Realization of chiral symmetry and the axial anomaly pole

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The paper shows that there is no vertex factorization for axial anomaly poles. This means that the Goldstone boson associated with spontaneous chiral-symmetry breaking cannot reproduce an axial anomaly pole. It also shows that introducing a massless Goldstone pseudoscalar boson requires introducing a massless pseudovector boson.

1. INTRODUCTION

The classical axial anomaly problem[1,2] is still in the focus of attention of theoreticians.

Recently I published a paper[3] giving the analytical form of the invariant amplitudes (free of kinematic singularities) of single-loop triangle diagrams (Fig. 1), representing the axial-vector current $\to \gamma(k_1)\gamma(k_2)$ transition at $k_1^2=0$ and $k_2^2\neq 0$. There I demonstrated that an axial anomaly pole[4] emerges only in the limit of massless fermions and only for real photons ($k_2^2=0$), in contrast to the rather prevailing opinion (see, e.g., Ref. 5).

In this paper I analyze the problem of spontaneous chiral-symmetry breaking, to which, I believe, the results published in Ref. 3 have given a new impetus.

Section 2 discusses in detail the calculation of triangle diagrams and gives the transformations that link invariant diagrams with and without kinematic singularities.

In Sec. 3, I show that the absence of an anomaly pole for $k_2^2\neq 0$ means that the pole's vertices are not factorizable. Hence, a massless pseudoscalar Goldstone boson related to spontaneous chiral-symmetry breaking (Fig. 2) cannot reproduce an axial anomaly pole. Here I also discuss the possibility of a "conspiracy" between a massless pseudoscalar Goldstone boson and a massless pseudovector boson as a result of which the pole in invariant amplitudes free from kinematic singularities disappears for $k_2^2\neq 0$.

In Sec. 4, I discuss problems that may arise.

2. THE AXIAL ANOMALY POLE

As is known,[6] the amplitude of the axial-vector-current $\to$ conserving-vector-current times conserving-vector-current transition (see Fig. 1) has the form

$$T_{\alpha\beta\mu}=\sum_i A_i t^i_{\alpha\beta\mu}=A_1 k_1{}^\sigma \varepsilon_{\sigma\alpha\beta\mu}+A_2 k_2{}^\sigma \varepsilon_{\sigma\alpha\beta\mu}+A_3 k_{1\beta} k_1{}^\delta k_2{}^\sigma \varepsilon_{\delta\sigma\alpha\mu}$$
$$+A_4 k_{2\beta} k_1{}^\delta k_2{}^\sigma \varepsilon_{\delta\sigma\alpha\mu}+A_5 k_{1\alpha} k_1{}^\delta k_2{}^\sigma \varepsilon_{\delta\sigma\beta\mu}+A_6 k_{2\alpha} k_1{}^\delta k_2{}^\sigma \varepsilon_{\delta\sigma\beta\mu}. \quad (1)$$

Gauge invariance (the condition for vector-current conservation),

$$k_1{}^\alpha T_{\alpha\beta\mu}=k_2{}^\beta T_{\alpha\beta\mu}=0 \quad (2)$$

is ensured if the following relations hold true:

$$A_1=k_2{}^2A_4+(k_1k_2)A_3,\quad A_2=k_1{}^2A_5+(k_1k_2)A_6. \quad (3)$$

Also,

$$A_3(k_1,\ k_2)=-A_6(k_2,\ k_1),\ A_4(k_1,\ k_2)=-A_5(k_2,\ k_1). \quad (4)$$

The invariant amplitudes A_3, A_4, A_5, and A_6 have no kinematic singularities and are well-defined.[6] At $k_1^2=0$ (or $k_2^2=0$) they can be calculated analytically.[3]

Note that A_4 and A_5 contribute nothing directly to physical quantities [not through Eqs. (3)] since $k_{1\alpha}$ and $k_{2\beta}$ in (1) convolute either with the polarization vectors, $[k_{1\alpha}e^\alpha(k_1)]=0$ and $[k_{2\beta}e^\beta(k_2)]=0$, or with conserving vector currents, $[k_{1\alpha}j^\alpha(k_1)]=0$ and $[k_{2\beta}j^\beta(k_2)]=0$. In the case at hand ($k_1^2=0$ and $k_2^2\neq 0$) the amplitude A_5 can be ignored completely, as Eqs. (3) show.

