Density of states of a disordered system in $d>4$ dimensions

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The Anderson model on d-dimensional cubic lattice with the Gaussian site energy distribution is analyzed. It is shown that for $d>4$ the mean density of states $N(E)$ may be calculated for all energies. It is a smooth function and has no singularity near the Anderson transition.

1. INTRODUCTION

The behavior of physical quantities in the vicinity of an Anderson transition is the problem of central concern in the theory of localization.[1–4] The analogy with the contemporary theory of critical phenomena[5,6] suggests that the Anderson transition might exhibit an (upper) critical space dimension d_{c2} above which the theory is considerably simplified and accordingly, and ε-expansion for a space of dimension $d_{c2}-\varepsilon$ can be hoped for.

We consider an Anderson model described by the discrete Schrödinger equation

$$\sum_{\mathbf{m}} J_{\mathbf{n}-\mathbf{m}}\Psi_{\mathbf{m}}+V_{\mathbf{n}}\Psi_{\mathbf{n}}=E\Psi_{\mathbf{n}}, \tag{1}$$

where $\mathbf{m}$ and $\mathbf{n}$ label the sites of a d-dimension cubic lattice with lattice constant a_0; the $J_{\mathbf{n}-\mathbf{n}}$ are the overlap integrals whose magnitude, of order J, falls off rapidly with $|\mathbf{m}-\mathbf{n}|$; and the $V_{\mathbf{n}}$ are independent, random site energies obeying the Gaussian distribution

$$P\{V\}\propto\exp\Big\{-\sum_{\mathbf{n}} V_{\mathbf{n}}^2/2W^2\Big\}. \tag{2}$$

We assume the disorder to be weak and we are interested in low energies lying in the range

$$W\ll J,\ |E|\ll J, \tag{3}$$

where E is measured from the bottom of the unperturbed band.

There is considerable evidence to indicate the special role of the dimension $d=4$ for the Gauss distribution model (1). The major items of evidence are as follows.

1. The condition for the absence of localization is given by the familiar Ioffe–Regel principle[2]

$$kl\gtrsim 1,\quad \text{or}\quad E\tau\gtrsim 1, \tag{4}$$

where E and k are the energy and the momentum of an electron, and l and τ its mean free path and lifetime, respectively. In the Born approximation the latter satisfies $\tau^{-1}(E)\sim a_0{}^d W^2 N_0(E)$, where $N_0(E)\sim(a_0{}^d J)^{-1}(E/J)^{(d-2)/2}$ is the perfect-lattice density of states, and for $d<4$ the condition (4) reduces to

$$E\gtrsim J\Big(\frac{W}{J}\Big)^{4/(4-d)}, \tag{5}$$

whereas for $d>4$ it holds for all values of E under the above assumptions.

2. For large negative values of E, the optimum fluctuation method[7,8] may be used to estimate the density of states of a disordered system. By (2), the probability of a fluctuation-induced potential well of depth V and radius R is of order

$$\exp\{-\text{const}\, V^2R^d/W^2a_0{}^d\}. \tag{6}$$

If there is a level $E=-|E|$ present in the well, V and R are related by $E=-V+aJ(a_0/R)^2$ and (6) reduces to

$$\exp\Big\{-\text{const}\,\frac{1}{W^2a_0{}^d}\Big[|E|+\alpha J\Big(\frac{a_0}{R}\Big)^2\Big]^2 R^d\Big\},\quad \alpha\sim 1. \tag{7}$$

For $d<4$, the density of states $N(E)$ is determined by the saddle point of (7), $R_c\propto|E|^{-1/2}$, leading to the familiar Lifshitz result

$$N(E)\propto\exp\{-\text{const}\,|E|^{(4-d)/2}\}\quad (d<4). \tag{8}$$

For $d>4$ the extremum of Eq. (7) is reached at the minimum R possible, i.e., $R\sim a_0$ giving

$$N(E)\propto\exp\Big\{-\frac{(\alpha_1|E|+\alpha_2 J)^2}{W^2}\Big\}\quad (d>4). \tag{9}$$

Because the extremum is reached on the boundary of the domain of definition, the R derivative of the exponent of (7) does not vanish; in the field-theoretical formulation this corresponds to the absence of classical (instanton) solutions in the continuum limit for $d>4$.

3. The large-momentum renormalizability of the theory may be analyzed by evaluating the power of momenta in the expression for an arbitrary diagram.[9] In an order-$2n$ diagram for the average Green's function for the model (1), this power is

$$r=(d-4)n+2, \tag{10}$$

and the theory is renormalizable for $d<4$; for $d>4$, large-momentum divergences are unavoidable and accordingly a cutoff must be introduced.

4. In the field-theoretical formulation, the calculation of the average Green's function for the model (1) reduces to

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the evaluation of a functional integral with a Ginzburg–Landau Lagrangian—but with the 'wrong' sign of the quartic term.[6,10] Wilson's renormalization group equations,[5,6] which are valid for any signs of the coefficients, imply the stability of the Gaussian fixed point for $d>4$, which is indicative of the relative simplicity of the theory.

The facts above naturally lead to the hypothesis (cf. Refs. 11 and 12, for example) that $d=4$ is the upper critical dimension in the problem. This hypothesis has been criticized in the work of Thouless.[13] Thouless argues that the special role of the dimension $d=4$ relates to the fact that in the Gaussian white noise limit (i.e., when the potential correlation length tends to zero), no localized states exist for $d>4$. Since states of this type appear whenever there is a minimum length scale in the analysis—in the Anderson model, for example—it follows that the condition $d>4$ generally does not allow any simplifications. Indirectly, Thouless' arguments are corroborated by the results from the Lloyd model,[14] in which the density of states is calculated exactly and the dimension $d=4$ plays no special role; on the other hand, the obvious weakness of Thouless' arguments is that the features we have listed above are already observable *before* the white-noise limit is taken. Since the work of Thouless,[13] arguments have been advanced that $d_{c2}=6$[15] or $d_{c2}=8$[16], that the values of d_{c2} are different in the strong and weak disorder limits,[17,18] or that there may be no special dimension at all in the range $2<d<\infty$ (Ref. 19).

The discussion above shows that as yet the problem of the upper critical dimension for the Anderson transition remains unanswered. On the one hand, there is no general agreement as to the value of d_{c2}; on the other, we are aware of no publications showing what (if any) simplifications the condition $d>d_{c2}$ may produce.[1)] In the present paper, we show that for $d>4$, the density of states $N(E)$ of a disordered system can be calculated for the entire energy range including the vicinity of the Anderson transition.

