Application of the method of dynamical quantization to the theory of gravity in (2+1)-dimensional space

S. N. Vergeles

L. D. Landau Institute of Theoretical Physics, Russian Academy of Sciences, Moscow
(Submitted 11 June 1992)
Zh. Eksp. Teor. Fiz. **102**, 1739–1761 (December 1992)

The Hamiltonian quantization of the $(2+1)$-dimensional theory of gravity coupled with a Dirac field is performed on the basis of the dynamical-quantization method developed previously by the author. The space of the regularized states of the system is constructed. A perturbation theory in terms of the gravitational constant is developed, and is used to solve the regularized Heisenberg equations.

1. INTRODUCTION

The idea of dynamical quantization of field theories was formulated by the author in Refs. 1. Here this method is applied to the quantization of $(2+1)$-dimensional gravity. The result obtained gives us grounds to suppose that the method of dynamical quantization can be used to construct a consistent quantum theory of gravity (and of other generally covariant field theories, such as, e.g., the theory of a relativistic string).

We now give a schematic outline of the idea of the method of dynamical quantization.

Consider a particular field theory. Assume that in this theory the physical degrees of freedom in the ultraviolet region can by classified by modes with the following properties:

a) The occupation numbers of these modes are conserved or are adiabatic invariants of the motion;

b) the corresponding creation operators a_N^+ and annihilation operators a_N are gauge-invariant.

The properties (a) and (b) are fundamental for the method of dynamical quantization. The regularized theory is developed by imposing the constraints of the second kind

$$a_N{}^+\approx 0, \quad a_N\approx 0 \tag{1.1}$$

for quantum numbers N from the ultraviolet tail. The constraints (1.1) cause the initial formal commutation relations (CR) to be replaced by the corresponding Dirac CR. Next one must solve the Heisenberg equations obtained by means of the Dirac CR. Thus, the theory becomes regularized with a definite number of physical degrees of freedom. Moreover, since the creation and annihilation operators in Eqs. (1.1) are gauge-invariant, the imposition of the constraints (1.1) does not destroy the gauge invariance.

We must draw attention to the difference between Feynman quantization and dynamical quantization of field theories. Feynman quantization is based on the hypothesis that the interaction can be switched on and off adiabatically. This hypothesis is equivalent to the assumption that the physical vacuum differs little from the naive vacuum (with the interaction switched off). This assumption leads to the Feynman rules. But, generally speaking, Feynman perturbation theory cannot be regularized in such a way that the number of physical degrees of freedom is well defined. As Gribov noted,[2] this nonconservation of the total number of degrees of freedom of the system leads to the appearance of gauge anomalies.

Consider, for example, Weyl electrodynamics, in which the gauge field interacts minimally with a right-handed (left-handed) Weyl field. Place this system in a spatial box. Let the Weyl field be expanded in plane waves or in some other modes, and let a_N and a_N^+ denote the corresponding annihilation and creation operators. Then regularization of the theory in accordance with (1.1) leads to the appearance of a gauge anomaly or to nonconservation of the fermion current. The reason for the appearance of the anomaly is that the equations of motion relate all the operators a_N and a_N^+, provided that the chosen modes do not satisfy special conditions. The dynamical coupling between the operators a_N and a_N^+ implies that the imposition of conditions of the form (1.1) is dynamically inconsistent. It follows from this that there is in the system a flow of particles from the region of the ultraviolet tail into the region of the physically accessible final energies, and vice versa. For this reason, the regularized fermion current is not conserved when the constraints (1.1) are imposed. The occurrence of a gauge anomaly in Feynman PT for any regularization indicates that in Feynman theory it is impossible to perform a regularization that is completely consistent with the dynamics of the system. The hypothesis of adiabatic switching on and off turns out to be extremely restrictive in this aspect. Here it should be noted that in Dirac massless electrodynamics the right and left Fermi currents are not conserved separately, but their sum is conserved in the Feynman theory.

If, however, the fermion modes considered above are not constants in time, but vary in accordance with the equations of motion in such a way that the annihilation and creation operators corresponding to them are conserved by virtue of the dynamical equations, then imposition of the constraints (1.1) is possible. Dynamical quantization is thereby realized. In this case the gauge anomaly is absent. This approach to the study of the gauge anomaly was applied in Ref. 3. A deficiency of Ref. 3 was the fact that the gauge field remained unquantized. In addition, in dynamical quantization it is impossible to state that stationary states exist in the theory. The latter circumstance is not important in gravitational theory.

It appears to us that, in generally covariant theories such as gravitational theory, upon dynamical quantization a consistent quantum theory free of "harmful" anomalies can be constructed in a natural manner.

We note also that the Feynman rules admit Wick rotation, as a result of which the equivalence between $[(D-1)+1]$-dimensional quantum field theories and D-

dimensional classical statistical models is established. Therefore, Euclidean quantization is automatically equivalent to Feynman quantization.

In this paper we present a dynamical quantum theory of gravity interacting with a Dirac massless field in $(2+1)$-dimensional space. The space of the regularized states of the system is constructed, and the regularized Heisenberg equations for the field operators are written out. Next, we develop a mathematically correct PT in the gravitational constant and use it to carry out certain calculations. The construction of the PT in the gravitational constant is the main achievement of this paper.

Note that the $(2+1)$-dimensional theory of gravity is close in a certain sense to the chiral Schwinger model studied previously by the method of dynamical quantization.[1] In fact, in both theories the gauge fields (the connection and the triads in the theory of gravity) do not have their own independent local degrees of freedom (the photon is absent for $D=1+1$, and the gravitation is absent for $D=2+1$).

2. THE LAGRANGIAN AND HAMILTONIAN OF THE SYSTEM

Let $x^\mu=(x^0,x^1,x^2)$ denote local coordinates in some three-dimensional metric space: the metric is represented in the form

$$ds^2=g_{\mu\nu}\,dx^\mu\,dx^\nu=e_\mu{}^a e_{a\nu}\,dx^\mu\,dx^\nu, \tag{2.1}$$

where e^a_μ are triads. The Latin indices $a, b, \ldots = 0, 1, 2$ pertain to local Lorentz frames, and the diagonal matrix η_{ab} is defined as follows: $\eta_{ab}=\mathrm{diag}(1,-1,-1)$. Let $g^{\mu\nu}$ be the inverse metric tensor, and

$$e_a{}^\mu=g^{\mu\nu}\eta_{ab}e_\nu{}^b.$$

Then

$$e_\mu{}^a e_b{}^\mu=\delta_b{}^a,\quad e_a{}^\mu e_\nu{}^a=\delta_\nu{}^\mu. \tag{2.2}$$

The connection 1-form is denoted by $\omega^a_b=\omega^a_{b\mu}\,dx^\mu$. Since the connection is consistent with the metric, we have

$$\omega_{ab\mu}=\eta_{ac}\omega_{b\mu}{}^c=-\omega_{ba\mu}. \tag{2.3}$$

Following Witten,[4] we shall use the equivalent quantity

$$\omega_\mu{}^c={}^1/_2\varepsilon^{abc}\omega_{ab\mu}. \tag{2.4}$$

In this notation the scalar curvature has the form

$$g^{1/2}R=\varepsilon^{\mu\nu\lambda}e_{c\lambda}(\partial_\mu\omega_\nu{}^c-\partial_\nu\omega_\mu{}^c-\varepsilon_{ab}{}^c\omega_\mu{}^a\omega_\nu{}^b),$$
$$g=\det g_{\mu\nu}. \tag{2.5}$$

As the Dirac matrices γ^a, satisfying the conditions

$$\gamma^a\gamma^b+\gamma^b\gamma^a=2\eta^{ab}, \tag{2.6}$$

we take the Pauli matrices

$$\gamma^0=\sigma_z,\quad \gamma^1=i\sigma_x,\quad \gamma^2=i\sigma_y. \tag{2.7}$$

We denote by ψ and $\bar\psi=\psi^+\gamma^0$ the Dirac two-component complex field. The covariant Dirac operator is represented as follows:

$$i\gamma^a\nabla_a=i\gamma^a e_a{}^\mu(\partial_\mu-{}^1/_2 i\omega_{b\mu}\gamma^b). \tag{2.8}$$

According to (2.5) and (2.8), the simplest generally covariant action has the form

$$A=\int d^3x\Big\{-\frac{1}{16\pi G}\varepsilon^{\mu\nu\lambda}e_{c\lambda}(\partial_\mu\omega_\nu{}^c-\partial_\nu\omega_\mu{}^c-\varepsilon_{ab}{}^c\omega_\mu{}^a\omega_\nu{}^b)$$
$$+{}^1/_2 i(\bar\psi\Gamma^\mu\nabla_\mu\psi-\nabla_\mu\bar\psi\Gamma^\mu\psi)\Big\}, \tag{2.9}$$

where

$$\nabla_\mu\psi=(\partial_\mu-{}^1/_2 i\omega_{a\mu}\gamma^a)\psi,$$
$$\nabla_\mu\bar\psi=\partial_\mu\bar\psi+{}^1/_2 i\bar\psi\gamma^a\omega_{a\mu}, \tag{2.10}$$

$$\Gamma^\mu=g^{1/2}e_a{}^\mu\gamma^a={}^1/_2\varepsilon^{\mu\nu\lambda}\varepsilon_{abc}e_\nu{}^b e_\lambda{}^c\gamma^a. \tag{2.11}$$

Note that the fermion part of the action (2.9) has been symmetrized to make the action Hermitian. In fact, the expression

$$\int d^3x i\bar\psi\Gamma^\mu\nabla_\mu\psi$$

is not Hermitian if the connection has torsion. But the equations of motion that follow from the action (2.9) lead to a connection with torsion.

