Kinetics of noninteracting fluxons in Josephson lattices

V. V. Bryksin

Physicotechnical Institute, Russian Academy of Sciences, Saint-Petersburg
(Submitted 12 May 1992)
Zh. Eksp. Teor. Fiz. **103**, 172–188 (January 1993)

An adiabatic theory of motion of solitary vortices in Josephson lattices, when the coupling with pinning centers is arbitrary, is suggested. The form of a fluxon can strongly depend on its position in the lattice with respect to pinning centers, markedly differing from the form of a standard soliton. The procedure is applicable to nonlinear systems described both by the sine-Gordon equation and by nonlinear Schrödinger and Korteweg-de Vries equations. Therefore it can be used also in nonlinear optics, electric circuits, plasma physics, in description of domain-wall motion, etc. Specific calculations of voltage relaxation for current flowing through an inhomogeneous Josephson junction are carried out. This process depends strongly on temperature and is characterized by long-time power-law tails, but is exponential for short times.

The motion of vortex structures (fluxons) in a Josephson medium with pinning centers (defects) is the subject of intensive studies (see, e.g., Refs. 1–3). A similar problem arises also in plasma physics, nonlinear optics, charge-density wave theory, hydrodynamics, magnetism, and other fields. Calculations are usually performed on the basis of perturbation theory in the framework of inverse-scattering problem.[4–6] The degrees of freedom are divided into radiative and proper soliton (adiabatic) ones. In this approach it is assumed that the form of a free (or renormalized) soliton is only slightly affected by defects, i.e., the interaction of a soliton with defects is considered to be fairly weak.

In the present paper we suggest a technique of calculation of fluxon motion for an arbitrary fluxon–defect interaction allowing for strong fluxon deformation in the process of its motion. In particular, the fluxon shape can differ strongly from the standard form given by a "free" nonlinear equation (defects are not taken into account). We restrict the discussion to the case of adiabatic approximation and neglect the interaction between fluxons. This approach allows, for example, to calculate the fluxon kinetics in an intersecting two-dimensional Josephson structure with a lattice constant larger than the Josephson length (the size of a fluxon). In this situation the form of a fluxon changes radically, when the latter moves with respect to the junction-intersection point which plays the role of a pinning center. The corresponding potential relief is a profile of narrow "canyons" which broaden abruptly near their intersection points.

We have also calculated the voltage relaxation in a one-dimensional Josephson structure with a random array of pinning centers. We show that if a step-like current is applied, the voltage relaxation has a long-time power-law character due to wells of a very small coupling rigidity. Since the probability of filling these wells depends strongly on temperature, the relaxation is also determined by temperature. In the language of frequency dependence of impedance, these relaxation-time tails correspond to $\omega^2 \ln \omega$ as $\omega \to 0$.

1. EQUATION OF FLUXON MOTION IN A JOSEPHSON LATTICE

We consider a two-dimensional Josephson lattice (medium), lying in the xy plane, in which there is a vortex state given by a vector potential $\mathbf{A}(x,y)$, $\mathbf{H} = \operatorname{curl} \mathbf{A}$. Accordingly, we have a magnetic flux

$$\Phi = \int dx\, dy\, \mathbf{H},$$

directed along the z axis (the vector $\mathbf{A}$ is in the xy plane). The system is assumed to be uniform along the z axis (i.e., vortex bending is not taken into account). The flux Φ is assumed to be an integral of motion, and for an ordinary vortex state $\Phi = \pi\hbar c/e$.

We introduce a vector $\mathbf{R}$, which gives the position of the center of gravity of the magnetic flux in the xy plane:

$$\mathbf{R} = \Phi^{-2}\left[\Phi \int dx\, dy\, \mathbf{A}\right]. \tag{1}$$

When a vortex (fluxon) moves in the xy plane, the potential $\mathbf{A}$ is time-dependent, and the function $\mathbf{R}(t)$ characterizes the fluxon motion (the flux Φ is time-independent!). Differentiating (1) with respect to the time t and taking into account the relation $\mathbf{E} = -c^{-1}\partial \mathbf{A}/\partial t$, we have

$$\int dx\, dy\, \mathbf{E} = \frac{i}{c}\,[\Phi\, d\mathbf{R}/dt]. \tag{2}$$

Here $\mathbf{E}$ is the electric field, and the scalar potential is set equal to zero.