2. THE IDEA OF THE METHOD

To obtain the density of states $N(E)$ requires the calculation of the average Green's function $G_k(E)$ for the model (1). The Green's function is determined by the diagram series shown in Fig. 1a,[24,25] where the vertices correspond to the scattering from individual lattice sites and the dashed lines link identical sites together; diagrams with more than two identical sites are absent in the Gaussian model (without a Born scattering assumption for an individual site).[6] By the standard argument it is found that

$$G_k(E)=\frac{1}{E-\varepsilon_0(k)-\sigma_k(E)}, \tag{11}$$

and our next step is then to calculate the self-energy $\sigma_k(E)$, containing irreducible diagrams only (Fig. 1b). For $d>4$, the k and E dependence of $\sigma_k(E)$ appears through the parameters ka_0 and E/J; for $E\cong 0$, the real part of σ may be incorporated into the renormalized spectrum without affecting its quadratic nature at small k (see Sec. 8). If the k integration is cut off at $\Lambda\sim a_0^{-1}$, the renormalized spectrum $\varepsilon(k)$ may be taken to be quadratic and $\Gamma=-\operatorname{Im}\sigma$ may be considered k-independent. Taking the edge of the renormalized spectrum as the zero of energy and considering the *retarded* Green's function as an example, we find

$$G_k(E)=\frac{1}{E-k^2/2m+i\Gamma(E)}, \qquad k<\Lambda. \tag{12}$$

In view of the above

$$\Gamma=-\operatorname{Im}\{\sigma_2+\sigma_4+\sigma_6+\ldots\}, \tag{13}$$

where σ_{2n} denotes the order-$2n$ perturbation theory self-energy contribution; since σ_{2n} is a functional of $G_k(E)$, the dependence $\Gamma(E)$ is given by

$$\Gamma=f(E,\Gamma). \tag{14}$$

To lowest (i.e., second) order in perturbation theory, (14) takes the form

$$\Gamma=f^{(0)}(E,\Gamma)=W^2a_0^d\int_0^J N_0(\varepsilon)\,d\varepsilon\,\frac{\Gamma}{(E-\varepsilon)^2+\Gamma^2}, \qquad J=\frac{\Lambda^2}{2m}. \tag{15}$$

It is readily seen that

$$\lim_{\Gamma\to 0} f^{(0)}(E,\Gamma)=\pi W^2a_0^dN_0(E), \quad \lim_{\Gamma\to 0}\frac{\partial f^{(0)}(E,\Gamma)}{\partial\Gamma}\sim\frac{W^2}{J^2}, \tag{16}$$

showing that for $E<0$ there is only a trivial solution $\Gamma=0$ (see Fig. 2), and since

$$N(E)=-\frac{1}{\pi}\operatorname{Im}\frac{1}{N}\sum_k G_k(E)=\int_0^J N_0(\varepsilon)\,d\varepsilon\,\frac{\Gamma}{(E-\varepsilon)^2+\Gamma^2}, \tag{17}$$

where N is the total number of lattice sites, we see that the density of states vanishes for $E<0$; consequently, it exhibits no fluctuation-induced tail in this approximation.

The same situation exists in any finite order of perturbation theory. To see this, note that for $E<0$, the real part of

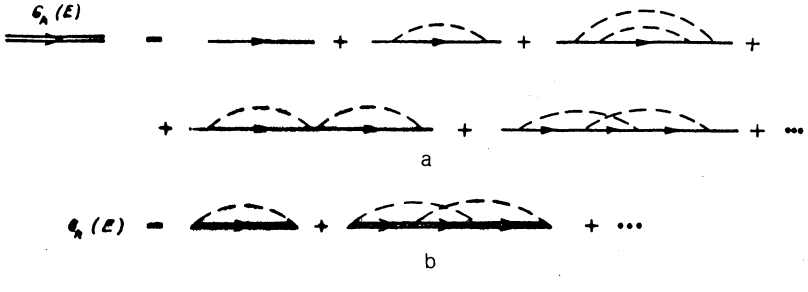

FIG. 1. Diagrams for the average Green's function (a) and for the self-energy (b) to the fourth order of perturbation theory.

the denominator of the G function (12) is nonzero, so that $\mathrm{Im}\, G_k(E) \propto \Gamma$ holds as $\Gamma \to 0$; consequently, both any individual diagram of Fig. 1b and a sum of any finite number of these diagrams have an infinitesimal imaginary part as $\Gamma \to 0$. As a result, $f(E,\Gamma) \to 0$ as $\Gamma \to 0$, and Eq. (14) has a root $\Gamma = 0$—indeed *the* (only) root because the estimate (16) for $\partial f/\partial \Gamma$ is true in the general case as well. A similar conclusion can be drawn from the estimate (9): since in the fluctuation region the density of states is exponentially small in $1/W^2$, the only way to obtain it is by summing the power series in W^2 to its very end, the term 'sum' departing from its conventional meaning in this context.

It is at this point that we are coming to the essence of the problem. It is easily shown that for $E \approx 0$, order-$2n$ self-energy diagrams are all of the same order of magnitude; because it is impossible to restrict oneself to any finite order of perturbation theory, it follows then that no selection of diagrams will suffice and so the series in its full should be summed up. There is, however, an important distinction of principle between the cases $d < 4$ and $d > 4$. For $d < 4$, not only different diagrams of the same order but also diagrams of different orders are of the same order of magnitude; for $d > 4$, an order-$(2n+2)$ diagram contains an extra small parameter W^2/J^2 as compared to its order-$2n$ counterpart.[2)] This latter feature is of crucial importance because it enables the right-hand side of (13) to be approximated by the first term σ_2 plus a sum of higher terms of the series; although small in magnitude, this sum is qualitatively important because it remains nonzero in the limit as $\Gamma \to 0$. Instead of (15) we have

$$\Gamma = W^2 a_0^d \int_0^J N_0(\varepsilon)\, d\varepsilon \frac{\Gamma}{(E-\varepsilon)^2+\Gamma^2} + \Gamma_0 \tag{18}$$

with the exponentially small quantity Γ_0

$$\Gamma_0 = cJ \left(\frac{J^2}{W^2}\right)^b \exp(-J^2/aW^2), \tag{19}$$

and we thus obtain a shift of the $f(E,\Gamma)$ curves in Fig. 2 (dashed line) and a fluctuation-induced tail for $E < 0$. Equation (18) describes a smooth transition between the asymptotic forms

$$\Gamma(E) = \begin{cases} \Gamma_0 + \pi W^2 a_0^d N_0(E), & E \gg \Gamma_0, \quad (20a) \\ \Gamma_0, & -E \ll \Gamma_0, \quad (20b) \end{cases}$$

and correspondingly for the density of states,

$$N(E) = \begin{cases} N_0(E) + \Gamma_0 \int_0^J \dfrac{N_0(\varepsilon)}{\varepsilon^2}\, d\varepsilon, & E \gg \Gamma_0, \quad (21a) \\ \dfrac{\text{const}}{Ja_0^d} \left(\dfrac{J^2}{W^2}\right)^b \exp(-J^2/aW^2), & -E \gg \Gamma_0. \quad (21b) \end{cases}$$