I have found it convenient to calculate the invariant amplitudes A_3, A_6, and A_4 by employing dispersion relations in $M^2=(k_1+k_2)^2$. To this end it has proved expedient to use the region where $k_1^2=0$, $Q^2=-k_2^2=-E^2>0$, and $W^2=-M^2=-(k_1+k_2)^2>0$.

The dispersion relations have the form

$$A_i=\frac{1}{\pi}\int_{4m_q^2}^{\infty}\frac{\operatorname{Im}A_i}{M'^2+W^2}dM'^2, \quad (5)$$

where the $\operatorname{Im}A_i$ are obtained by "cutting" the diagrams in Fig. 1 in the axial-vector channel (the M^2-channel):

$$\operatorname{Im}A_3=-\operatorname{Im}A_6=\frac{1}{2\pi}\frac{1}{(Q^2+M'^2)^2}\times\left(\frac{Q^2}{Q^2+M'^2}\rho'-2m_q^2\ln\frac{1+\rho'}{1-\rho'}\right),$$

$$\operatorname{Im}A_4=-\frac{1}{2\pi}\frac{1}{Q^2+M'^2}\rho', \quad (6)$$

where $\rho'=1-4m_q^2/M'^2$.

The dispersion integrals (5) can be evaluated analytically. The result of these calculations is

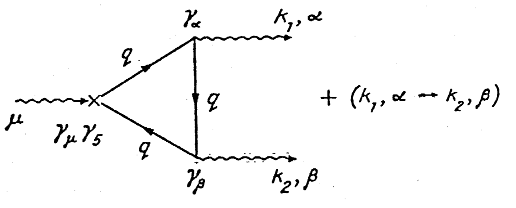

FIG. 1.

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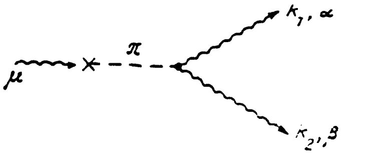

FIG. 2.

$$A_3=-A_6=-\frac{1}{2\pi^2}\frac{1}{Q^2-W^2}\left(\frac{Q^2}{Q^2-W^2}L_1+\frac{m_q^2}{Q^2-W^2}L_2-1\right),$$

$$A_4=-\frac{1}{2\pi^2}\frac{1}{Q^2-W^2}L_1, \tag{7}$$

where

$$L_1=-\rho\ln\frac{\rho+1}{\rho-1}+\lambda\ln\frac{\lambda+1}{\lambda-1},$$
$$L_2=-\ln^2\frac{\rho+1}{\rho-1}+\ln^2\frac{\lambda+1}{\lambda-1},$$
$$\rho^2=1+\frac{4m_q^2}{W^2},\qquad \lambda^2=1+\frac{4m_q^2}{Q^2}. \tag{8}$$

The amplitudes A_1 and A_2 are obtained from Eqs. (3):

$$A_1=\frac{1}{4\pi^2}\left(\frac{Q^2}{Q^2-W^2}L_1-\frac{m_q^2}{Q^2-W^2}L_2+1\right),$$
$$A_2=\frac{1}{4\pi^2}\left(\frac{Q^2}{Q^2-W^2}L_1+\frac{m_q^2}{Q^2-W^2}L_2-1\right). \tag{9}$$

The functions L_1 and L_2 are analytically continued into the other regions of $M^2=-W^2$ and $E^2=-Q^2$ as follows:

a) $0<-W^2=M^2<4m_q^2$:

$$\rho\to i(-\rho^2)^{1/2},\qquad \frac{1}{2}\ln\frac{\rho+1}{\rho-1}\to -i\,\mathrm{arctg}\frac{1}{(-\rho^2)^{1/2}},$$

$2m_q<M$:

$$(-\rho^2)^{1/2}\to -i\rho,\qquad \mathrm{arctg}\frac{1}{(-\rho^2)^{1/2}}\to\frac{\pi}{2}+\frac{i}{2}\ln\frac{1+\rho}{1-\rho}. \tag{10}$$

b) $0<-Q^2=E^2<4m_q^2$:

$$\lambda\to i(-\lambda^2)^{1/2},\qquad \frac{1}{2}\ln\frac{\lambda+1}{\lambda-1}\to -i\,\mathrm{arctg}\frac{1}{(-\lambda^2)^{1/2}},$$