Everywhere below we shall use the letters $x, y, \ldots$ to denote the set of spatial coordinates (x^1,x^2), (y^1,y^2), ..., the time argument x^0 will be omitted, and a dot over a symbol will denote the derivative $\partial/\partial x^0$. In addition, we introduce the notation

$$d^2x=dx^1dx^2,\quad \delta^{(2)}(x)=\delta(x^1)\delta(x^2).$$

In the derivation of the Hamiltonian from the action (2.9) the following difficulty arises. Since the action (2.9) contains the quantities ψ and $\bar\psi$, both the fields ψ and $\bar\psi$ must be regarded as coordinate variables. However, the corresponding momentum variables π_ψ and $\pi_{\bar\psi}$ are proportional to these same fields ψ and $\bar\psi$. Therefore, the following constraints hold:[1)]

$$\bar\tau=\pi_\psi-{}^1/_2\bar\psi\Gamma^0\approx0,$$
$$\tau=\pi_{\bar\psi}-{}^1/_2\Gamma^0\psi\approx0. \tag{2.12}$$

Thus, the Hamiltonian dynamical variables are combined into the following pairs:

$$(\psi,\pi_\psi),\quad(\bar\psi,\pi_{\bar\psi}),\quad(\omega_i{}^a,P_a{}^i),$$
$$P_a{}^i=-\frac{1}{8\pi G}\varepsilon_{ij}e_{aj}.$$

The Latin indices $i, j, \ldots$ are spatial indices, taking the values 1, 2; $\varepsilon_{ij}=-\varepsilon_{ji}$. The fields $\omega^a_0=\omega^a$ and $e^a_0=e^a$ play the role of Lagrange multipliers.

We shall need the parity function α, defined on uniform fields or operators with values in the group Z_2. By definition,

$$\alpha(\omega^a)=\alpha(e^a)=\alpha(\omega_i{}^a)=\alpha(P_a{}^i)=0,$$
$$\alpha(\psi)=\alpha(\pi_\psi)=\alpha(\bar\psi)=\alpha(\pi_{\bar\psi})=1.$$

If the function α is defined on operators A and B, we set $\alpha(AB)=[\alpha(A)+\alpha(B)]/\mathrm{mod}\,2$. Throughout, by the commutator of two homogeneous operators A and B we shall mean the expression

$$[A,B]=AB-BA(-1)^{\alpha(A)\alpha(B)}. \tag{2.13}$$

The initial nonzero single-time CR are written as follows:

$$[\omega_i{}^a(x),P_b{}^j(y)]=i\delta_{ij}\delta_b{}^a\delta^{(2)}(x-y),$$
$$[\psi(x),\ \pi_\psi(y)]=1\cdot\delta^{(2)}(x-y),$$
$$[\bar\psi(x),\pi_{\bar\psi}(y)]=1\cdot\delta^{(2)}(x-y).$$

The Hamiltonian of the system has the form

$$H=H_\chi+H_\varphi,$$

$$H_\chi=-\int d^2x\,\omega^a\chi_a, \tag{2.14a}$$

$$H_\varphi=\frac{1}{8\pi G}\int d^2x\,e_a\varphi^a,$$

where

$$\chi_a=\partial_i P_a{}^i-{}^1/_2\varepsilon_{ab}(\omega_i{}^b P_c{}^i+P_c{}^i\omega_i{}^b)+{}^1/_2\bar\psi g^{1/2}e_a{}^0\psi, \tag{2.14b}$$

$$\begin{aligned}\varphi^a={}&{}^1/_2\varepsilon_{ij}(\partial_i\omega_j{}^a-\partial_j\omega_i{}^a-\varepsilon_{bc}{}^a\omega_i{}^b\omega_j{}^c)\\&+{}^1/_2 i(8\pi G)^2\varepsilon_b{}^{ca}(\bar\psi\gamma^b P_c{}^i\partial_i\psi-\partial_i\bar\psi P_c{}^i\gamma^b\psi)\\&+{}^1/_4(8\pi G)^2\varepsilon_b{}^{ca}\bar\psi\{P_c{}^i\omega_{di}[s\gamma^b\gamma^d+(1-s)\gamma^d\gamma^b]\\&+\omega_{di}P_c{}^i[(1-s)\gamma^b\gamma^d+s\gamma^d\gamma^b]\}\psi.\end{aligned} \tag{2.14c}$$

The quantity $g^{1/2}e_a^\mu$ must be expressed in terms of the canonical variables:

$$\begin{aligned}g^{1/2}e_a{}^0&={}^1/_2(8\pi G)^2\,\varepsilon_a{}^{bc}\varepsilon_{ij}P_b{}^iP_c{}^j,\\ g^{1/2}e_a{}^i&=-e_c(8\pi G)\varepsilon_a{}^{bc}P_b{}^i.\end{aligned} \tag{2.15}$$

In (2.14c) s is a free real parameter. The existing arbitrariness in the choice of the parameter s implies arbitrariness in the ordering of the operators in the Hamiltonian (2.14). This arbitrariness will be eliminated below.

Up to the end of Sec. 2 we shall ignore the noncommutativity of the operators, and, therefore, we shall not keep track of their ordering. At the same time we shall regard the bracket [..., ...] as a classical Poisson bracket.

According to the general scheme for the construction of the Hamiltonian mechanics for systems that are invariant under arbitrary coordinate transformations, the Hamiltonian should be a constraint of the first kind.[6] In our case this implies that

$$\begin{aligned}&\tilde\chi_a\approx0,\quad \tilde\varphi^a\approx0,\\ &[\tilde\chi_a,\tau]\approx0,\quad [\tilde\varphi^a,\tau]\approx0,\quad [\tilde\chi_a,\tilde\chi_b]\approx0,\\ &[\tilde\chi_a,\tilde\varphi^b]\approx0,\quad [\tilde\varphi_a,\tilde\varphi_b]\approx0.\end{aligned} \tag{2.16}$$

Here the quantities $\tilde\chi^a$ and $\tilde\varphi_a$ can differ from χ^a and φ_a by a linear combination of the constraints τ and $\bar\tau$. From the definition (2.12) and the initial CR the value of the Poisson bracket is easily derived:

$$[\bar\tau_\sigma(x),\tau_\rho(y)]=-\delta^{(2)}(x-y)\Gamma_{\rho\sigma}{}^0(x), \tag{2.17}$$

where $\sigma, \rho=1, 2$ are spinor indices. Using (2.13) and (2.17), it is not difficult to establish that the quantities

$$\tilde\chi_a=\chi_a+{}^1/_2\bar\psi\gamma_a\tau+{}^1/_2\bar\tau\gamma_a\psi,$$
$$\tilde\varphi^a=\varphi^a+i(8\pi G)^2\varepsilon_b{}^{ca}P_c{}^i\{\bar\tau(\Gamma^0)^{-1}\gamma^b\nabla_i\psi-\nabla_i\bar\psi^b(\Gamma^0)^{-1}\tau\}$$

satisfy the conditions (2.16). The relations (2.16) and (2.17) imply that on the classical level the constraints (2.12) can be regarded as constraints of the second kind. This gives the possibility of going over to the corresponding Dirac commutation relations, after which the constraints (2.12) can be set equal to zero in the strong sense.

3. FORMAL COMMUTATION RELATIONS

We proceed to the Dirac commutation relations that arise when the constraints (2.12) of the second kind are imposed.

Obviously, the quantity

$$-\delta^{(2)}(y-z)(\Gamma^0)_{\sigma\tau}{}^{-1}(z)$$

is the inverse of the quantity (2.17). (The definition of the quantity Γ^μ is given by (2.11) and (2.15).) Let $A, B, C, \ldots$ denote the homogeneous fundamental fields ω_i^a, P_a^i, ψ, $\bar\psi$. The Dirac CR are defined as follows:

$$\begin{aligned}[A,B]^*={}&[A,B]+\int d^2z\{[A,\bar\tau_\sigma(z)](\Gamma^0)_{\sigma\rho}{}^{-1}(z)[\tau_\rho(z),B]\\&-(-1)^{\alpha(A)\alpha(B)}[B,\bar\tau_\sigma(z)](\Gamma^0)_{\sigma\rho}{}^{-1}(z)[\tau_\rho(z),A]\}.\end{aligned} \tag{3.1}$$

Let us clarify the meaning of the definition (3.1). Since τ and $\bar\tau$ are Fermi operators it is useful to follow the transition to the Dirac commutation relations in a system containing only one pair of fermion creation and annihilation operators a^+ and a, which should be set to zero. The CR

$$[a^+,a]=1, \tag{3.2}$$

and the constraints

$$a^+\approx0,\quad a\approx0. \tag{3.3}$$

hold. For arbitrary operators the following representation is valid:

$$\begin{aligned}A&=K_{00}+K_{10}a^++K_{01}a+K_{11}a^+a,\\ B&=L_{00}+L_{10}a^++L_{01}a+L_{11}a^+a.\end{aligned} \tag{3.4}$$

The operators K_{ij} and L_{ij} ($i, j=0, 1$) in (3.4) can depend on any of the boson and fermion operators except the operators a^+ and a.

We define the Dirac CR analogous to (3.1) in the case when the constraints (3.3) are imposed:

$$[A,B]^*=[A,B]-\{(A,a^+][a,B]-(-1)^{\alpha(A)\alpha(B)}[B,a^+][a,A]\}. \tag{3.5}$$

To avoid ambiguities it is necessary to adopt the following rule:

Rule 1: Before the constraints (3.3) [or (2.12)] are imposed, the operators a^+ and a (or $\bar\tau$ and τ) should be normally ordered, i.e., the operators a^+ (or $\bar\tau$) should stand to the left of the operators a (or τ).