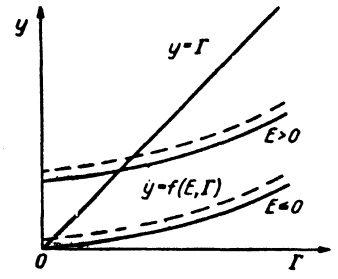

FIG. 2. Graphical solution of equation (14).

The decay of $N(E)$ at large negative values of E is controlled by the slow variation of a,

$$a(E/J) = a(0) + a'(0)E/J \tag{22}$$

[cf. (9)], the scale for the decay being given by W^2/J. In fact, equation (20a) is accurate to within

$$\frac{W^2}{J^2}\left(\frac{\Gamma_0}{J}\right)^{(d-4)/2} \tag{23}$$

uniformly in E, the exact result differing only in the smoothing (on a scale of Γ_0) of the singularity in $N_0(E)$ at $E = 0$. For $|E| \lesssim \Gamma_0$, the second term in (20a) is small in the parameter (23) and $N(E)$ is given by (17) with $\Gamma = \Gamma_0$. It thus follows (Fig. 3) that the density of states of a $d > 4$ disordered system crosses, over an exponentially small energy interval, from the perfect-crystal $N_0(E)$ behavior to exponential decay on a scale of W^2/J, without showing any singularities in the vicinity $E \sim \Gamma_0$ around the Anderson transition.[3)]

At this point, the origin of the last term in (18) should be explained in some detail. Consider the chain of equalities

$$\sum_{n=0}^{\infty} n! z^n = \sum_{n=0}^{\infty} z^n \int_0^{\infty} dx e^{-x} x^n = \int_0^{\infty} dx \frac{e^{-x}}{1-xz} = \int_0^{\infty} \frac{dy}{y(y-z)} e^{-1/y}, \tag{24}$$

describing the summation, in the Borel sense, of a divergent factorial series.[27] Setting $z = g + iO(g > 0)$ and taking the imaginary part yields

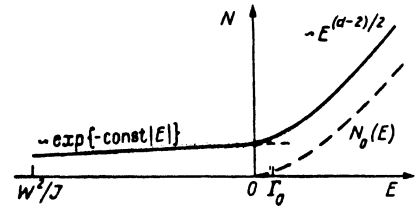

FIG. 3. Density of states near the edge of the unperturbed band in $d > 4$ dimensions.

$$\mathrm{Im}\sum_{n=0}^{\infty} n!\,(g+i0)^n = \frac{\pi}{g}\,e^{-1/g}, \tag{25}$$

showing how the infinitesimal increment $+iO$ transforms, when the divergent series is summed, into an exponentially small (for $g \ll 1$) but still finite quantity.

Intuitively, the applicability of the above arguments to a perturbation theory series follows from the fact that $2n$-order self-energy diagrams increase in number factorially with n while being all of the same order of magnitude; for $g \sim W^2/J^2$ [which is the expansion parameter in (13)], a result of the type (19) is obtained.[4)] To show that the imaginary addition $+iO$ appears in exactly the necessary combination in each term calls for a more detailed treatment which can be carried out by the statistical analysis of the remote terms of the perturbation expansion; that the remote terms alone may be taken follows from the fact that (25) is also true when the summation starts from an arbitrary finite n_0 instead of $n = 0$. The quantities a, b, and c appear as phenomenological parameters in the statistical analysis, however, they can be determined by recognizing that the domian of validity of our approach overlaps with those of the optimum fluctuation and (more sophisticated) instanton methods.[28–30] To see this, recall that in view of (9) the value of the exponent in the case $d > 4$ remains large down to $E = 0$, and the applicability of the above two methods is not controlled by the saddle-point method but only depends on whether it is possible to neglect the interaction of deep fluctuations via the wave function tails $\exp(-\varkappa r)$ ($\varkappa = \sqrt{2m|E|}$) which extend to infinity as $|E| \to 0$. Because the concentration of deep fluctuations is exponentially small, their interaction at $E < 0$ is weak outside the exponentially narrow vicinity of the point $E = 0$. Since the quantity Γ_0 may be considered energy-independent for $\Delta \lesssim W^2/J$, the parameters a, b, and c can be determined by matching to the results of the instanton method.

3. RELATION BETWEEN THE ORDER-2n AND ORDER-(2n+2) DIAGRAMS

An order-$2n$ diagram involves n integrations over the momenta $q_1, q_2, \ldots q_n$ of the dashed (impurity) lines and $(2n-1)$ Green's functions whose momenta are determined by linear combinations of the q_i and the external momentum k; the momentum q_i only appears in the arguments of those G-functions covered by the ith dashed line. Replacing q_i with $q_i\Lambda$ to nondimensionalize the integrals, we can write the contribution from an order-$2n$ diagram in the form

$$\sigma^{(2n)} = J\frac{W^{2n}}{J^{2n}} f\left(\frac{E}{J}, \frac{\Gamma}{J}, \frac{k^2}{2mJ}\right). \tag{26}$$

For $d > 4$, the function f is finite at the point $E = \Gamma = k^2/2m = 0$ and so are its first derivatives with respect to all its arguments; this follows from the nature of the small-momentum convergence at $E = \Gamma = k^2/2m = 0$. Suppose l dashed lines have a small momentum $\sim q$, and the remaining $l' = n - l$ lines have a large momentum $\sim\Lambda$, which enters the arguments of no less than $2l'$ Green's functions (at a minimum, l' dashed lines cover $2l' - 1$ Green's functions but in this particular case the corresponding part of the diagram turns out to be a self-energy insertion); the small momentum contribution is then given by the integral

$$\int_0^{q_0} \frac{q^{dl-1}\,dq}{q^{2(2l-1)}} \sim q^{2+(d-4)/l}\Big|_0^{q_0}, \tag{27}$$

which is convergent for all $l \geqslant 1$ on its lower limit. Differentiation of the function f with respect to one of its arguments reduces the power of q by 2, but the lower-limit convergence survives. It follows from (27) that the main contribution into the diagram comes from the region $q_i \sim \Lambda$. Since the above argument shows $\sigma^{(2n)}$ to vary slowly with E, Γ, and k, we set $E = 0$ and $k = 0$; we consider the parameter Γ/J to be finite because this is the only source for the imaginary part of $\sigma^{(2n)}$.