$2m_q<E$:

$$(-\lambda^2)^{1/2}\to -i\lambda,\qquad \mathrm{arctg}\frac{1}{(-\lambda^2)^{1/2}}\to\frac{\pi}{2}+\frac{i}{2}\ln\frac{1+\lambda}{1-\lambda}. \tag{11}$$

Equations (7)–(11) show that on the physical sheets defined by (8), (10), and (11) the amplitudes A_i have no singularities except dynamic cuts for $4m_q^2\leqslant M^2<\infty$ and $4m_q^2\leqslant E^2<\infty$ caused by intermediate $q\bar{q}$-states.

As $m_q\to 0$ (the chiral limit), we get the following formulas:

$$A_1=\frac{1}{4\pi^2}\left(\frac{Q^2}{Q^2-W^2}\ln\frac{Q^2}{W^2}+1\right),$$
$$A_2=\frac{1}{4\pi^2}\left(\frac{Q^2}{Q^2-W^2}\ln\frac{Q^2}{W^2}-1\right),$$
$$A_3=-A_6=-\frac{1}{2\pi^2}\frac{1}{Q^2-W^2}\left(\frac{Q^2}{Q^2-W^2}\ln\frac{Q^2}{W^2}-1\right),$$
$$A_4=-\frac{1}{2\pi^2}\frac{1}{Q^2-W^2}\ln\frac{Q^2}{W^2}, \tag{12}$$

which are valid for $0<Q^2=-E^2$ and $0<W^2=-M^2$.

The analytic continuation into the other regions of M^2 and E^2 is done as follows:

a) $0<-Q^2=E^2$: $\ln Q^2\to -i\pi+\ln E^2$.

b) $0<-W^2=M^2$: $\ln\frac{1}{W^2}\to i\pi+\ln\frac{1}{M^2}$. (13)

Thus, in the massless limit, the physical sheets of the amplitudes A_i given by (12) and (13) have cuts for $0\leqslant E^2<\infty$ and $0\leqslant M^2<\infty$ when $k_2^2\neq 0$. And only when $Q^2\to 0$ does a pole appear in A_3 and A_6 at $M^2=0$:

$$A_3=-A_6=\frac{2}{M^2}A_1=-\frac{2}{M^2}A_2=\frac{1}{2\pi^2}\frac{1}{M^2}. \tag{14}$$

Before we begin to discuss the result, here are some useful identities:

$$k_1^\sigma\varepsilon_{\sigma\alpha\beta\mu}=\frac{k_1^2}{M^2k^2}[k_{2\mu}k_1^\delta k_2^\sigma\varepsilon_{\delta\sigma\beta\alpha}+k_{2\beta}k_1^\delta k_2^\sigma\varepsilon_{\delta\sigma\alpha\mu}+k_{2\alpha}k_2^\delta k_1^\sigma\varepsilon_{\delta\sigma\beta\mu}]$$
$$-\frac{(k_1k_2)}{M^2k^2}[k_{1\mu}k_1^\delta k_2^\sigma\varepsilon_{\delta\sigma\beta\alpha}+k_{1\beta}k_1^\delta k_2^\sigma\varepsilon_{\delta\sigma\alpha\mu}+k_{1\alpha}k_2^\delta k_1^\sigma\varepsilon_{\delta\sigma\beta\mu}], \tag{15}$$

$$k_2^\sigma\varepsilon_{\sigma\alpha\beta\mu}=\frac{(k_1k_2)}{M^2k^2}[k_{2\mu}k_1^\delta k_2^\sigma\varepsilon_{\delta\sigma\beta\alpha}+k_{2\beta}k_1^\delta k_2^\sigma\varepsilon_{\delta\sigma\alpha\mu}+k_{2\alpha}k_2^\delta k_1^\sigma\varepsilon_{\delta\sigma\beta\mu}]$$
$$-\frac{k_2^2}{M^2k^2}[k_{1\mu}k_1^\delta k_2^\sigma\varepsilon_{\delta\sigma\beta\alpha}+k_{1\beta}k_1^\delta k_2^\sigma\varepsilon_{\delta\sigma\alpha\mu}+k_{1\alpha}k_2^\delta k_1^\sigma\varepsilon_{\delta\sigma\beta\mu}], \tag{16}$$

where k is the momentum in the reference frame in which $\mathbf{k}_1+\mathbf{k}_2=0$, with

$$k^2=\frac{M^2}{4}\left[1-\frac{k_1^2+k_2^2}{M^2}+\frac{(k_1^2-k_2^2)^2}{M^4}\right].$$