Taking this rule into account, using the CR (3.2), it is easy to find that

$$[A,B]^*=[K_{00},L_{00}]. \tag{3.6}$$

Thus, the Dirac CR (3.5), after imposition of the constraints (3.3), can be expressed in terms of ordinary CR in accordance with (3.4) and (3.6). Obviously, we can also assert that the Dirac CR (3.1), after imposition of the constraints (2.12), are expressed in terms of CR of the form (2.13). It follows from this that the Dirac CR (3.1) satisfy all the general properties of arbitrary CR; namely, for arbitrary complex numbers x and y the following equalities hold:

$$\begin{aligned}&[A,B]^*=-[B,A]^*(-1)^{\alpha(A)\alpha(B)},\\ &[xA+yB,C]^*=x[A,C]^*+y[B,C]^*,\end{aligned} \tag{3.7}$$

which is directly obvious from the definition (3.1). In addition,

$$[A, BC]^{*}=[A, B]^{*}C+B[A, C]^{*}(-1)^{\alpha(A)\alpha(B)},$$
$$[A, [B, C]^{*}]^{*}(-1)^{\alpha(A)\alpha(C)}+[B, [C, A]^{*}]^{*}(-1)^{\alpha(A)\alpha(B)}+[C, [A, B]^{*}]^{*}(-1)^{\alpha(B)\alpha(C)}=0. \quad (3.8)$$

We now write out the nonzero single-time Dirac commutation relations (3.1) for the fundamental fields:

$$[\psi_\sigma(x), \bar{\psi}_\rho(y)]=\delta^{(2)}(x-y)(\Gamma^0)_{\sigma\rho}{}^{-1}(y),$$
$$[P_a{}^i(x), \omega_j{}^b(y)] = -i\delta_{ij}\delta_a{}^b\delta^{(2)}(x-y),$$
$$[\omega_i{}^a(x), \psi(y)] = {}^1/_2 i\delta^{(2)}(x-y)(8\pi G)^2\varepsilon_{ij}\varepsilon_b{}^{ca}(P_c{}^j(\Gamma^0)^{-1}\gamma^b\psi)(y), \quad (3.9)$$
$$[\omega_i{}^a(x), \bar{\psi}(y)] = {}^1/_2 i\delta^{(2)}(x-y)(8\pi G)^2\varepsilon_{ij}\varepsilon_b{}^{ca}(\bar{\psi}\gamma^b(\Gamma^0)^{-1}P_c{}^j)(y),$$
$$[\omega_i{}^a(x), \omega_j{}^b(y)] = \frac{i}{2}\delta^{(2)}(x-y)(8\pi G)^2\varepsilon_{ij}\left(\bar{\psi}\frac{e^{0a}e^{0b}}{e_c{}^0e^{0c}}\psi\right)(y).$$

The quantities e_a^0 are given by (2.15). Here and below, we omit the asterisk in the symbol of the Dirac CR (3.1); everywhere below, [..., ...] denotes the Dirac CR (3.1), and the constraints (2.12) are set equal to zero in the strong sense. The formulas (3.7)–(3.9) inductively determine the Dirac commutation relations for any functionals of the fundamental fields ω_i^a, P^i_a, ψ, $\bar{\psi}$.

4. EXTRACTION OF THE GAUGE-INVARIANT DEGREES OF FREEDOM AND REGULARIZATION

To perform the quantization we must construct the states that are annihilated by the operators (2.14) and solve the Heisenberg equations $i\dot{A} = [A, H]$, where A is any of the fundamental fields (or a functional of them) and H is given by (2.14).

However, this problem can be solved only jointly with the problem of extracting the gauge-invariant degrees of freedom and regularizing the theory. In this section the latter problem is solved.

The regularization of the theory is performed in Sec. 4.2. Therefore, at first sight, the account in Sec. 4.1 has a heuristic character. However, all results needed in this subsection are rigorously justified below. These justifications are based in essence on the fact that the regularized Heisenberg equations for the fundamental fields have the same form as the formal equations. Therefore, the proposed construction is self-consistent.

4.1. Extraction of the conserved gauge-invariant fermion variables

Let the Heisenberg equation for the Dirac field have the form

$$i\dot{\psi}=\mathcal{O}\psi,$$
$$\mathcal{O} = -{}^1/_2\omega_a\gamma^a-i(\Gamma^0)^{-1}\Gamma^i\nabla_i \quad (4.1)$$

and let $\{\psi_N(x)\}$ be a complete set of boson spinor fields satisfying (4.1). This implies that at any moment of time the following formulas hold:

$$\int d^2x\bar{\psi}_M\Gamma^0\psi_N=\delta_{MN}, \quad (4.2a)$$
$$\sum_N \psi_N(x)\bar{\psi}_N(y)=\delta^{(2)}(x-y)(\Gamma^0)^{-1}(y). \quad (4.2b)$$

The set of functions $\{\psi_N\}$ can be obtained as follows. Let $\{\varkappa_N\}$ be a complete set of c-number spinor boson fields, satisfying the conditions

$$\int d^2x\varkappa_M{}^+\varkappa_N=\delta_{MN}. \quad (4.3)$$

Then for the fields

$$\psi_N=(\Gamma^0(t_0))^{-1/2}(\gamma^0)^{1/2}\varkappa_N \quad (4.4)$$

the conditions (4.2) are fulfilled at the time t_0.

If the fields ψ_N satisfy Eq. (4.1) and the quantity Γ^0 varies in accordance with the Heisenberg equations, the conditions (4.2) remain valid with the passage of time (as shown below, both in the formal theory and in the regularized theory). Therefore, we must solve (4.1) with the initial conditions (4.3) and (4.4). The problem of the determination of the complete set of spinor fields with the properties (4.2) is thereby solved.

It turns out that the fields constructed in this way possess a further important property: The right-hand side of Eq. (4.1) for the fields ψ_N can be represented in the form of the Heisenberg commutator $[\psi_N, H]$. The proof of this fact is given in the Appendix.

Thus, (4.1) can also be written in the form

$$i\dot{\psi}_N=\mathcal{O}\psi_N=[\psi_N, H]. \quad (4.5)$$

From the relations (4.5) it follows that

$$[\psi_M, \psi_N]=[\bar{\psi}_M, \psi_N]=0. \quad (4.6)$$

In fact, the CR (4.6) are valid at $t = t_0$ as a consequence of the definition (4.4) and the CR (3.9). But it follows from (4.5) that the evolution in time reduces to a unitary transformation. Therefore, the CR (4.6) is preserved in time.

Now we can define conserved fermion creation operators a_n^+ and annihilation operators a_N with the commutation properties

$$[a_M{}^+, a_N]=\delta_{MN}, \quad [a_M, a_N]=0,$$
$$[a_N, \psi_M]=[a_N{}^+, \psi_M]=0 \quad (4.7)$$

and

$$[a_N, \chi_a]=0, \quad [a_N, \varphi_a]=0. \quad (4.8)$$

It follows from (4.8) and (4.5) that all the CR are dynamically reproduced.

The relations (4.2) and the CR (4.6), (4.7) make it possible to represent the Fermi fields in the form

$$\psi(x)=\sum_N a_N\psi_N(x),$$
$$\bar{\psi}(x)=\sum_N a_N{}^+\bar{\psi}_N(x). \quad (4.9)$$

It is obvious that the fields (4.9) satisfy the correct CR (3.9). From what has been said it also follows that the creation operators can be represented in the form

$$a_N{}^+=\int d^2x\bar{\psi}\Gamma^0\psi_N. \quad (4.10)$$

(If particular formulas are written out for the quantities a_N or ψ, the analogous formulas for a_N^+ or $\bar{\psi}$ are obtained by Hermitian conjugation.)

Remark: The outcome of Sec. 4.1 is the extraction of gauge-invariant dynamical variables. In a number of gauge theories this problem is easily solved. For example, consider in flat space a set of modes $\{\psi_N\}$ of the Dirac operator $-i\gamma^i\nabla_i$, where $\nabla_i = \partial_i - iA_i$. Then in the expansion (4.9) the operators a_N and a_N^+ are gauge-invariant. However, in contrast with an ordinary gauge theory (such as electrodynamics), in gravitational theory the gauge transformations exhaust the entire dynamics of the system.

4.2. The imposition of regularizing constraints

As has been elucidated by Witten,[4] in the absence of matter quantum gravity in (2 + 1)-dimensional space is not only a renormalizable theory but is even finite (in the framework of Feynman theory). However, the introduction of the Dirac field makes Feynman quantization impossible, since in the Witten variables (P_a^i, ω_i^a) the zeroth approximation for the Dirac field is absent. The reason for this is that, according to Witten, the variables P_a^i and ω_i^a fluctuate about the classical values $P_a^i = 0$ and $\omega_i^a = 0$. This point of view is also adopted in our work. But it can be seen from (2.14) that in this case the fermion contribution to the Hamiltonian contains only contributions of higher than quadratic order in the fundamental fields. Therefore, the fermion part of the Hamiltonian can be taken into account only as a whole by perturbation theory, which is impossible in the framework of Feynman theory.

The picture is different for dynamical quantization. Dynamical quantization permits one to preserve in the theory any (including a finite) number of fermion degrees of freedom. In this case the remaining fermion degrees of freedom are eliminated completely from the dynamics. The theory thereby becomes regularized.[2)] Moreover, since the "number" of conserved fermion degrees of freedom can be "sufficiently small," the development of a PT by expansion of the Hamiltonian in its fermion part becomes possible.

For example, let the surface $x^0 = \text{const}$ be a compact Riemann surface of genus g, which we denote by Σ. We introduce on the surface Σ a certain metric and a connection that is without torsion and consistent with the metric. In the local coordinates x^i $(i = 1, 2)$ let $e_{ai}(x)$ $(a = 1, 2)$ denote dyads, so that

$$g_{ij}=e_{ai}e_{aj},\quad \delta_{ab}=e_a{}^ie_{bi}, \tag{4.11}$$

where g_{ij} is the metric tensor in the coordinates x^i. The connection can be represented as $\omega_{abi} = \varepsilon_{ab}\omega_i$. In this case the quantities (2.11) have the form

$$\Gamma^i=\sum_{a=1,2}\Gamma_a{}^i\gamma^a=\sum_{a=1,2}g^{1/2}e_a{}^i\gamma^a=\sum_{a,b,j}\varepsilon_{ij}\varepsilon_{ab}e_{bj}\gamma^a,\qquad g^{1/2}=\det(g_{ij}).$$

The covariant Dirac operator on the surface Σ can be written as follows:

$$-i\Gamma^i\nabla_i=-i\Gamma_a{}^i\gamma^0(\gamma^a\partial_i-{}^1/_2\omega_i\varepsilon_{ab}\gamma^b). \tag{4.12}$$

The absence of torsion is expressed by the equality

$$\partial_i\Gamma_a{}^i-\varepsilon_{ab}\omega_i\Gamma_b{}^i=0. \tag{4.13}$$

As a consequence of (4.13) the operator (4.12) is self-adjoint. Therefore, the operator (4.12) has a complete set of eigenmodes $\{\varkappa_N(x)\}$:

$$-i\gamma^0\Gamma^i\nabla_i\varkappa_N=\lambda_N\varkappa_N,\qquad \sum_N\varkappa_N(x)\varkappa_N{}^+(y)=\delta^{(2)}(x-y). \tag{4.14}$$

The c-number boson spinor fields $\varkappa_N$ found in this way will be used in Eq. (4.4).