An order-$(2n+2)$ diagram may be obtained from an order-$2n$ diagram by "suspending" a dashed (impurity) line as shown in Fig. 4; the contributions from the diagrams of Figs. 4a and 4b are of the form

$$\sigma^{(2n)} = \sum_{k'}\sum_{q_1\ldots q_m} F(k', q_1 \ldots q_m)\, G_{k'} G_{k'+q_1} \ldots G_{k'+q_1+\ldots q_m}, \tag{28}$$

$$\sigma^{(2n+2)} = \sum_{k'}\sum_{q_1\ldots q_m} F(k', q_1 \ldots q_m)\,\frac{W^2}{N} \times \sum_{k_1} G_{k'} G_{k_1} G_{k_1+q_1} \ldots G_{k_1+q_1+\ldots q_m} G_{k'+q_1+\ldots q_m}, \tag{29}$$

or, in a more compact notation,

$$\sigma^{(2n)} = \sum_Q H(Q), \qquad \sigma^{(2n+2)} = \sum_Q H(Q)\,h(Q), \tag{30}$$

$$Q \equiv (k', q_1, \ldots, q_m),$$

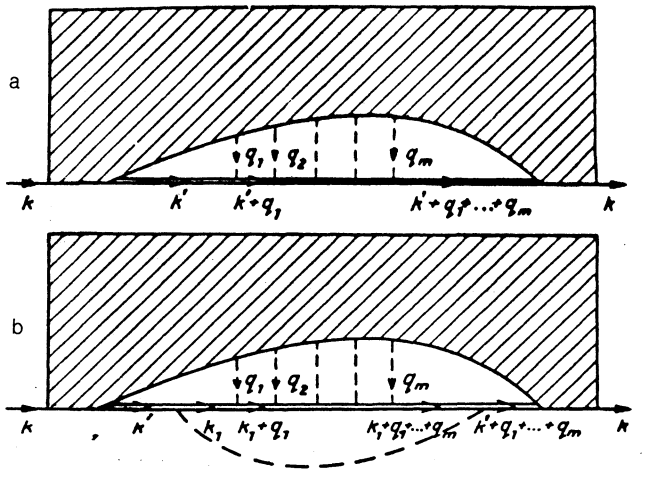

FIG. 4. An order-$(2n+2)$ diagram (b) is obtained by "suspending" a dashed (impurity) line on a certain order-$2n$ diagram.

$$h(Q)=h(k', q_1 \dots q_m)$$

$$=\frac{W^2}{N}\frac{\sum_{k_1} G_{k_1}G_{k_1+q_1}\dots G_{k_1+q_1+\dots+q_m}}{G_{k'+q_1}\dots G_{k'+q_1+\dots q_{m-1}}}. \tag{31}$$

Now one and the same order-$(2n+2)$ diagram may be obtained from different order-$2n$ diagrams. To remove this ambiguity, let us require that the suspended dashed line be of the smallest length possible; then there is an overwhelming probability that the number of vertices it covers is ~ 1 (see Appendix 1). Taking $k', q_i \sim \Lambda$ and $m \sim 1$ in (31), we obtain the estimates

$$h' \sim \frac{W^2}{J^2}, \quad h'' \sim \frac{W^2}{J^2}\frac{\Gamma}{J}, \tag{32}$$

where a single (double) prime denotes a real (imaginary) part. Taking the real and imaginary parts of (30)

$$\begin{aligned}\sigma'^{(2n+2)}&=\sum_Q h'(Q)H'(Q)-\sum_Q h''(Q)H''(Q),\\ \sigma''^{(2n+2)}&=\sum_Q h''(Q)H'(Q)+\sum_Q h'(Q)H''(Q)\end{aligned} \tag{33}$$

and making use of (32) to obtain dimensional estimates, the relation between the order-$2n$ and order-$(2n+2)$ diagrams follows as

$$\begin{vmatrix}\sigma'^{(2n+2)}\\ \sigma''^{(2n+2)}\end{vmatrix}=\begin{vmatrix}\sim W^2/J^2 & \sim(-W^2\Gamma/J^3)\\ \sim W^2\Gamma/J^3 & \sim W^2/J^2\end{vmatrix}\begin{vmatrix}\sigma'^{(2n)}\\ \sigma''^{(2n)}\end{vmatrix}. \tag{34}$$

By appropriate introduction of constants we now express (34) in the exact form

$$\begin{vmatrix}x_{n+1}\\ y_{n+1}\end{vmatrix}=g\begin{vmatrix}A_n & -\mu D_n\\ \mu B_n & C_n\end{vmatrix}\begin{vmatrix}x_n\\ y_n\end{vmatrix}, \tag{35}$$

where for the sake of brevity we have defined

$$x_n=\sigma'^{(2n)}, \quad y_n=\sigma''^{(2n)}, \quad g=W^2/J^2, \quad \mu=\Gamma/J. \tag{36}$$

We can divide the suspended dashed lines into certain "classes" each of which is characterized by a certain narrow interval of the values of A_n, B_n, C_n, and D_n. If we successively suspend dashed lines of only one particular class to the first diagram of Fig. 1b, we note that in the limit $n\to\infty$, the resulting diagrams constitute only an infinitesimal portion of the total number of self-energy diagrams. In a typical order-$2n$ diagram, the class of a dashed line is chosen randomly at each step. This allows one to invoke statistical concepts and to treat A_n, B_n, C_n, and D_n as random quantities.

