\delta}(2i)^{1/2}\mu(\varkappa_n),\quad \varkappa_1=0.$$

To a first nonvanishing approximation we obtain the result

$$\varkappa=\pm\frac{l}{\delta}\frac{1+i}{(1-\alpha/3)^{1/2}},$$

accurate to l^3/δ^3. We have thus shown that, for l/δ values sufficiently small, the function $F(\varkappa)$ has precisely two zeros in D, which we are free to denote as $\pm\varkappa_0$.

It remains to show that varying the parameter l/δ does not alter the number of zeros of $F(\varkappa)$. Assuming the opposite we would be obliged to admit that as l/δ tends to some critical value, l_0/δ_0, at least one of the zeros must either go to infinity or to one of the rims (that is, to the boundary of D). The former possibility is excluded in view of the estimate (A48). In what follows, the latter possibility is also ruled out.

Suppose $F(\varkappa)=0$ at, say, $\varkappa=it$ ($t>1$, $\arg\varkappa=\pi/2$). But then $\operatorname{Re}\mu(\varkappa)=0$ in view of (A47) and setting

$$\bar{K}(it)=K_1(t)+iK_2(t),$$

we find that

$$\frac{3}{\alpha}=K_1+\frac{K_2^2}{K_1}. \tag{A49}$$

From (A46),

$$K_1(t)=\frac{3}{4}\left(1-\frac{1}{t^2}\right)\frac{1}{t}\ln\frac{t+1}{t-1}+\frac{3}{2t^2}, \tag{A50}$$

$$K_2(t)=-\frac{3\pi}{4}\left(1-\frac{1}{t^2}\right). \tag{A51}$$

The function $K_1(t)$ varies in the range

$$\frac{3}{t^2}\left(1-\frac{1}{2t^2}\right)\leqslant K_1(t)\leqslant\frac{3}{t^2},\quad t>1, \tag{A52}$$

which follows from the strict inequalities

$$\sigma<\frac{1}{2}\ln\frac{1+\sigma}{1-\sigma}<\frac{\sigma}{1-\sigma^2},\quad 0<\sigma<1.$$

Equation (A49) fails if $\alpha<0$ and we are therefore left with the case $\alpha>0$. Clearly, in this case (A49) is false unless

$$K_1+\frac{K_2^2}{K_1}\geqslant 3, \tag{A53}$$

which is equivalent to

$$K_1(3-K_1)<K_2^2.$$

According to (A52), the left-hand side of this last inequality is greater than

$$\frac{3}{t^2}\left(1-\frac{1}{2t^2}\right)\left(3-\frac{3}{t^2}\right).$$

On the other hand, using (A51) gives

$$\frac{3}{t^2}\left(1-\frac{1}{2t^2}\right)\left(3-\frac{3}{t^2}\right)>K_2^2(t)$$

in contradiction with (A53).

Thus none of the zeros of the function $F(\varkappa)$ can possibly reach the edges of the cuts, which means that whatever the value of l/δ, the number of the zeros is the same and hence is 2.

We note, finally, that for $\varkappa\in(-i,i)$ the imaginary part of the function $F(\varkappa)$ is negative and hence cannot be zero, with the consequence that whatever the value of l/δ, neither of the zeros $\pm\varkappa_0$ ever leaves its quadrant.

[1)] We employ the following common notation: $\mathbf{p},\mathbf{v}=\mathbf{p}/m$, ε, m, and e are respectively the momentum, velocity, energy, mass, and charge of an electron; the subscript F refers to the Fermi electrons; $\sigma=ne^2l/p_F$ is the conductivity; n the concentration of conduction electrons; l the mean free path; $\tau=l/v_F$ the electron lifetime; $\delta=c/(2\pi\sigma\omega)^{1/2}$ is the ordinary skin-layer depth; ω the frequency of the electromagnetic wave; c the speed of light.

[2)] Since the wave frequency ω is only moderately high ($\omega\tau\ll1$), we are justified in having dropped the term $\partial\chi/\partial t$ in (6). If $\omega\tau\gtrsim1$, $1/l$ should be replaced by $1/l-i\omega/v_F$ throughout.

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