Since Σ is a compact surface, it may be assumed that the set of indices N coincides with the set Z.

The theory is regularized by imposing an infinite series of second-class constraints:

$$a_N{}^+=0,\quad a_N=0\quad \text{for}\ |N|>N_0\in Z. \tag{4.15}$$

The specific choice of the metric (4.11) and of the modes $\varkappa_N$ for $|N| < N_0$, and also the way of filling the "vacuum" (see below), implies a choice of initial physical conditions.

5. DIRAC COMMUTATION RELATIONS

The Dirac CR corresponding to the constraints (4.15) are determined in analogy with the CR (3.1) and (3.5).

We define for the fundamental fields A, B, ... the quantity

$$[A,B]_T=\sum_{|N|>N_0}\{[A,a_N{}^+][a_N,B]-(-1)^{\alpha(A)\alpha(B)}[B,a_N{}^+][a_N,A]\}. \tag{5.1}$$

Then the nonzero Dirac CR for the Fermi fields have the following form:

$$[\psi(x),\bar\psi(y)]^*=[\psi(x),\bar\psi(y)]-[\psi(x),\bar\psi(y)]_T=\sum_{|N|<N_0}\psi_N(x)\bar\psi_N(y). \tag{5.2}$$

In the derivation of the CR (5.2) we took Eqs. (4.6) and (4.7) into account.

If we take the point of view that (see the Appendix) the fundamental boson variables are the generators χ_a, φ_a of gauge transformations and the local coordinates (η) in the gauge group $\mathcal{G}$, while the fermion variables are the operators a_N^+ and a_N for $|N| < N_0$, the picture becomes rather simple. In fact, as a consequence of (4.8) and the definition (5.1) we have

$$[\psi(x),\chi_a(y)]_T=0,\quad [\psi(x),\varphi_a(y)]_T=0, \tag{5.3}$$

and so

$$[\psi(x),H]^*=[\psi(x),H],\quad [\bar\psi(x),H]^*=[\bar\psi(x),H]. \tag{5.4}$$

Analogously, we obtain the following result:

$$[\chi_a(x),H]^*=[\chi_a(x),H]-[\chi_a(x),H]_T=[\chi_a(x),H],\qquad [\varphi_a(x),H]^*=[\varphi_a(x),H]. \tag{5.5}$$

According to Sec. 3, the Dirac CR defined in this way satisfy all the necessary properties of the arbitrary CR (3.7)–(3.9). Therefore, the Dirac CR for composite fields (such as ω_i^a and P_a^i) can be defined with the aid of Eqs. (3.7)–(3.9) with allowance for the CR (5.2)–(5.5). Thus, we obtain

$$[\omega_i{}^a(x),H]^*=[\omega_i{}^a(x),H],\qquad [P_a{}^i(x),H]^*=[P_a{}^i(x),H]. \tag{5.6}$$

The equations (5.4)–(5.6) show that the regularized Heisenberg equations for the fields ω_i^a, P_a^i and $\bar{\psi}$, ψ coincide with the formal equations, and the regularized algebra of the gauge transformations coincides with the formal algebra.

It is possible to arrive at the same result by working directly with the fields ω_i^a, P_a^i, $\bar{\psi}$, ψ. To shorten the expressions we introduce some notation:

$$\hat{\nabla}_i P_a^{\ i}=\partial_i P_a^{\ i}-{}^1/_2\varepsilon_{ab}^{\ \ c}(\omega_i^{\ b}P_c^{\ i}+P_c^{\ i}\omega_i^{\ b}),$$
$$\chi_a=\hat{\nabla}_i P_a^{\ i}+T_a,$$
$$F^a={}^1/_2\varepsilon_{ij}(\partial_i\omega_j^{\ a}-\partial_j\omega_i^{\ a}-\varepsilon_{bc}^{\ \ a}\omega_i^{\ c}\omega_j^{\ c}),$$
$$\varphi^a=F^a+R^a. \tag{5.7}$$

Thus, the quantities T_a and R_a are the fermion parts of the operators χ_a and φ_a, respectively.

Since in (5.7) the operators $\hat{\nabla}_i P_a^i$, F^a, etc., are complicated, Dirac CR of the form $[F^a(x), \psi(y)]^*$ can be decomposed into Dirac CR of the fundamental fields, in accordance with (3.8). Therefore, we must define the Dirac CR for the fundamental fields. Suppose that, by definition, Eqs. (5.4) hold. Then the relations (5.2) and (5.4) define the Dirac CR

$$[\psi(x), \bar{\psi}(y)]^*, \quad [\psi(x), P_a^{\ i}(y)]^*, \quad [\psi(x), \omega_i^{\ a}(y)]^*.$$

The last two CR can be determined in practice by expanding in the parameter G. Since expanding in G is equivalent to expanding in the quantities T_a and R_a in (5.7), which are the fermion parts of the constraints χ_a and φ_a, while the number of fermion degrees of freedom is bounded, for a sufficiently small value of the parameter G this expansion is mathematically correct. In Sec. 7 this expansion is discussed in more detail.

In the next section it is shown that the right-hand sides of Eqs. (4.1) and (5.4) coincide. Suppose that in the expansion (4.9) for the Dirac fields the summation is bounded by those N for which $|N| < N_0$ holds. In addition, the CR (4.7) hold. Then the Fermi operators a_N and a_N^+ are integrals of the motion. In fact, it follows from what has been said that

$$\sum_{N<N_0}\psi_N(x)[a_N, H]^*=0,$$

whence, by virtue of the linear independence of the functions $\psi_N(x)$, it follows that

$$[a_N, H]^*=0, \quad |N|<N_0. \tag{5.8}$$

As a consequence of (4.5) the relation (4.2b) is conserved in time. Consequently, the relation (4.2a) is also conserved. Therefore, the conserved operators (a_N^+, a_N) can be represented at any moment of time in the form (4.10).

Comparison of (5.8) with the analogous equations in an ordinary gauge theory [see (3.17) in Ref. 1] shows that the application of the method of dynamical quantization to generally covariant theories is the most natural. This is because in the given case the entire dynamics reduces to gauge transformations, and the essence of the method consists in singling out gauge-invariant operators of the type a_N^+ and a_N. As Ref. 1 shows, in ordinary gauge theories the analogous gauge-invariant operators near the momentum cutoff acquire a phase factor in the process of the dynamics.

We now define the remaining Dirac CR. In the definition of Dirac commutation relations of boson variables, of the form $[P_a^i(x), P_b^j(y)]^*$ etc., the difficulty is that the quantity [see (5.1)] $[P_a^i(x), P_b^j(y)]$ cannot be determined directly, since the commutators $[-P_a^i(x, a_N]$ are unknown. However, these commutators can be found by means of Eqs. (4.8). (The analogous commutators in Ref. 1 were found in the same way.) For example, in place of the quantity

$$[\hat{\nabla}_i P_a^{\ i}(x), a_N]$$

we must everywhere substitute

$$-[T_a(x), a_N].$$

In addition, in place of the quantity

$$[T_a(x), T_b(y)]_T$$

we shall everywhere substitute

$$[T_a(x), T_b(y)]-[T_a(x), T_b(y)]^*.$$

The latter quantity is more convenient, since in its calculation we can use the usual rules (3.7)–(3.9), expanding it in the CR of the fundamental fields.

Thus, by definition,

$$[\hat{\nabla}_i P_a^{\ i}(x), \hat{\nabla}_j P_b^{\ j}(y)]^* = [\hat{\nabla}_i P_a^{\ i}(x), \hat{\nabla}_j P_b^{\ j}(y)] - [T_a(x), T_b(y)] + [T_a(x), T_b(y)]^*, \tag{5.9}$$

$$[\hat{\nabla}_i P_a^{\ i}(x), T_b(y)]^* = [\hat{\nabla}_i P_a^{\ i}(x), T_b(y)] + [T_a(x), T_b(y)] - [T_a(x), T_b(y)]^*. \tag{5.10}$$

Combining (5.9) and (5.10), we find

$$[\hat{\nabla}_i P_a^{\ i}(x), \chi_b(y)]^* = [\hat{\nabla}_i P_a^{\ i}(x), \chi_b(y)]. \tag{5.11}$$

Analogously, we arrive at the following equations:

$$[\hat{\nabla}_i P_a^{\ i}(x), F_b(y)]^* = [\hat{\nabla}_i P_a^{\ i}(x), F_b(y)] - [T_a(x), R_b(y)] + [T_a(x), R_b(y)]^*, \tag{5.12}$$

$$[\hat{\nabla}_i P_a^{\ i}(x), R_b(y)]^* = [\hat{\nabla}_i P_a^{\ i}(x), R_b(y)] + [T_a(x), R_b(y)] - [T_a(x), R_b(y)]^*, \tag{5.13}$$

$$[\hat{\nabla}_i P_a^{\ i}(x), \varphi_b(y)]^* = [\hat{\nabla}_i P_a^{\ i}(x), \varphi_b(y)], \tag{5.14}$$

$$[T_a(x), H]^* = [T_a(x), H]. \tag{5.15}$$

Note that the definitions (5.4) and (5.10), (5.13) are mutually consistent. In fact, following our rule for the construction of CR, which leads to the definitions (5.10) and (5.13), we have

$$[\hat{\nabla}_i P_a^{\ i}(x), \psi(y)]^*$$
$$=[\hat{\nabla}_i P_a^{\ i}(x), \psi(y)] + [T_a(x), \psi(y)]_T$$
$$=[\hat{\nabla}_i P_a^{\ i}(x), \psi(y)] + [T_a(x), \psi(y)] - [T_a(x), \psi(y)]^*$$

etc. The equalities obtained in this way are equivalent to Eqs. (5.4).