If a uniform electric current $\mathbf{j}$ flows in the xy plane, then a fluxon interacts with it with energy (per unit fluxon length)

$$H_{int} = \frac{i}{c}\int dx\, dy\, (\mathbf{jA}) = \frac{i}{c}\mathbf{R}[\Phi \mathbf{j}]. \tag{3}$$

The second equality here is derived with the help of the definition (1).

Let us now derive the equation of motion of a fluxon. For this purpose we introduce the notion of Josephson lattice energy H. Minimization of this energy for a given flux Φ gives a vortex state localized near some points $\mathbf{R}_i$ and having a certain configuration of magnetic field, $\mathbf{A}_{0i}(x,y)$, which differs from that of a free soliton. Substituting $\mathbf{A}_{0i}(x,y)$ into (1), we find localization centers $\mathbf{R}_i$. When a fluxon moves under the action of external force (e.g., current $\mathbf{j}$), it deviates from equilibrium and, generally speaking, changes its form, if pinning is not very weak. To describe the fluxon deviation

 1063-7761/93/010094-09$10.00

from equilibrium, we introduce a notion of energy $H(\mathbf{R})$, which is a minimized lattice energy H for a given value $\mathbf{R}(1)$. For this purpose we make use of the Legendre transformation

$$\tilde{H}=H+\mathbf{FR}, \tag{4}$$

where $\mathbf{F}$ is a generalized force, and $\mathbf{R}$ is defined in (1). Minimizing $\tilde{H}$, we find $\tilde{H}(\mathbf{F})$. We have

$$\mathbf{R}(\mathbf{F})=d\tilde{H}(\mathbf{F})/d\mathbf{F}. \tag{5}$$

Equation (5) allows to find $\mathbf{F}(\mathbf{R})$, after which we get

$$H(R)=\tilde{H}[\mathbf{F}(\mathbf{R})]-\mathbf{F}(\mathbf{R})\mathbf{R}. \tag{6}$$

The quantity $-dH(\mathbf{R})/d\mathbf{R}$ has the meaning of restoring force and, according to (6), it equals $\mathbf{F}(\mathbf{R})$. In the approximation of viscous friction the equation of motion for a fluxon in the presence of external current [see (3)] can be written in the form

$$\eta\frac{d\mathbf{R}}{dt}=\mathbf{F}(\mathbf{R})+\frac{1}{c}[\mathbf{j}\mathbf{\Phi}], \tag{7}$$

where η is the friction coefficient, which is, generally speaking, a tensor. With the help of (2) we can show that η is related to the electric resistivity of the medium, ρ, by a well-known expression

$$\eta=\Phi^2 n/\rho c^2, \tag{8}$$

where n is the fluxon density in the xy plane, so that the induction $\mathbf{B}=4\pi n\mathbf{\Phi}$. The resistivity ρ is defined by the relation $\mathbf{E}=\rho\mathbf{j}$ for $\mathbf{j}\to\infty$, when we can omit $\mathbf{F}(\mathbf{R})$ in (7), and $\mathbf{E}$ is the average electric field in the plane:

$$\bar{\mathbf{E}}(t)=n\int dx\,dy\,\mathbf{E}(x,y,t). \tag{9}$$

The found relations can be easily rewritten for a single junction lying in the xz plane. The current $\mathbf{j}$ and vector potential $\mathbf{A}$ are directed along y, the magnetic field along z, and the vortex moves along the x axis. Equation (7) takes the form

$$\eta\frac{dX}{dt}=F(x)+\frac{\Phi}{c}j. \tag{10}$$

In Eq. (8) η is the fluxon density per unit junction length, and ρ is the resistance of a unit junction area. Simultaneously, instead of the electric field (9), we should consider the voltage drop U across the junction:

$$U=\frac{\Phi n}{c}\frac{dX}{dt}. \tag{9a}$$

As usual, the equations of type (7) or (10) can be generalized, if the fluxon "mass" is to be taken into account. For this purpose, we must add the second derivative with respect to time to the left-hand side of the equation:

$$m\frac{d^2\mathbf{R}}{dt^2}+\eta\frac{d\mathbf{R}}{dt}=\mathbf{F}(\mathbf{R})+\frac{1}{c}[\mathbf{j}\mathbf{\Phi}]. \tag{11}$$