4. ANALYSIS OF EQUATION (35)

Equations of the type (35) are studied in the theory of one-dimensional disordered systems[8,31] and are in principle amenable to a thorough investigation. For our purposes, it is sufficient to analyze the evolution of the first and second moments of x_n and y_n, which is governed by

$$\begin{vmatrix}\overline{x_{n+1}}\\ \overline{y_{n+1}}\end{vmatrix}=g\begin{vmatrix}\bar A & -\mu\bar D\\ \mu\bar B & \bar C\end{vmatrix}\begin{vmatrix}\bar x_n\\ \bar y_n\end{vmatrix}, \tag{37}$$

$$\begin{vmatrix}\overline{x_{n+1}^2}\\ \overline{x_{n+1}y_{n+1}}\\ \overline{y_{n+1}^2}\end{vmatrix}=g^2\begin{vmatrix}\overline{A^2} & -2\mu\overline{AD} & \mu^2\overline{D^2}\\ \mu\overline{AB} & \overline{AC}-\mu^2\overline{BD} & -\mu\overline{CD}\\ \mu^2\overline{B^2} & 2\mu\overline{BC} & \overline{C^2}\end{vmatrix}\begin{vmatrix}\overline{x_n^2}\\ \overline{x_ny_n}\\ \overline{y_n^2}\end{vmatrix}. \tag{38}$$

Solutions to (37) are of the form $\overline{x_n}, \overline{y_n}\sim(g\chi)^n$, where

$$\chi^{\pm}={}^1\!/_2[(\bar A+\bar C)\pm[(\bar A-\bar C)^2-\mu^2\bar B\bar D]^{1/2}]. \tag{39}$$

Within the phenomenological analysis framework, two possibilities can arise,

$$(a)\ \ (\bar A-\bar C)^2>\mu^2\bar B\bar D \ \text{ и } \ (б)\ \ (\bar A-\bar C)^2<\mu^2\bar B\bar D. \tag{40}$$

The following argument shows, however, that case (a) is not, in fact, realizable. Suppose the energy E has an imaginary part $\operatorname{Im} E>0$ sufficiently large that $|E|\gg J$; the quantity h then does not depend on momenta and is given by

$$h=\frac{W^2}{E^2}=\frac{W^2}{|E|^2}e^{-2i\varphi}, \tag{41}$$

which implies that $\chi^{\pm}=|h|e^{\mp 2i\varphi}$, $\operatorname{Im}\chi^{\pm}\neq 0$ and hence case (b) is realized. The assumption that case (a) is obtained for $\operatorname{Im} E\to +0$ implies that at finite $\operatorname{Im} E$, a transition from (b) to (a) takes place; as a result, the Green's function exhibits a jump-like (discontinuous) singularity, which is inconsistent with its analyticity in the upper half-plane. For case (b), the general solution of (37) is

$$\begin{aligned}\overline{x_n}&=CJ\left(\frac{a}{2}\frac{W^2}{J^2}\right)^n\left(\frac{4\bar D}{\bar B}\right)^{1/2}\cos(\tilde\mu n+\varphi_0),\\ \overline{y_n}&=CJ\left(\frac{a}{2}\frac{W^2}{J^2}\right)^n\cos(\tilde\mu n+\varphi_0+\varphi_1),\end{aligned} \tag{42}$$

where

$$\begin{aligned}&a=\bar A+\bar C, \quad \tilde\mu=a^{-1}(\mu^2\bar B\bar D-(\bar A-\bar C)^2)^{1/2},\\ &\varphi_0=\arccos[(\bar C-\bar A)/\mu(\bar B\bar D)^{1/2}],\end{aligned} \tag{43}$$

and C and φ_1 are determined from the initial conditions; their exact calculation from $\sigma^{(2)}$ is meaningless because the distribution of the quantities $A_n, B_n, \dots$ is clearly time-dependent for small n. Noting that $\operatorname{Re}\sigma^{(2n)}\to\text{const}$., $\operatorname{Im}\sigma^{(2n)}\sim\Gamma$, and $\operatorname{Im}\sigma^{(2n)}<0$ for $\Gamma\to 0$, we find that $C\sim 1$ and $\varphi_0+\varphi_1=\pi/2+O(\mu)$; neglecting the term $O(\mu)$,

$$\overline{y_n}=-CJ\left(\frac{a}{2}\frac{W^2}{J^2}\right)^n\sin\tilde\mu n \qquad (C>0). \tag{44}$$

The solution to (38) is

$$\overline{x_n^2}, \ \overline{x_ny_n}, \ \overline{y_n^2}\propto(\chi_1 g)^{2n}, \tag{45}$$

where

$$\chi_1^2=\max\{\overline{A^2}, \overline{AC}, \overline{C^2}\}+O(\mu^2). \tag{46}$$

It is readily seen that $\chi_1>|\chi^{\pm}|$, that is, the variances of x_n and y_n grow faster than their means; which implies that the

contribution from an individual order-$2n$ diagram is a strongly fluctuating quantity.

5. TOTAL ORDER-$2n$ CONTRIBUTION

The total order-$2n$ perturbation theory contribution σ_{2n} is a sum of contributions $\sigma_i^{(2n)}$ from individual diagrams and is determined, within the statistical context, by the average $\langle \ldots \rangle_{M_n}$ over a finite-size sample of M_n elements,

$$\sigma_{2n}=\sum_{i=1}^{N_{2n}}\sigma_i^{(2n)}\approx N_{2n}\langle\sigma_i^{(2n)}\rangle_{M_n}, \tag{47}$$

where N_{2n} denotes the number of order-$2n$ self-energy diagrams; it is easy to show (see Appendix 2) that

$$(2n-1)!!>N_{2n}>(2n-3)!!. \tag{48}$$

To determine σ_{2n} with an accuracy ξ_n requires that M_n, the size of the sample, be of order

$$M_n\sim\frac{1}{\xi_n^2}\left(\frac{2\chi_1}{a}\right)^{2n}, \tag{49}$$

where we have used (42) and (45) and assumed the $\sigma_i^{(2n)}$ to be statistically independent. Let $\sigma_i^{(2n)}$ and $\sigma_j^{(2n)}$ be two elements of the sample; introducing the notation $\hat{S}$ for the matrix in (35), we have

$$\sigma_i^{(2n)}=\hat{S}_{n-1}\hat{S}_{n-2}\ldots\hat{S}_2\hat{S}_1\sigma^{(2)},$$

$$\sigma_j^{(2n)}=\hat{S}'_{n-1}\hat{S}'_{n-2}\ldots\hat{S}_2'\hat{S}_1'\sigma^{(2)}. \tag{50}$$

Some of the matrices $\hat{S}_k$, $\hat{S}'_k$ may be identical; the probability for this to happen is only noticeable for $k\lesssim k_0$, where k_0 is given by the condition

$$N_{2k_0}\sim M_n. \tag{51}$$

The correlation coefficient for the elements $\sigma_i^{(2n)}$ and $\sigma_j^{(2n)}$ is of order $(a/2\chi_1)^{2(n-k_0)}$. Owing to the factorial growth of N_{2n}, the quantities M_n can be selected so that we have $\xi_n\to 0$ and $k_0/n\to 0$ as $n\to\infty$; this ensures an arbitrarily high accuracy of σ_{2n} as the correlation between the elements of the sample is unboundedly weakened.