Equations (15) and (16) make it possible to go over to other invariant amplitudes:

$$T_{\alpha\beta\mu}=\sum B_i u^i_{\alpha\beta\mu}=B_1k_{1\mu}k_1^\delta k_2^\sigma\varepsilon_{\delta\sigma\beta\alpha}$$
$$+B_2k_{2\mu}k_1^\delta k_2^\sigma\varepsilon_{\delta\sigma\beta\alpha}+B_3k_{1\beta}k_1^\delta k_2^\sigma\varepsilon_{\delta\sigma\alpha\mu}$$
$$+B_4k_{2\beta}k_1^\delta k_2^\sigma\varepsilon_{\delta\sigma\alpha\mu}+B_5k_{1\alpha}k_1^\delta k_2^\sigma\varepsilon_{\delta\sigma\beta\mu}+B_6k_{2\alpha}k_1^\delta k_2^\sigma\varepsilon_{\delta\sigma\beta\mu}, \tag{17}$$

where

$$B_1=-\frac{1}{M^2k^2}[(k_1k_2)A_1+k_2^2A_2],$$
$$B_2=\frac{1}{M^2k^2}[(k_1k_2)A_2+k_1^2A_1],$$
$$B_3=A_3+B_1,\ B_4=A_4+B_2,\ B_5=A_5-B_1,\ B_6=A_6-B_2. \tag{18}$$

It must be noted, however, that the amplitudes B_i generally have kinematic singularities at $M^2=0$ and $k^2=0$, as Eqs. (18) and (3) show. Only at $k_1^2=k_2^2=0$ are the amplitudes B_i free from kinematic singularities, and we have

$$T_{\alpha\beta\mu}=A_6(k_1+k_2)_\mu k_1^\delta k_2^\sigma \varepsilon_{\delta\sigma\beta\alpha} + (A_4+A_6)(k_{2\beta}k_1^\delta k_2^\sigma\varepsilon_{\delta\sigma\alpha\mu}-k_{1\alpha}k_1^\delta k_2^\sigma\varepsilon_{\delta\sigma\beta\mu}). \tag{19}$$

where

$$A_6=-\frac{1}{2\pi^2}\frac{1}{W^2}\left(1-\frac{m_q^2}{W^2}\ln^2\frac{\rho+1}{\rho-1}\right),$$

$$A_4=\frac{1}{2\pi^2}\frac{1}{W^2}\left(2-\rho\ln\frac{\rho+1}{\rho-1}\right). \tag{20}$$

Obviously, from the standpoint of physics the second term in (19) is insignificant. Equation (19) shows that at $k_1^2=k_2^2=0$ only pseudoscalar intermediate states (0^-) are possible in the axial-vector channel. Thus, the axial anomaly pole that appears at $k_1^2=k_2^2=0$ in the massless fermion limit appears as a contribution of some massless pseudoscalar boson.

In Sec. 3 we will need transformations that are the inverses of (15) and (16):

$$k_{1\mu}k_1^\delta k_2^\sigma\varepsilon_{\delta\sigma\beta\alpha}=k_1^2k_2^\sigma\varepsilon_{\sigma\alpha\beta\mu}-(k_1k_2)k_1^\sigma\varepsilon_{\sigma\alpha\beta\mu} -k_{1\beta}k_1^\delta k_2^\sigma\varepsilon_{\delta\sigma\alpha\mu}-k_{1\alpha}k_2^\delta k_1^\sigma\varepsilon_{\delta\sigma\beta\mu},$$

$$k_{2\mu}k_1^\delta k_2^\sigma\varepsilon_{\delta\sigma\beta\alpha}=(k_1k_2)k_2^\sigma\varepsilon_{\sigma\alpha\beta\mu}-k_2^2k_1^\sigma\varepsilon_{\sigma\alpha\beta\mu} -k_{2\beta}k_1^\delta k_2^\sigma\varepsilon_{\delta\sigma\alpha\mu}-k_{2\alpha}k_2^\delta k_1^\sigma\varepsilon_{\delta\sigma\beta\mu}. \tag{21}$$