In fact, the necessary Dirac CR can be extracted from (5.2), (5.4), and (5.9)–(5.15) by expanding in the constant G. In order to begin this expansion we must establish the order of the following quantities in the parameter G:

$$\psi,\ \bar{\psi}\sim 1, \quad \omega_i^{\ a}\sim 1, \quad P_a^{\ i}\sim (G)^{-1},$$
$$[\psi, \bar{\psi}]^*\sim 1, \quad [P_a^{\ i}, P_b^{\ j}]^*\sim 1, \quad [\omega_i^{\ a}, \omega_j^{\ b}]^*\sim G^2,$$
$$[\psi, P_a^{\ i}]^*\sim 1, \quad [\psi, \omega_i^{\ a}]^*\sim G, \quad [P_a^{\ i}, \omega_j^{\ b}]^*\sim 1. \tag{5.16}$$

The commutator $[\psi, \bar{\psi}]^*$ is known exactly by (5.2).

From the equation

$$[\psi(x),\ \chi_a(y)]^{*}=[\psi(x),\ \chi_a(y)]$$

[see (5.4)] we extract the contribution of zeroth order in G. For this we must neglect the quantities

$$[\psi,\ \omega_i^{a}]^{*}\sim G,\quad [\psi,\ g^{1/2}e_a^{\ 0}]^{*}\sim G$$

[see (2.25) and (5.16)]. We find

$$\nabla_i([\psi(x),P_a^{\ i}(y)]^{*})^{(0)}=\frac{1}{2}\sum_{|N|>N_0}\psi_N((x)(\bar{\psi}_N g^{1/2}e_a^{\ 0}\psi)(y). \tag{5.17}$$

Here the superscripts (0), (1), ... denote the order of the quantity in the parameter G.

From the equation

$$[\psi(x),\ \varphi_a(y)]^{*}=[\psi(x),\ \varphi_a(y)]$$

we extract the contribution of order G. Taking (5.16), (2.14c), and (3.9) into account, we obtain

$$\begin{aligned}&\varepsilon_{ij}\nabla_i(y)([\psi(x),\omega_j^{a}(y)]^{*})^{(1)}\\&\quad=\varepsilon_{ij}\nabla_i(y)[\psi(x),\omega_j^{a}(y)]+[\psi(x),R_a(y)]_T^{(1)}.\end{aligned} \tag{5.18}$$

The last term in (5.18) is obtained from $R_a(y)$ by replacing $\bar{\psi}(y)$ by the quantity

$$[\psi(x),\bar{\psi}(y)]_T=\sum_{|N|>N_0}\psi_N(x)\bar{\psi}_N(y).$$

Analogously, we also find

$$\begin{aligned}&\nabla_i(x)\nabla_j(y)([P_a^{\ i}(x),P_b^{\ j}(y)]^{*})^{(0)}\\&\quad=-\frac{1}{4}\sum_{|N|>N_0}\{(\bar{\psi}g^{1/2}e_a^{\ 0}\psi_N)(x)\\&\qquad\cdot(\bar{\psi}_N g^{1/2}e_b^{\ 0}\psi)(y)-(\bar{\psi}g^{1/2}e_b^{\ 0}\psi_N)(y)(\bar{\psi}_N g^{1/2}e_a^{\ 0}\psi)(x)\},\end{aligned} \tag{5.19}$$

$$\begin{aligned}&\varepsilon_{jk}\nabla_i(x)\nabla_j(y)([P_a^{\ i}(x),\omega_{bk}(y)]^{*})^{(1)}\\&\quad=-\frac{1}{4}(8\pi G)^2\varepsilon_c^{\ db}\sum_{|N|>N_0}\{(\bar{\psi}g^{1/2}e_a^{\ 0}\psi_N)(x)\\&\qquad\cdot[i(\bar{\psi}_N\gamma^c P_d^{\ i}\partial_i\psi-\partial_i\bar{\psi}_N P_d^{\ i}\gamma^c\psi)\\&\qquad+{}^1/_2\bar{\psi}_N P_d^{\ i}\omega_{gi}(s\gamma^c\gamma^g+(1-s)\gamma^g\gamma^c)\psi\\&\qquad+{}^1/_2\bar{\psi}_N\omega_{gi}P_d^{\ i}((1-s)\gamma^c\gamma^g+s\gamma^g\gamma^c)\psi](y)\\&\qquad-[i(\bar{\psi}\gamma^c P_d^{\ i}\partial_i\psi_N-\partial_i\bar{\psi}P_d^{\ i}\gamma^c\psi_N)\\&\qquad+{}^1/_2\bar{\psi}P_d^{\ i}\omega_{gi}(s\gamma^c\gamma^g+(1-s)\gamma^g\gamma^c)\psi_N\\&\qquad+{}^1/_2\bar{\psi}\omega_{gi}P_d^{\ i}((1-s)\gamma^c\gamma^g+s\gamma^g\gamma^c)\psi_N](y)\\&\qquad\cdot(\bar{\psi}_N g^{1/2}e_a^{\ 0}\psi)(x)\}.\end{aligned} \tag{5.20}$$

In view of its unwieldiness, we do not write out the expression for the quantity

$$([F_a(x),\ F_b(y)]^{*})^{(2)}. \tag{5.21}$$

The equations (5.18)–(5.21) make it possible to determine the CR (5.16) to lowest order in the parameter G. We can then develop the iteration in G, using (5.9), (5.12), (5.15), and the initial values for the commutators (5.18)–(5.21) and (5.2).

It can be seen from (5.18)–(5.21) that the Dirac CR of the fundamental fields are not uniquely determined by these equations. Therefore, the process of determining the Dirac CR should be supplemented by the principle that Eqs. (5.6) are valid. The latter are consistent with Eqs. (5.11), (5.14), and (5.15).

It is now not difficult to establish the following: *Assertion*: The Dirac CR defined in this section possess all the necessary properties (3.7), (3.8).

In fact, according to the definition (5.2) and the relation (3.6), the fermion CR (5.2) are ordinary commutators. Furthermore, all the remaining Dirac CR are expanded in a series in the gravitational constant. The first nonzero terms of these expansions contain only fermion Dirac commutation relations, and so are ordinary commutators. Since in the expansion of the Dirac CR in the gravitational constant each subsequent term is expressed in terms of the preceding terms, all the terms in this expansion are ordinary commutators. Consequently, the above assertion is valid.

6. THE HEISENBERG EQUATIONS AND STATE VECTORS

Equations (5.4)–(5.6) show that the regularized Heisenberg equations can be obtained using the formal commutation relations (3.9).

Here two questions arise: 1) How does one define a product of operator fields at one spatial point x? 2) How does one arrange the operator fields at one spatial point x in the Heisenberg equations obtained [since the CR (3.9) differ from the Dirac CR]?

The answer to the first question is as follows: As a consequence of the imposition of the constraints (4.15) the Dirac fields ψ and $\bar{\psi}$ are smooth, and so their product (in either order, and, possibly, at the same point x) is regular.

If in the theory under consideration the fermion degrees of freedom were absent, there would be no local degrees of freedom. In this case there would be only global degrees of freedom, associated with the fundamental group of a spatial surface. If the spatial surface is a compact Riemann surface of genus g, its fundamental group has $2g$ generators a_i, b_j $(i,\ j=1,\ldots,g)$, with the single relationship

$$a_1b_1a_1^{-1}b_1^{-1}\ldots a_gb_ga_g^{-1}b_g^{-1}=1. \tag{6.1}$$

We introduce the notation

$$U_i(V_i)=\hat{T}\exp\left[\frac{i}{2}\oint_{\gamma_i(\delta_i)}\omega_{ai}\gamma^a\,dx^i\right]\subset SO(2,1), \tag{6.2}$$

where γ_i (δ_i) is some particular representative of the class a_i (b_i), and $\hat{T}$ is the operator of ordering along the contour of integration. Since, in the absence of fermions, $F_a=0$, the quantities (6.2) do not depend on representatives of the classes a_i (b_i). The quantities (6.2) satisfy the relation (6.1):

$$U_1V_1U_1^{-1}V_1^{-1}\ldots U_gV_gU_g^{-1}V_g^{-1}=1. \tag{6.3}$$

The relation (6.3) is a natural restriction for the quantities (6.2) (see Ref. 4). The wave functions of the system in the absence of the Fermi fields $\Psi_0(U_i,\ V_j)$ depend only on the quantities (6.2); the Ψ_0 are defined on the hypersurface (6.3) and are invariant under the transformations

$$U_i\to E^{-1}U_iE,\quad V_j\to E^{-1}V_jE, \tag{6.4}$$

where E is a certain element from the group $SO(2,1)$. We also have the restrictions

$$\hat{\nabla}_iP_a^{\ i}\Psi_0=0,\quad F_a\Psi_0=0. \tag{6.5}$$

It follows from what has been said that in the absence of fermions small-scale fluctuations of the fields ω_i^a and P_a^i do not play any role, and can be "discarded." By small-scale fluctuations we mean fluctuations with wavelengths $\sim\lambda$ for which

$$\lambda<\lambda_0\ll s_i,\ r_i\ \ (i=1,\ \dots,\ g), \tag{6.6}$$

where s_i, r_i are the characteristic lengths of the cycles γ_i, δ_i in any particular metric. The "discarding" of the small-scale fluctuations of the fields ω_i^a and P_a^i implies that in the corresponding commutator in (3.9) the δ-function is replaced by a smoothed δ-function, which we denote by δ_{λ_0}. The actual way in which the δ-function is smoothed is of no importance here. It is important only that δ_{λ_0} acts on smooth functions (that vary appreciably on scales much greater than λ_0) in the same way as the δ-function.

In our case, in the presence of the fermion degrees of freedom, one must proceed analogously. Let λ_F be a length of the order of the largest wavelengths of the functions ψ_N for $|N|<N_0$. Then the condition (6.6) for λ_0 should be supplemented by the condition

$$\lambda_0\ll\lambda_F, \tag{6.7}$$

after which the scheme of regularization of the fields ω_i^a and P_a^i remains in force. Thus, we have given the answer to the first question.

The answer to the second question will be given after we have carried out certain formal calculations in the unregularized theory.

We say that a given system is formally quantized if the formal algebra of the operators $(\chi_a,\ \varphi_a)$ is closed and the structure functions are positioned all to the left (or right) of the generators $(\chi_a,\ \varphi_a)$, as in Sec. 1.