The "mass" m is related to the imaginary part of the medium impedance (capacitance) in the applicability range of Ohm's law $\mathbf{j}=(i\omega C+1/\rho)\bar{\mathbf{E}}$ by the expression

$$m=\Phi^2 n/Cc^2. \tag{12}$$

The equation of motion in the form (11) allows us to write the quantum-mechanical Hamiltonian of a vortex state in external "electric" field $e\mathbf{E}=[\mathbf{j}\mathbf{\Phi}]/c$ as

$$H=-\frac{\hbar^2}{2m}\Delta_{\mathbf{R}}+H(\mathbf{R})-e\mathbf{ER}. \tag{13}$$

In this formulation dissipation in the system is ignored, and the term $\eta\dot{\mathbf{R}}$ in the equation of motion can be restored only if the fluxon interaction with a random field (e.g., with random currents or phonons) is taken into account. For a periodic array of "defects" in the structure $H(\mathbf{R})=H(\mathbf{R}+\mathbf{K})$, where $\mathbf{K}$ is the translation vector of the defect lattice, and Eq. (13) fully coincides with the one-dimensional Hamiltonian of the crystal lattice in an external electric field. Note that the idea of the Stark quantization of Josephson vortices due to noncommutation of the particle phase and number operators was put forward in Refs. 7 and 8 and experimentally confirmed for granulated tin.[9]

Equation (11) has the simplest form for an ordered one-dimensional structure, if the fluxon size is larger than the defect lattice constant a. In this case

$$H(x)={}^1/_2\varepsilon_p\cos(2\pi X/a),$$

where ε_p is the pinning energy, which can be found if we calculate $H(X)$ as described above. For such $H(X)$ Eq. (11) takes the form

$$\frac{\hbar}{2e}na\left\{C\frac{d^2\varphi}{dt^2}+\frac{1}{\rho}\frac{d\varphi}{dt}\right\}=j-j_c\sin\varphi. \tag{14}$$

Here we have introduced the following notation: $\varphi=2\pi X/a$, $j_c=e\varepsilon_p/\hbar a$, and the magnetic flux $\Phi=\pi hc/e$. According to (9a), we also have $U=\hbar na\dot{\varphi}/2e$. Equation (14) formally coincides with the expression for a current flowing through a defectless Josephson junction,[10] with a critical current equal to $\varepsilon_p e/\hbar a$ and effective charge $e^*=e/na$. Therefore all the characteristics of a defectless junction in the absence of magnetic field (current-voltage characteristics, hysteresis, the Shapiro steps, etc.) are valid also for the junctions with pinning centers, when the size of moving vortices is larger than the distance between the centers.

The one-dimensional equation of motion (14) is easily generalized to the case of two-dimensional periodic lattices of defects. In particular, for rectangular lattices

$$2H(\mathbf{R})=-\varepsilon_{px}\cos(2\pi X/a_x)-\varepsilon_{py}\cos(2\pi Y/a_y),$$

and the equations of motion for $\varphi_x=2\pi X/a_x$ and $\varphi_y=2\pi Y/a_y$ separate:

$$\frac{\hbar}{2e}na_\alpha\left\{C\frac{d^2\varphi_\alpha}{dt^2}+\frac{1}{c}\frac{d\varphi_\alpha}{dt}\right\}=\tilde{j}_\alpha-j_{c\alpha}\sin\varphi_\alpha, \tag{14a}$$

where $\alpha=x,y$, $j_{c\alpha}=\varepsilon_{p\alpha}e/\hbar a_\alpha$, $\tilde{j}_x=j_y$, and $\tilde{j}_y=-j_x$. According to (9), we have

$$\bar{E}_x=-\frac{\hbar}{2e}na_x\dot{\varphi}_x,\quad \bar{E}_y=\frac{\hbar}{2e}na_y\dot{\varphi}_y. \tag{15}$$