3. SPONTANEOUS CHIRAL-SYMMETRY BREAKING

The conclusion that the vertices of an axial anomaly pole are not factorizable suggests itself immediately. Let us prove, however, that this conclusion is valid. We allow for the contribution of a massless Goldstone boson in the axial-vector channel (see Fig. 2):

$$T^G_{\alpha\beta\mu}=\frac{f_\pi g_{\pi\gamma\gamma}}{M^2}(k_1+k_2)_\mu k_1^\delta k_2^\sigma\varepsilon_{\delta\sigma\beta\alpha}. \tag{22}$$

Factorization of the vertices in (22) is obvious. We use Eq. (21) and go back to Eq. (1):

$$A_1^G=-\frac{f_\pi g_{\pi\gamma\gamma}}{M^2}[(k_1k_2)+k_2^2],\quad A_2^G=\frac{f_\pi g_{\pi\gamma\gamma}}{M^2}[(k_1k_2)+k_1^2],$$

$$A_3^G=A_4^G=-A_5^G=-A_6^G=-\frac{f_\pi g_{\pi\gamma\gamma}}{M^2}. \tag{23}$$

We see that the amplitudes A_i contain a Goldstone pole not only at $k_1^2=k_2^2=0$.

Hence, a massless pseudoscalar Goldstone boson cannot reproduce an axial anomaly pole.

't Hooft has proposed an elegant principle[7] according to which a compound particle must reproduce the axial anomaly of all its fermion components.

But does our result mean that an axial anomaly pole is incompatible with spontaneous chiral-symmetry breaking? Generally speaking, no. However, there is a price to pay for introducing a Goldstone boson. To understand how high the price is, we identify a Goldstone pole with an axial anomaly pole at $k_1^2=k_2^2=0$.

Comparison of (22) and (19) at $m_q=0$ yields

$$f_\pi g_{\pi\gamma\gamma}=-\frac{1}{2\pi^2}. \tag{24}$$

Now let us consider the differences of amplitudes (12) and (23), allowing for (24):

$$\bar A_1=A_1-A_1^G=\frac{1}{4\pi^2}Q^2\left(\frac{1}{Q^2+M^2}\ln\frac{Q^2}{-M^2}+\frac{1}{M^2}\right)=\bar A_2=A_2-A_2^G,$$

$$\bar A_6=A_6-A_6^G=-\bar A_3=-A_3+A_3^G =\frac{1}{2\pi^2}\frac{Q^2}{Q^2+M^2}\left(\frac{1}{Q^2+M^2}\ln\frac{Q^2}{-M^2}+\frac{1}{M^2}\right), \tag{25}$$

where we have retained only physically meaningful amplitudes $(M^2<0)$.

We see that the invariant amplitudes $\bar A_i$ have a pole at $M^2=0$ when $k_2^2\neq0$ and that the pole appears as a contribution of a massless boson.

One can easily see that the amplitude

$$\bar T_{\alpha\beta\mu}=\sum_i \bar A_i t^i_{\alpha\beta\mu} \tag{26}$$

is transverse in the axial-vector channel:

$$(k_1+k_2)^\mu \bar T_{\alpha\beta\mu}=0, \tag{27}$$

that is, in the axial-vector channel of $\bar T_{\alpha\beta\mu}$ only pseudovector intermediate states (1^+) are possible. Hence, in the chiral limit with $k_2^2\neq0$, the amplitude $\bar T_{\alpha\beta\mu}$ has a pole that appears as a contribution of a massless pseudovector boson.

4. CONCLUSION

Thus, within the chiral limit in the case of spontaneous chiral-symmetry breaking (or, if you like, a nonlinear realization of this symmetry), a pseudoscalar massless Goldstone boson can reproduce an axial anomaly pole only as a result of a "conspiracy" with a pseudovector massless boson, a "conspiracy" that leads at $k_2^2\neq0$ to the disappearance of their total contribution to the invariant amplitudes A_i free from kinematic singularities.

Is this a sufficient price? I feel that introducing a pseudovector massless (1^+)-boson will require introducing the vector massless chiral partner (1^-). This, however, is already "small talk."

I realize that the problem of higher radiative corrections has been ignored, but this has become a sort of tradition in such studies.

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