We note next that the lower and upper bounds for N_{2n} [see Eq. (48)] obey the recursion relations

$$N_{2n+2}^{\text{up}}=(2n+1)N_{2n}^{\text{up}},\quad N_{2n+2}^{\text{low}}=(2n-1)N_{2n}^{\text{low}} \tag{52}$$

and we assume that a similar recursion relation holds for N_{2n} proper,

$$N_{2n+2}=(2n+2\beta)N_{2n},\quad 1\geqslant 2\beta\geqslant -1. \tag{53}$$

Then

$$N_{2n}=\text{const}\,2^n\gamma(n+\beta), \tag{54}$$

where $\gamma(x)$ is the gamma function, and by (44) and (47),

$$\text{Im}\,\sigma_{2n}=-\text{const}\,J\gamma(n+\beta)\left(a\frac{W^2}{J^2}\right)^n\sin\tilde{\mu}n,\quad \text{const}>0. \tag{55}$$

6. EQUATION FOR Γ

If we assume the statistical description to be sufficiently accurate for $n>n_0$, Eqs. (13) and (55) yield

$$\Gamma=\text{Im}\left\{-\sigma_2-\sigma_4-\ldots-\sigma_{2n_0}+\text{const}\sum_{n=n_0+1}^{\infty}\gamma(n+\beta)(ag)^n(1+i\tilde{\mu})^n\right\}. \tag{56}$$

The finite-order contributions σ_2, σ_4,... have vanishingly small imaginary parts as $\Gamma\to 0$,

$$\text{Im}\,\sigma_{2n}\sim\Gamma\frac{W^{2n}}{J^{2n}}, \tag{57}$$

and it suffices to keep only the first of these contributions. The sum over n in (56) is calculated similarly to (24) and has a finite imaginary part in the $\Gamma\to 0$ limit. Noting that $\tilde{\mu}\sim\mu\ll 1$ holds we arrive at (18) and (19) with $a=\overline{A}+\overline{C}$, $b=\beta$, although the second of these equalities is of only minor importance, though (see Sec. 7).

7. EFFECTS OF DISTRIBUTION

In the above discussion, a number of assumptions concerning the statistical properties of the coefficients A_n, B_n, C_n, and D_n have been used (such as the existence of means; steadiness of the distribution over n, etc.) which are difficult to justify mathematically. This justification problem may be circumvented, however, by recognizing that the domain of validity of the present approach overlaps with that of the instanton method (see Sec. 2).

Remaining within the phenomenological analysis framework, we can modify the statistical hypotheses underlying the above calculations; the consequences may be as follows:

a) the functional structure of Γ_0, Eq. (19), remains unchanged, although the meaning of the parameters a, b, and c is generally altered; this modification is of no importance for our further discussion;

b) the functional structure of Γ_0 *does* change; this modification should be rejected as inconsistent with the results of the instanton method (see Sec. 9).

As an illustration, the (possible) nonstationarity in the distribution of the matrix coefficients in (35) will be analyzed. Let $\overline{A}_n$, $\overline{B}_n$,... depend on n; if we assume this dependence to be weak and use a quasiclassical-type approximation, (55) becomes

$$\text{Im}\,\sigma_{2n}=-\text{Im}\,CJ(2g)^n\gamma(n+\beta)\exp\left\{\sum_{k=0}^{n-1}\lambda_k+i\mu\sum_{k=0}^{n-1}\eta_k\right\}, \tag{58}$$

where the λ_n and η_n are expressed through the means $\overline{A}_n$, $\overline{B}_n$,... . If $\lambda_n\to\lambda_\infty$, $\eta_n\to\eta_\infty$ as $n\to\infty$,

$$\sum_{k=0}^{n-1}\lambda_k=\lambda_\infty n+o(n),\qquad \sum_{k=0}^{n-1}\eta_k=\eta_\infty n+o(n). \tag{59}$$

It turns out that the above sums cannot in fact grow faster than linearly: the proof goes through exactly as that for the falsehood of the inequality $(\overline{A}-\overline{B})^2>\mu^2\overline{B}\,\overline{D}$ (Sec. 4).

Generally speaking, the rate of convergence of the λ_n and η_n to their stationary values is of considerable importance. For example, let

$$\lambda_n=\lambda_\infty+\frac{\lambda_1}{n}, \quad \eta_n=\eta_\infty+\frac{\eta_1}{n} \quad (n\to\infty), \tag{60}$$

(which corresponds to a regular $1/n$ expansion); then, by (58) and Stirling's formula,

$$\begin{aligned}\operatorname{Im}\sigma_{2n}&=-\operatorname{Im}CJ(2g)^n\gamma(n+\beta)\exp\{\lambda_\infty n+\lambda_1\ln n\\&\quad+i(\eta_\infty n+\eta_1\ln n)+\text{const}\}\\&=-\tilde{C}J\operatorname{Im}(2g\exp(\lambda_\infty+i\eta_\infty))^n\gamma(n+\beta+\lambda_1+i\mu\eta_1)\end{aligned} \tag{61}$$

which, on performing the sum in (56), leads to (19) with the exponent

$$b=\beta+\lambda_1. \tag{62}$$

Thus the exponent b cannot generally be identified with the combinatorial constant β.

If the λ_n and η_n converge to λ_∞ and η_∞ faster than $1/n$, the distribution time dependence only figures in redefining the constant c in (19). For convergence rates lower than $1/n$, no finite power of J^2/W^2 is capable of representing the preexponential factor in (19); the implication is that this latter undergoes a radical change in its functional form—in contradiction to the results of the instanton method.