Note that, according to (14a), the effective critical current in a regular two-dimensional lattice depends on the angle between j and the x axis. For $0<\vartheta<\vartheta_0$

$$j_c^*=j_{cy}/\cos\vartheta,$$

and for $\vartheta_0<\vartheta<\pi/2$

$$j_c^*=j_{cx}/\sin\vartheta,$$

where $\tan\vartheta_0 = j_{cx}/j_{cy}$. The critical current reaches maximum, for $\vartheta=\vartheta_0$, when $j_c^* = (j_{cx}^2 + j_{cy}^2)^{1/2}$.

In disordered Josephson structures the potential energy $H(\mathbf{R})$ has the character of a random geodetic relief with a constant gradient due to $\mathbf{j}$, so that for the current along the y axis the potential energy has the form $H(X,Y) - \Phi jX/c$. The critical current corresponds to a profile slope such that the first infinitely long trajectory of the steepest descent appears. The critical trajectory has at least one point at which

$$\partial\{H(X,Y)-\Phi Xj/c\}/\partial l=0$$

(we differentiate along the trajectory). If the current decreases the system becomes a superconducting state at $j_c' > j_c$ owing to fluxon inertia, i.e., the current-voltage characteristic has hysteresis.[10]

Smooth geodesic relief is typical of "dense" Josephson structures, when the fluxon size exceeds the characteristic distance between defects. In the opposite case the relief is like a system of intersecting canyons (see below).

2. PINNING ON AN ISOLATED DEFECT

As the simplest example of the use of the method suggested above, we consider a junction with a linear defect at $x=0$. The energy $\tilde{H}$ defined in Eq. (4) has, in this case, the following form

$$\tilde{H}=\frac{1}{8}E_J\int_{-\infty}^{\infty}dx\left(A-\cos\vartheta+\frac{1}{2}\vartheta'^2-l\delta(x)\vartheta'^2\right)+\frac{F\delta}{2\pi}\int_{-\infty}^{\infty}dx\,x\vartheta', \tag{16}$$

where x is the dimensionless coordinate (in the units of the Josephson length δ), $\vartheta(x)$ is the phase difference across the junction, $\vartheta' = d\vartheta/dx$, $E_J = 4\hbar j_c\delta/e$ is the soliton energy, j_c is the critical current, and l is the dimensionless parameter of the pinning force. For a defect in the form of a cavity of area S, for example, $l = S/2d\delta$, where $d = d' + 2\lambda$ is the effective thickness of the junction, d' is its thickness, and λ is the London length. The constant A ensures convergence of the integral as $x\to\pm\infty$ and is defined below. Recall that the magnetic field is $H = \Phi\vartheta'/2\pi d\delta$.

Minimizing (16) with respect to ϑ, we find the following equation

$$\vartheta''=\sin\vartheta-\sin\alpha+2l\frac{d}{dx}\delta(x), \tag{17}$$

where the notation

$$\sin\alpha=4F\delta/\pi E_J \tag{18}$$

is introduced. Since the dimensionless "force" $4F\delta/\pi E_J$ should be less than unity, we have denoted it by $\sin\alpha$. The fluxon magnetic flux is assumed to be equal to a flux quantum, so that the phase difference across the junction is 2π, and we have the following boundary conditions for Eq. (17):

$$\vartheta|_{x\to-\infty}=\alpha;\quad \vartheta|_{x\to\infty}=2\pi+\alpha. \tag{19}$$

Hence the constant A in (16) equals $\cos\alpha$.

At the point $x=0$ the quantity ϑ' is continuous, and the phase has discontinuity. Introducing the notation

$$\vartheta_\pm={}^1/_2(\vartheta_{0>}\pm\vartheta_{0<}),$$

where $\vartheta_{\vartheta>}$ ($0_{0<}$) is the phase on the right (left) of the defect, we find the relation

$$\vartheta_-=l\vartheta_0', \tag{20}$$

where $\vartheta_0' = d\vartheta/dx|_{x=0}$ is the field at the defect.