We shall carry out formal calculations of the necessary commutators, using relations of the form

$$\begin{gathered}\psi_\sigma(x)\bar\psi_\rho(x)=-\bar\psi_\rho(x)\psi_\sigma(x)+\delta^{(2)}(0)\,(\Gamma^0)_{\sigma\rho}{}^{-1}(x),\\ \omega_i{}^a(x)P_b{}^j(x)=P_b{}^j(x)\,\omega_i{}^a(x)+\delta^{(2)}(0)\,i\delta_b{}^a\delta_i{}^j\end{gathered} \tag{6.8}$$

etc., which follow formally from the CR (3.9). This approach makes it possible to calculate the (operator) coefficient of the symbol $\delta^{(2)}(0)$, which arises as a result of permutations of the field operators in various expressions. In this way we arrive at the conclusion that the system (2.14) is formally quantized only for the value

$$s={}^5/_8. \tag{6.9}$$

Thus, the requirement of formal quantizability of the system uniquely fixes the ordering of the operator fields in the Hamiltonian. The formulas given below are valid for $s=5/8$.

6.1. The equations of motion

The formal calculations can be carried out in stages. In the first stage we find the Heisenberg equations $i\dot A=[A,\ H]$ for the fundamental fields:

$$\begin{gathered}i\Gamma^\mu\nabla_\mu\psi={}^1/_2 i\,(8\pi G)\,\varepsilon_a{}^{bc}e_c\gamma^a\psi\chi_b,\\ i\nabla_\mu\bar\psi\Gamma^\mu={}^1/_2 i\,(8\pi G)\,\varepsilon_a{}^{bc}e_c\chi_b\bar\psi\gamma^a,\\ \dot P_a{}^i-\varepsilon_{ab}{}^c\omega^bP_c{}^i+\frac{1}{8\pi G}\,\varepsilon_{ij}(\partial_je_a-\varepsilon_{ab}{}^c\omega_j{}^be_c)\\ +{}^1/_2(8\pi G)\,\varepsilon_a^{bc}\,e_c(\bar\psi P_b{}^i\psi)=0,\\ i\dot\omega_i{}^a=i\,(\partial_i\omega^a-\varepsilon^a{}_{bc}\omega_i{}^b\omega^c)\\ +{}^1/_2(8\pi G)\,\varepsilon_b{}^{ca}e_c(\bar\psi\gamma^b\nabla_i\psi-\nabla_i\bar\psi\gamma^b\psi)\\ +{}^1/_2 i\,(8\pi G)^2\varepsilon_b{}^{ca}\varepsilon_{ij}(\bar\psi P_c{}^j\gamma^b[\psi,\ H_\varphi]\\ -[\bar\psi,\ H_\varphi]\gamma^bP_c{}^j\psi).\end{gathered} \tag{6.10}$$

In the classical limit all the equations (6.10) with the exception of the Dirac equations coincide with Euler–Lagrange equations that are obtained by variation of the action (2.9). The equations (6.10) for the Dirac fields differ insignificantly from the Dirac equations, by only a term that is equal to zero in the weak sense.

6.2. The algebra of generators of the gauge group

Let A be any particular fundamental field. We have the Jacobi identity

$$\begin{gathered}[[\varphi_a(x),\ \varphi_b(y)],\ A]\\ =[\varphi_a(x),\ [\varphi_b(y),\ A]]-[\varphi_b(y),\ [\varphi_a(x),\ A]].\end{gathered} \tag{6.11}$$

The right-hand side in (6.11) is calculated with the aid of Eqs. (6.10). Knowing the right-hand side of the expression (6.11), and also the classical Poisson bracket $[\varphi_a(x),\ \varphi_b(y)]$, as a result of lengthy calculations we can establish the quantum formal commutator $[\varphi_a(x),\ \varphi_b(y)]$. Here, in all the calculations, we use only (6.10) and the formal algebra (6.8). Note that in all the formal calculations described the symbol $\delta^{(2)}(0)$ is encountered to a power no higher than the first, and symbols of the form $\delta'(0)$ are absent in the calculations. Also, the final results do not contain the indeterminate symbol $\delta^{(2)}(0)$.

The answer is

$$[\chi_a(x),\ \chi_b(y)]=-i\delta^{(2)}(x-y)\,\varepsilon_{ab}{}^c\chi_c(x), \tag{6.12a}$$

$$[\chi_a(x),\ \varphi^b(y)]=-i\delta^{(2)}(x-y)\,\varepsilon_{ca}{}^b\varphi^c(x), \tag{6.12b}$$

$$\begin{gathered}[\varphi^a(x),\ \varphi^b(y)]={}^1/_2 i\delta^{(2)}(x-y)\,(8\pi G)^2\varepsilon_c{}^{ab}(\bar\psi\psi)\varphi^c\\ +{}^1/_2\delta^{(2)}(x-y)\,(8\pi G)^4(\varepsilon_d{}^{ab}\varepsilon_g{}^{hc}-\varepsilon^{abc}\varepsilon_{dg}{}^h)\cdot\\ \cdot\{\bar\psi P_c{}^i\gamma^g(\Gamma^0)^{-1}\gamma^d\nabla_i\psi-\nabla_i\bar\psi\gamma^d(\Gamma^0)^{-1}\gamma^gP_c{}^i\psi\}\chi_h\\ +{}^1/_4\delta^{(2)}(x-y)\,(8\pi G)^4(\varepsilon^{abh}\varepsilon_{gd}{}^f-\varepsilon_g{}^{ab}\varepsilon_d{}^{fh})\cdot\\ \cdot\{A\chi_h\bar\psi\gamma^g(\Gamma^0)^{-1}\gamma^d\psi\chi_f+B\chi_f\bar\psi\gamma^g(\Gamma^0)^{-1}\gamma^d\psi\chi_h\\ +C\bar\psi\gamma^g\chi_h(\Gamma^0)^{-1}\gamma^d\psi\chi_f+D\bar\psi\gamma^g\chi_f(\Gamma^0)^{-1}\gamma^d\psi\chi_h\\ +F\bar\psi\gamma^g(\Gamma^0)^{-1}\gamma^d\chi_h\psi\chi_f+Q\bar\psi\gamma^g(\Gamma^0)^{-1}\gamma^d\chi_f\psi\chi_h\\ +H\bar\psi\gamma^g(\Gamma^0)^{-1}\gamma^d\psi\chi_h\chi_f\\ +(1-A-B-C-D-F-Q-H)\cdot\\ \cdot\bar\psi\gamma^g(\Gamma^0)^{-1}\gamma^d\psi\chi_f\chi_h\}.\end{gathered} \tag{6.12c}$$

Here A, B, C, D, F, Q, and H are certain numerical real parameters, satisfying the following equations:

$$D+F+2=0,\quad B+H=1,$$
$$2A+D-F=0,$$
$$2B+3D-F+2Q=0.$$

All the operators in the right-hand sides of (6.12) are taken at the point x. The right-hand side of (6.12c) is formally anti-Hermitian.

6.3. The regularized Heisenberg equations

The theory described above gives us the possibility of formulating:

Rule 2: The regularized Heisenberg equations for the fundamental fields coincide with (6.10); the values of the commutators

$$[\chi_a(x), \chi_b(y)]^*, \quad [\chi_a(x), \varphi^b(y)]^*, \quad [\varphi^a(x), \varphi^b(y)]^*$$

coincide with the right-hand sides of Eqs. (6.12a), (6.12b), and (6.12c), respectively.

It follows from Rule 2 that the right-hand side of (6.12c) is anti-Hermitian in the regularized theory as well. It is possible that Rule 2 completely fixes the Dirac CR. However, this question is not answered here.

6.4. The state vectors

First we shall correct the Heisenberg equations (5.8) for the operators a_N, since the Dirac equations (6.10) differ from Eqs. (4.1) for the functions $\psi_N(x)$ in the weak sense. In fact, the conclusion reached above that the quantity (4.10) is conserved would be rigorously true if the equation for the functions $\psi_N(x)$ coincided strictly with the Dirac equation for the field $\psi(x)$. Therefore, in our case we have for the operator (4.10)

$$[H, a_N^+]^* = \int d^2x (\alpha_N{}^a \chi_a + \beta_N{}^a \varphi_a), \quad |N| < N_0. \tag{6.13}$$

We work in the representation in which the action of the operators a_N^+ reduce to multiplication of the vectors by a Grassmann number, denoted by the same symbol a_N^+. Then the operators a_N can be represented in the form

$$a_N = \partial/\partial a_N^+, \quad |N| < N_0. \tag{6.14}$$

By definition, for the wave functions Ψ_0 [see the relations (6.5)] we have

$$a_N \Psi_0 = 0, \quad |N| < N_0. \tag{6.15}$$

From (6.5) and (6.15) it follows that

$$\varphi_a \Psi_0 = 0, \quad \chi_a \Psi_0 = 0. \tag{6.16}$$

We now have all the means for the construction of the state vectors. We write out the state vectors with a well defined occupation of the vacuum:

$$\Psi_{N_1 N_2 \dots N_s} = a_{N_1}^+ a_{N_2}^+ \dots a_{N_s}^+ \Psi_0, \quad |N_i| < N_0, \quad s = 0, 1, \dots . \tag{6.17}$$

Any state is a superposition of the states (6.17). From the relations (6.13) and (6.16) it follows quickly that

$$\chi_a \Psi_{N_1 N_2 \dots N_s} = 0, \quad \varphi_a \Psi_{N_1 \dots N_s} = 0. \tag{6.18}$$

Remark: Since in the theory under consideration the problem of the normal modes of the Dirac operator has no meaning, it is impossible to classify the operators a_N and a_N^+ by their energy. Therefore, the problem of the occupation of the physical vacuum remains unsolved here. A criterion permitting one to distinguish the ground and excited states, or the degree of excitation of the states is completely absent.

7. PERTURBATION THEORY

We show now that in the model under consideration dynamical quantization makes it possible to construct a mathematically correct PT in the gravitational constant.