8. THE k AND E DEPENDENCE OF σ

Comparing (26) with (55) or (61) and noting that the quantity Γ_0 in (18) is determined by passing to the limit $\Gamma\to 0$, we find that

$$a=a\left(\frac{E}{J},\frac{k^2}{2mJ}\right), \quad b=b\left(\frac{E}{J},\frac{k^2}{2mJ}\right), \quad c=c\left(\frac{E}{J},\frac{k^2}{2mJ}\right), \tag{63}$$

where all the functions are finite and singly differentiable at the point $E=k^2/2m=0$. According to the results of the instanton method (Sec. 9), which are valid for $-E\gg\Gamma$, the parameters a and b are both independent of k, and b is independent of E:

$$a=a(E/J), \quad b=\text{const}, \quad c=c(E/J,\ k^2/2mJ). \tag{64}$$

Because of the differentiability with respect to E, these properties also hold for $|E|\ll J$. Thus the k dependence of Γ_0 only appears through the coefficient c and is weak. For a specific choice of the overlap integrals J_{n-n} in (1), this dependence can be determined by the instanton method (using the Fourier transform of Eq. (77) of Sec. 9); for the cutoff model (12), the introduction of the $c(k)$ dependence implies going beyond the accuracy of the calculation. The energy dependence of Γ_0 mainly comes through the parameter a, whose energy dependence is given by (22) because of the differentiability property.

The real part of σ, associated with remote perturbation theory terms, is analytic in a, b, and c [cf. (24)]—hence its slow variation with energy and its differentiability with respect to k^2 (Sec. 2).

With the approximations adopted, the density of states $N(E)$ is given by expressions (17) through (19) and turns out to be a smooth function of E. The following argument shows that this result also holds for the exact $N(E)$: For finite values of Γ, the contributions from an individual diagram (26) is analytic as a function of E and Γ. Comparison with (55) and (61) shows that the parameters a, b, c, and μ are analytic as functions of E and Γ, and therefore, so is the sum of the remote perturbation theory terms, which is an analtyical function of these parameters [cf. (24)]. As a result, the function $f(E,\Gamma)$ in (14) is analytic for finite Γ. Since the root of equation (14) is nonzero for all values of E, it follows that $N(E)$ is a smooth function of E.

9. CALCULATION OF $N(E)$ FOR $E<0$ BY THE INSTANTON METHOD

Because the instanton method is usually applied to the case $d<4$ (Refs. 28–30)—and noting that Harris and Lubensky's[18] result is clearly incorrect—we consider the case $d>4$ in this section; we follow the review article by Sadovskii[30] and focus on the specifics of the case $d>4$ while omitting unnecessary details.

Application of the replica trick to the discrete Gaussian model of (1) yields

$$G_{n_1n_2}=\lim_{n\to 0}\int D\Phi^\alpha\Phi_{n_1}^{(1)}\Phi_{n_2}^{(1)}\exp\{-S[\Phi]\}, \tag{65}$$

$$\begin{aligned}S[\Phi]&=\tfrac{1}{2}\sum_{\alpha=1}^{n}\sum_{nn'}(J_{nn'}-E\delta_{nn'})\Phi_n^\alpha\Phi_{n'}^\alpha\\&\quad-\tfrac{1}{8}W^2\sum_n\left(\sum_{\alpha=1}^{n}(\Phi_n^\alpha)^2\right)^2.\end{aligned} \tag{66}$$

The classical (or instanton) solution is of the form

$$(\Phi_n^\alpha)_{cl}=\Phi_n^{cl}u^\alpha, \tag{67}$$

where u^α denotes a component of the unit vector in the replica space and Φ_n^{cl} solves the equation

$$\sum_m J_{n-m}\Phi_m^{cl}-E\Phi_n^{cl}-\frac{W^2}{2}(\Phi_n^{cl})^3=0. \tag{68}$$

For $d>4$, an optimum-fluctuation estimate (see Sec. 1) implies that the instanton is localized on the interatomic scale, $|\vec{n}|\sim 1$; the term $E\Phi$ in (68) is then negligible in comparison with the first term $\sim J\Phi$. For $|\vec{n}|\gg 1$, the function Φ_n^{cl} is small in magnitude and slow, which enables one to neglect the term $\sim\Phi^3$ and to expand the first term in gradients (note that $\Sigma_m J_{n-m}\equiv 0$). We obtain

$$\Phi_n^{cl}=C|n|^{-\nu}K_\nu(\varkappa|n|), \quad \nu=(d-2)/2, \quad \varkappa=(|E|/J)^{1/2}, \tag{69}$$

where $K_\nu(x)$ is the modified Bessel function of the second kind and the constant C is determined by matching to the region $|\vec{n}|\sim 1$ and is of order $\varkappa^\nu J^{1/2}/W$. It is easily checked that the region $|\vec{n}|\gg 1$ contributes negligibly to the action: the opposite conclusion was reached in Ref. 18 from using in this limit a solution parametrization technique analogous to that for $d<4$; this led to $C\sim\varkappa^{-1}$ and an overpredicted instanton tail contribution. Neglecting the energy dependence we set $E=0$ and nondimensionalize by writing

$$\Phi_n^{cl}=\left(\frac{2J}{W^2}\right)^{1/2}\chi_n, \quad J_{n-m}=JI_{n-m}, \tag{70}$$

where $I_{n-m}\sim 1$ and χ_n satisfies the equation

$$\sum_m I_{n-m}\chi_m-\chi_n^3=0. \tag{71}$$

The classic action then takes the form

$$S[\Phi^{cl}]=\frac{J^2}{aW^2}, \quad a^{-1}=1/2\sum_n \chi_n^4. \tag{72}$$

For a slightly nonclassical solution such that

$$\Phi_n^\alpha=\Phi_n^{cl}u^\alpha+\varphi_n^\alpha \tag{73}$$

the action is written in the form

$$S[\Phi]=S[\Phi^{cl}]+1/2\sum_n\sum_{\alpha\beta}\varphi_n^\alpha\hat{M}_{\alpha\beta}\varphi_n^\beta, \tag{74}$$

where

$$\begin{aligned}\hat{M}_{\alpha\beta}&=\hat{M}_L u^\alpha u^\beta+\hat{M}_T(\delta_{\alpha\beta}-u^\alpha u^\beta),\\ \hat{M}_L f_n&=J\Big[\sum_m I_{n-m}f_m-3\chi_n^2 f_n\Big],\\ \hat{M}_T f_n&=J\Big[\sum_m I_{n-m}f_m-\chi_n^2 f_n\Big].\end{aligned} \tag{75}$$

Because of the atomic-scale character of instanton localization the discrete nature of the model is of principal importance for $d>4$ (there are no instantons in the continuum limit) and leads to the absence of translational zero modes. Separating out *rotational* zero modes, nondimensionalizing the eigenvalues of the operators $\hat{M}_L$ and $\hat{M}_T$,

$$\lambda_s^L=J\mu_s^L, \quad \lambda_s^T=J\mu_s^T \tag{76}$$

and passing to the limit $n\to 0$, we obtain

$$\operatorname{Im} G_{nn'}=\text{const}\,\frac{1}{J}\left(\frac{J}{W}\right)\exp(-J^2/aW^2)\sum_{n_0}\chi_{n+n_0}\chi_{n'+n_0}, \tag{77}$$

which yields the density of states of the form (21b) with the exponent

$$b=1/2. \tag{78}$$

The energy dependence of the parameter a is obtained from (68) by iterating on E/J in the instanton "core" region:

$$a^{-1}(E)=\frac{1}{2}\sum_n\chi_n^4+\frac{2E}{J}\sum_n\chi_n^3\zeta_n, \quad \zeta_n=J\hat{M}_L^{-1}\chi_n, \tag{79}$$

which determines the quantity $a'(0)$ in (22).

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APPENDIX 1. ESTIMATE OF THE LENGTH OF A SUSPENDED DASHED LINE

We define the length of a dashed line as the number of the sites it covers.

1. We want to prove that the shortest dashed line has a length ~ 1 however large n is.

Let us evaluate the probability for finding a diagram with no dashed lines shorter than x. Suppose a diagram is constructed by successively connecting by dashed lines arbitrarily chosen pairs of sites. Neglecting boundary effects, the probability that the first dashed line does not terminate at a distance x from its beginning is

$$1-\frac{2x}{2n-1}.$$

Since the earlier occupied sites are distributed randomly, the same estimate holds for all later dashed lines as well. The probability that none of the lines is shorter than x is given by

$$p(x)=\left(1-\frac{2x}{2n-1}\right)^n\approx e^{-x},$$

indicating that there is an overwhelming probability that a dashed line of length ~ 1 does exist.

2. Consider the shortest possible dashed line in an irreducible order-$(2n+2)$ diagram (as discussed in part 1, its length is ~ 1). Let us demonstrate that for large n, the removal of this line with an overwhelming probability leaves the diagram irreducible.

Suppose the removal of this line makes the diagram reducible, that is, breaks it up into two *disconnected* blocks with $2n'$ and $2n''$ vertices such that $n'+n''=n$ (one of the blocks may be a self-energy insertion within the other). By (54), the probability for the separation into two blocks with fixed boundaries is

$$\frac{N_{2n'}N_{2n''}}{N_{2n}}=\frac{2^{n'}\gamma(n'+\beta)\,2^{n''}\gamma(n''+\beta)}{2^{n'+n''}\gamma(n'+n''+\beta)}$$

and is of order $1/n''$ for small n''. Separation of a (minimum-size) 2-site block has the highest probability, equal to $1/n$; since $\sim n$ locations are available for such a block, the separation probability becomes ~ 1, ensuring that the number of irreducible diagrams differs considerably from the total number of diagrams. We may also consider blocks having a boundary at the position of the eliminated line, but the probability of separation into such blocks is of order $1/n$ an is negligibly small in the $n\to\infty$ limit.

It thus follows from parts 1 and 2 that an order-$(2n+2)$ diagram may be obtained from a certain order-$2n$ diagram by suspending a dashed (impurity) line of length ~ 1.

APPENDIX 2. ESTIMATE OF THE NUMBER OF SELF-ENERGY DIAGRAMS

The upper bound for N_{2n} is given by the total number of order-$2n$ diagrams for the G function; this number is $(2n-1)!!$: a dashed line connects the first vertex with any one of the remaining $(2n-1)$ vertices, then the next free vertex is connected to one of the remaining $(2n-3)$, and so forth.

The lower bound is obtained as follows: the first vertex is connected with any other except for the second and the last [$(2n-3)$ possibilities], then the first free vertex is connected with any one of the free vertices, except for the second and the last [$(2n-5)$ possibilities], and so forth; the two last vertices are connected with each other. Let us prove that the resulting $(2n-3)!!$ diagrams are all of the self-energy type.

(a) All the above diagrams are compact in the sense that they cannot be dissected by a vertical without simultaneously crossing dashed (impurity) lines. Suppose a dashed

line terminates in the vertex k at a given step in the above construction; since, by construction, each dashed line passes above one free vertex at least, there are free vertices between the first and k th vertices. For the same reason, these free vertices cannot join together: the last of them will necessarily give rise to a dashed line passing above the k th vertex, so that the compactness of the diagram cannot break down at the k th vertex. Repeating the above argument for the next dashed line etc., we are led to the conclusion that the compactness should not be broken at all.

(b) The diagrams we have constructed contain no self-energy insertions. The first dashed line does not terminate at the last vertex and the remainder of the diagram does not represent its self-energy insertion. Suppose that immediately below the line there is an insertion containing some portion of the remaining sites. Then the removal of the dashed line would render the diagram noncompact—but this is impossible in view of (a) because $2n-2$ vertices that remained when the first dashed line was removed were filled by the same algorithm used for the original $2n$ vertices. Similar arguments show the absence of self-energy insertions under the second and further dashed lines.

[1)]An attempt of this kind has been reported by Harris and Lubensky,[18] whose analysis is clearly unphysical in predicting the density of states in $d>4$ to go to zero at a certain point: a consequence, in fact, of the use of incorrectly constructed lattice instanton, see Sec. 9. The self-consistent localization theory,[20,21] while yielding kinks in critical exponents at $d=4$, employs uncontrollable approximations unlikely to be valid for any space dimension; in particular, the theory breaks down completely if a spatially dispersive diffusion coefficient is considered. A disagreement between the value of the critical exponent of conductivity and its $d=\infty$ counterpart[22,23] should also be noted.

[2)]For $d<4$, the relevant integrals are dominated by the small values of momenta, and the Green's function $G_k(E)$ should be considered of order Γ^{-1}. For $d>4$, the main contribution comes from momenta of order Λ, giving $G_k(E)\sim J^1$.

[3)]One usually employs exact Lloyd model solutions to justify the latter result. Results from hierarchical models show, however,[26] that potentials with infinite dispersion belong to a different universality class and Lloyd model results cannot generally be applied to the Gaussian model (1).

[4)]In a somewhat different context, Brezin and Parisi[28] analyzed the relation between the fluctuation-induced tail and the factorial divergence of the perturbation theory series.

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