For simplicity we assume that the spatial surface Σ is a compact Riemann surface of genus 1, or a two-dimensional torus.[3] In this case all the field variables should be doubly periodic:

$$A(x^1, x^2) = A(x^1 + a, x^2) = A(x^1, x^2 + a). \tag{7.1}$$

In the given case the geometric characteristics of the surface Σ can be chosen to be trivial:

$$g_{ij} = \delta_{ij}, \quad e_{ia} = \delta_{ia}, \quad \omega_i = 0.$$

Therefore, the Dirac operator (4.12) $-i\gamma^0\gamma^i\partial_i$ is easily diagonalized. We have

$$\varkappa_{N\pm}(x) = \frac{1}{2^{1/2} a |k_N|} \begin{pmatrix} \pm(ik_1 + k_2)_N \\ |k_N| \end{pmatrix} \exp(ik_N x),$$

$$(k_1, k_2)_N = \frac{2\pi}{a}(n_1, n_2), \quad n_i = 0, \pm 1, \dots ,$$

$$|k_N| = \frac{2\pi}{a}(n_1^2 + n_2^2)^{1/2}, \tag{7.2}$$

$$-i\gamma^0\gamma^i\partial_i \varkappa_{N\pm} = \pm k_N \varkappa_{N\pm}.$$

The Heisenberg fields can be expanded in a series in the constant G:

$$A = A^{(0)} + A^{(1)} + \dots ,$$

where $A^{(0)}$ is the zeroth approximation, $A^{(1)}$ is the first approximation, etc. It is obvious that the Dirac CR $[A^{(i)}, B^{(j)}]^*$ should be calculated in the $(i+j)$th approximation.

Let $P_a^{i(0)}(x)$ and $\omega_i^{a(0)}(x)$ be certain operator fields, having in the zeroth approximation in the gravitational constant the following (nonzero) commutation relations:

$$[\omega_i^{a(0)}(x), P_b^{j(0)}(y)]^* = i\delta_i^j \delta_b^a \delta^{(2)}(x-y). \tag{7.3}$$

According to (4.4),

$$\psi_N(t_0, x) = (\Gamma^0(t_0))^{-1/2} (\gamma^0)^{1/2} \varkappa_N(t_0, x).$$

We assume that the index $N = (n_1, n_2)$ obeys the condition that $|N| < N_0$ if $|n_i| < n_0$. The regularized Dirac field at the time t_0 can be represented in the form

$$\psi^{(0)}(x) = \sum_{|N| < N_0} a_N \psi_N(t_0, x). \tag{7.4}$$

The constant operators $\{a_N, a_N^+\}$ satisfy the CR (4.7).

We extract the zeroth approximation from (6.10):

$$\dot{P}_a^{i(0)} - \varepsilon_{ab}{}^c \omega^b P_c^{i(0)} + \frac{1}{8\pi G} \varepsilon_{ij} (\partial_j e_a - \varepsilon_{ab}{}^c \omega_j^{b(0)} \omega^c) = 0,$$

$$\dot{\omega}_i^{a(0)} = \partial_i \omega^a - \varepsilon_{bc}{}^a \omega_i^{b(0)} \omega^c, \tag{7.5}$$

$$\nabla_0 \psi^{(0)} = 0, \quad \nabla_0 \overline{\psi}^{(0)} = 0.$$

The equations (7.5) are easily solved. We introduce the notation

$$\omega = {}^1/_2 \omega_a \gamma^a, \quad e = {}^1/_2 e_a \gamma^a, \quad \omega_i = {}^1/_2 \omega_{ai} \gamma^a, \quad P^i = {}^1/_2 P_a{}^i \gamma^a. \tag{7.6}$$

Let the c-number matrix field satisfy the equation

$$\partial U/\partial t = i\omega U, \quad U(t_0, x) = 1. \tag{7.7}$$

From group-theoretical arguments it is obvious that the operators χ_a generate gauge transformations, which are easily eliminated. For this we introduce fields with the tilde symbol:

$$\omega_i = U\tilde{\omega}_i\bar{U} + iU\partial_i\bar{U},$$
$$P^i = U\tilde{P}^i\bar{U}, \quad \psi = U\tilde{\psi}, \tag{7.8}$$
$$\tilde{e}(t,x) = \bar{U}(t,x)\,e(t,x)\,U(t,x),$$

where $\bar{U} = \delta^0 U^+ \gamma^0$. We note that $\bar{U}U = 1$. In the adopted notation, the solutions of Eqs. (7.5) have the form

$$\tilde{\omega}_i^{(0)}(t) = \omega_i^{(0)}(t_0),$$

$$\tilde{P}^{i(0)}(t) = P^{i(0)}(t_0) - \frac{\varepsilon_{ij}}{8\pi G}\int_{t_0}^{t} dt'\,(\partial_j\tilde{e} - i[\tilde{\omega}_j^{(0)}, \tilde{e}])(t'), \tag{7.9}$$

$$\tilde{\psi}^{(0)}(t) = \psi^{(0)}(t_0).$$

In (7.9) all the fields are taken at the point x.

Let us write out the equations for the first-order corrections to the Heisenberg fields (all the fields carry the tilde symbol, although, to simplify the writing, it is absent in the formulas):

$$i\dot{\omega}^{a(1)} = {}^1/_2(8\pi G)\,\varepsilon_b{}^{ca}e_c(\bar{\psi}\gamma^b\nabla_i\psi - \nabla_i\bar{\psi}\gamma^b\psi)$$
$$+{}^1/_2(8\pi G)^2\,\varepsilon_b{}^{ca}\varepsilon_{ij}\cdot(\bar{\psi}P_c{}^j\gamma^b(\Gamma^0)^{-1}\Gamma^i\nabla_i\psi - \nabla_i\bar{\psi}\Gamma^i(\Gamma^0)^{-1}\gamma^b P_c{}^j\psi), \tag{7.10a}$$

$$\dot{P}_a^{i(1)} = \frac{1}{8\pi G}\varepsilon_{ij}\varepsilon_{ab}{}^c\omega_j^{b(1)}e_c - \frac{1}{2}(8\pi G)\varepsilon_a{}^{bc}e_c\bar{\psi}^{(0)}P_b^{i(0)}\psi^{(0)}. \tag{7.10b}$$

$$i\dot{\psi}^{(1)} = -i(\Gamma^0)^{-1}\Gamma^i\nabla_i\psi + {}^1/_2 i(8\pi G)\varepsilon_a{}^{bc}e_c(\Gamma^0)^{-1}\gamma^a\psi\chi_b. \tag{7.10c}$$

In (7.10a) and (7.10c) all the fields in the right-hand side are taken in the zeroth approximation as in (7.9). The solutions of Eqs. (7.10) are obvious. As in the zeroth approximation, we first solve (7.10a) for the connection and then solve (7.10b) for the triads.

One must also correct the Dirac CR (7.3) so that these CR will be valid to first order inclusive in the gravitational constant. This problem is solved in Sec. 5.

Thus, the chain of regularized recursion equations does not differ formally from the analogous unregularized equations. In the essence of the matter, however, in our theory a correctly defined chain of recursion equations arises. This is a consequence of the regularization of the Dirac field in accordance with (7.4) and the replacement of the formal CR (3.9) by Dirac CR.

We note that the set of fields $\omega_i^{(0)}(t_0)$, $P^{i(0)}(t_0)$, $\psi^{(0)}(t_0)$, and $\bar{\psi}^{(0)}(t_0)$ is precisely the field $\Phi^{(0)}$ (see the Appendix).

Finally, we consider one exactly solvable model. Let the system contain only one fermion degree of freedom. This implies that from the set of modes $\{\psi_N\}$ we choose a certain mode $\psi_0(x)$ and impose the infinite series of constraints (4.15) with $|N|>0$. With allowance for the constraints (4.15) the Fermi fields have the form

$$\psi(x) = a_0\psi_0(x), \quad \bar{\psi}(x) = \bar{\psi}_0(x)a_0^+.$$

From this it can be seen that

$$\psi(x)\psi(y) = 0, \quad \bar{\psi}(x)\bar{\psi}(y) = 0. \tag{7.11}$$

The relations (7.11) permit us to find the Dirac CR exactly and to solve the Heisenberg equations exactly. It is easy to see that Eqs. (7.8)–(7.10) give the exact solution of the Heisenberg equations. In fact, subsequent iterations in the gravitational constant lead to corrections that contain the fields ψ or $\bar{\psi}$ to powers higher than the first. As a consequence of (7.11), all these corrections vanish. Terms containing ψ (or $\bar{\psi}$) to the power 2 or higher vanish even in the case when these fields are separated by operators ω_i^a or P_a^i. In this case one of the fields ψ ($\bar{\psi}$) must be moved so that it stands alongside the other field ψ ($\bar{\psi}$). According to (7.11), such a term vanishes. In any case, the commutators (of the field ψ ($\bar{\psi}$) with the boson variables) that arise in this move increase the order in the gravitational constant by unity. It is clear, therefore, that by continuing this process of permutations we shall obtain either zero or a term of infinitely high degree in the gravitational constant. Here, everywhere in Eqs. (7.10), by the symbols ψ and $\bar{\psi}$ we mean the exact values of the Dirac fields which satisfy Eq. (7.10c). From this it can be seen that in the given case the expansion of the Heisenberg equations in the gravitational constant terminates on Eqs. (7.10).

It is not difficult to convince oneself that in the case of one fermion degree of freedom the expansions for the Dirac CR in the gravitational constant are also truncated.

8. CONCLUSION

Thus, the theory of dynamical quantization developed here is found to be adequate for the construction of a quantum theory of gravity in $(2+1)$-dimensional space. The theory possesses the following necessary properties:

a) The theory is unitary and causal; the set of evolution operators forms a group (this property is not obvious in Witten's theory[4]);

b) the algebra (6.12) of gauge transformations holds;

c) there exists a mathematically correct perturbation theory in the gravitational constant G.

It appears to us that the theory of dynamical quantization can be applied successfully to construct a quantum theory of gravity in $(3+1)$-dimensional space. The main difference between that theory and the present one is that it is necessary to identify not only fermion but also boson gauge-invariant creation and annihilation operators. It makes sense to study the supersymmetric variant of the theory. Dynamical quantization should not break the supersymmetry, in analogy with the fact that dynamical quantization preserves the gauge symmetry (with regard to the absence of a gauge anomaly in dynamical quantization, see Ref. 1). Therefore, the occupation of the vacuum can be implemented in such a way that the contributions from the boson and fermion zero-point oscillations to the energy-momentum tensor of matter cancel out. It is evident that in this case a perturbation theory in the gravitational constant analogous to the PT considered in Sec. 7 will be valid. Thus, it appears to us that the method of dynamical quantization gives the possibility of constructing a quantum theory of gravity in a space of any dimensionality with the necessary properties (a)–(b).

APPENDIX

Let the spinor boson field $\psi_N(t)$ be the solution of Eq. (4.1) with certain initial conditions. We must establish that the equation for the field $\psi_N(t)$ can be represented in the form of a Heisenberg equation as in (4.6).

To establish this fact the following arguments are necessary.

The set of generators $\chi_a(x)$ and $\varphi_a(x)$ in the classical limit form an algebra, i.e., the Poisson brackets of the form $[\chi_a, \varphi_b]$ are proportional to these generators. For example,

$$[\varphi_a(x), \varphi_b(y)] = \delta^{(2)}(x-y)\{f_{ab}{}^c(x)\varphi_c(x) + g_{ab}{}^c(x)\chi_c(x)\}. \tag{A1}$$

Here the structure functions $f_{abc}(x)$ and $g_{abc}(x)$ depend on the dynamical variables. In quantum mechanics regularization should preserve the algebra of the generators (χ_a, φ_a). This means that the regularized CR should preserve the form of (A1), and the structure functions $f_{abc}(x)$ and $g_{abc}(x)$ should stand to the left of the generators φ_c and χ_c in (A1). This is necessary, since the quantities $f_{abc}(x)$ and $g_{abc}(x)$ depend on the Heisenberg operators. In this case we can speak of a group $\mathscr{G}$ of gauge transformations, the generators of which are the constraints (χ_a, φ_a).

We go over to new dynamical variables. We denote by η certain local coordinates in the group $\mathscr{G}$. By the symbol $\Phi(t)$ we denote the set of Heisenberg fields ω_i^a, P_a^i, ψ, and $\bar{\psi}$ at the time t, and by the symbol $\xi(t)$ the set of c-number fields $\omega_a(t)$ and $e_a(t)$. It is required that we express the fields $\Phi(t)$ in terms of the coordinates η, the vector fields on the group $\mathscr{G}$ (i.e., the differential operators of first order on the group $\mathscr{G}$), and the fields $\Phi^{(0)}$ that do not depend on the group $\mathscr{G}$. Thus, the fields $\Phi^{(0)}$ are integrals of the motion. This problem has a solution.

In fact, the fields ω_i^a and P_a^i constitute independent boson degrees of freedom of the system. At each point x there are 6 degrees of freedom ω_i^a and 6 degrees of freedom P_a^i. On the other hand, at the point x there are 6 degrees of freedom of the operators $\chi_a(x)$ and $\varphi_a(x)$ and 6 degrees of freedom of the parameters $\xi(x) = [\omega_a(x), e_a(x)]$. It can be seen from Eqs. (2.14) that the operators χ_a depend in an essential way on the longitudinal parts of the fields P_a^i, while the operators φ_a depend on the transverse parts of the fields ω_i^a. The gauge transformations (which are generated completely by the Heisenberg equations) give essential increments of the transverse parts of the fields P_a^i and of the longitudinal parts of the fields ω_i^a. Therefore, locally (in the space of the fields), we can take as the independent boson dynamical variables the operators χ_a and φ_a, and also the coordinates η. In fact, the fields Φ can be expressed in terms of the field η and $\chi_a(x)$, $\varphi_a(x)$ by expanding in the gravitational constant G and simultaneously integrating in the group $\mathscr{G}$ over the variables η. The fields Φ here will also depend on the constant fields $\Phi^{(0)}$, so that $\Phi(\eta) \to \Phi^{(0)}$ for $\eta \to 0$.

The fields Φ, expressed in terms of the variables η, φ_a, χ_a, and $\Phi^{(0)}$, will be denoted by $\tilde{\Phi}$. It is convenient to choose the variables η in such a way that

$$\dot{\eta} = -i[\eta, H]^{\cdot} = \xi, \quad \eta \to 0. \tag{A2}$$

Equation (A2) shows that as $\eta \to 0$ the Hamiltonian operator can be represented in the form of the following differential operator:

$$\tilde{H} = -i\int d^2x\, \xi(x)\frac{\delta}{\delta\eta(x)} + h. \tag{A3}$$

To find the operator h we can make use of the following device. Let $\{\eta\} = \{te_a, t\omega_a\}$ be a set of local coordinates in the group $\mathscr{G}$; this set can also be regarded as a complete set of commuting boson variables. The operator (2.14), expressed in terms of the fields $\Phi^{(0)}$ and multiplied by t, will be denoted by $H_\eta^{(0)}$. We shall represent an element of the group $\mathscr{G}$ with coordinates η in the form

$$U = \exp(-iH_\eta^{(0)}).$$

The product of two elements of the group $\mathscr{G}$ can be represented in an analogous way by means of the Campbell–Hausdorff formula:

$$\exp(-iH_\eta^{(0)})\cdot\exp(-iH_\lambda^{(0)}) = \exp(-iH_\Sigma^{(0)}).$$

The operator $H_\Sigma^{(0)}$ is expressed in terms of the quantity $\Sigma(\eta, \lambda, \Phi^{(0)})$ in the same way as the operator $H_\eta^{(0)}$ is expressed in terms of the field $\eta(x)$. Here, in the quantity $H_\Sigma^{(0)}$ the operators $\varphi_a^{(0)}$ and $\chi_a^{(0)}$ stand to the right of the operators $\Sigma(x)$; this is a consequence of the algebra (6.12).

If the theory is regularized, application of the Campbell–Hausdorff formula is legitimate.

Now the vector field (A3) is obtained by differentiation of the quantity Σ:

$$\tilde{H} = -i\int d^2x \int d^2y\, \xi(x) \frac{\delta\Sigma(y)}{\delta\lambda(x)}\bigg|_{\lambda=0} \frac{\delta}{\delta\eta(y)}. \tag{A3'}$$

The quantity (A3') or (A3) is a Heisenberg operator in the new variables. Using the explicit form of the Campbell–Hausdorff expansion, it is easy to establish that

$$\Sigma(x) = \eta(x) + \lambda(x) + \Pi(\eta, \lambda),$$

where $\Pi(\eta, \lambda) \to 0$ if $\eta \to 0$ or $\lambda \to 0$. From this it follows that in (A3)

$$h \to 0 \quad \text{as} \quad \eta \to 0. \tag{A4}$$

Since the operator $\mathscr{O}$ (4.1) depends linearly on ξ, it is useful to note this with the aid of the corresponding notation

$$\mathscr{O} = \xi\mathscr{O}_\xi.$$

Let $\psi_N(t)$ and $\tilde{\psi}_N(t)$ be known near the point $\eta = 0$, which can be any point in the group $\mathscr{G}$. We shall trace the change of these fields at the point $\eta = 0$. Let $\delta t \to 0$. From (4.1) we have

$$\psi_N(t+\delta t) = \exp(-i\delta t\xi\mathscr{O}_\xi)\psi_N(t). \tag{A5}$$

From this, and from the definitions given above, it follows that

$$\tilde{\psi}_N(t+\delta t) = \exp(-i\eta\mathscr{O}_\xi)\tilde{\psi}_N(t). \tag{A6}$$

Here $\eta = \delta t\xi$ are the coordinates of a point infinitesimally close to the point $\eta = 0$. The field $\psi_N(t)$ depends only on the fields $\Phi^{(0)}$. As a consequence of (A3) and (A4) we have

$$[\tilde{H}, \Phi^{(0)}] \to 0 \quad \text{as} \quad \delta t \to 0. \tag{A7}$$

Let $\varepsilon \to 0$. We shall consider the "encased" (A6) (to simplify the writing we omit the tilde):

$$e^{i\varepsilon H}\psi_N(t+\delta t)e^{-i\varepsilon H} = [\exp(-ie^{i\varepsilon H}\eta\mathscr{O}_\xi e^{-i\varepsilon H})]e^{i\varepsilon H}\psi_N e^{-i\varepsilon H}. \tag{A8}$$

We go over to the limit $\delta t \to 0$ in (A8). As a consequence of (A7) and the relation

$$e^{i\varepsilon H}\eta e^{-i\varepsilon H}=\eta+\varepsilon\xi$$

(A8) goes over into the equality

$$e^{i\varepsilon H}\psi_N e^{-i\varepsilon H}=[\exp(-i\varepsilon\mathcal{O})]\psi_N(t), \qquad (A9)$$

which is valid at any point in the space of the fields. Combining (A9) and (A6), we obtain the required equality (4.5).

Remark: Since in the regularized theory the Heisenberg equations and the algebra of the operators φ_a, χ_a have the same form as in the formal theory, the derivation in this Appendix remains valid in the regularized theory.

1) This question is also discussed in Ref. 5.

2) Here we are concerned with fermion degrees of freedom only, since in our case local boson degrees of freedom are completely absent. In more complicated theories it is necessary to perform an analogous analysis for the boson degrees of freedom as well.

3) The problem of the calculation of the transition amplitudes with change of the topology of the space is not considered here.

[1] S. N. Vergeles, Pis'ma Zh. Eksp. Teor. Fiz. **49**, 375 (1989) [JETP Lett. **49**, 427 (1989)]; Zh. Eksp. Teor. Fiz. **96**, 445 (1989) [Sov. Phys. JETP **69**, 251 (1989)]; Yad. Fiz. **50**, 1476 (1989) [Sov. J. Nucl. Phys. **50**, 917 (1989)]; Int. J. Mod. Phys. A **5**, 2117 (1990).

[2] V. N. Gribov, Preprint No. KFKI-66, Budapest (1981).

[3] S. N. Vergeles, Zh. Eksp. Teor. Fiz. **95**, 397 (1989) [Sov. Phys. JETP **68**, 225 (1989)].

[4] E. Witten, Nucl. Phys. **B311**, 46 (1988); **B323**, 113 (1989).

[5] R. de A. Campos and J. M. L. Fisch, Int. J. Mod. Phys. A **3**, 2371 (1988).

[6] P. A. M. Dirac, *Lectures on Quantum Mechanics* (Yeshiva University, New York, 1964) [Russ. Transl., Mir, Moscow, 1968].

Translated by P. J. Shepherd