The first integral of Eq. (17) has the form

$$^1/_2\vartheta'^2=\begin{cases}\cos\alpha-\cos\vartheta-(\vartheta-\alpha)\sin\alpha={}^1/_2\vartheta_<'^2, & x<0\\ \cos\alpha-\cos\vartheta-(\vartheta-2\pi-\alpha)\sin\alpha={}^1/_2\vartheta_>'^2, & x>0\end{cases}. \tag{21}$$

Allowing for the continuity of ϑ' at $x=0$, we find

$$^1/_2\vartheta_0'^2=\cos\alpha-\cos\vartheta_+\cos\vartheta_--\sin\alpha(\vartheta_+-\pi-\alpha), \tag{21a}$$

$$\sin\vartheta_+\sin\vartheta_-=(\vartheta_--\pi)\sin\alpha. \tag{21b}$$

The system of three equations (20) and (21) defines the quantities ϑ_0', ϑ_+ and ϑ_- as functions of α (i.e., of the force F).

Now, with the help of (5), we can find the function $F(X)$. First, substituting (21) into (16), we find $\tilde{H}(\alpha)$ [i.e., $\tilde{H}(F)$]. In doing so, we must take it into account that due to discontinuity of ϑ at $x=0$ the derivative ϑ', apart from a regular contribution, has a singularity $\vartheta_s^2 = 4l\vartheta_0'^2\delta(x)$. As a result we have

$$\tilde{H}=\frac{1}{8}E_J\left\{l\vartheta_0'^2+2^{1/2}\int_{\vartheta_{0>}-2\pi}^{\vartheta_{0<}}du\,[\cos\alpha-\cos u-(u-\alpha)\sin\alpha]^{1/2}\right\},$$

where ϑ_0', $\vartheta_{0<}$, and $\vartheta_{0>}$ depend on α in accordance with (21). Now we find from (5)

$$X=\frac{4\delta}{\pi E_J}\frac{d\tilde{H}}{d\sin\alpha}=\frac{\delta}{2\pi}\int_{\vartheta_{0>}-2\pi}^{\vartheta_{0<}}\frac{(\alpha-\vartheta)\,d\vartheta}{2^{1/2}[\cos\alpha-\cos\vartheta-(\vartheta-\alpha)\sin\alpha]^{1/2}}. \tag{22}$$

The relation (22) gives the required dependence $X(\alpha)$, i.e., Eq. (18) taken into account, the function $F(X)$. In the general form the latter can be found only numerically. However in the limit of weak pinning, $l\ll 1$, the problem becomes much easier. In this limit $\alpha\ll 1$ and $\vartheta\ll 1$, and, according to (21a) and (21b), we have

$$\sin\vartheta_+\simeq-\pi\alpha/\vartheta_-=-\pi\alpha/l\vartheta_0';\quad \vartheta_0'=2\sin(\vartheta_+/2).$$

Integrating in (22) over ϑ for α and $\vartheta\ll 1$, we find

$$X=-\delta\ln\mathrm{tg}(\vartheta_+/4).$$

Using this relation to express ϑ_+ in terms of α (i.e., F), we get

$$F=-E_J\frac{l}{\delta}\frac{\operatorname{sh}(X/\delta)}{\operatorname{ch}(X/\delta)}. \tag{23}$$

The vortex energy in the field of the defect has the form

$$H(X)=E_J\left\{1-\frac{l}{2\operatorname{ch}^2(X/\delta)}\right\}. \tag{24}$$

The last expression, valid in the limit of weak pinning, can be found by a much simpler perturbation-theory procedure.[1,11] The simplest realization of this procedure consists in substituting into the Hamiltonian (16) the solution for ϑ in the form of an isolated undeformed soliton with the center of gravity at the point X:

$$\vartheta=4\operatorname{arc\,tg}\left[\exp\left(x-\frac{X}{\delta}\right)\right],$$

which immediately gives (24).

The suggested method leads to nontrivial results in the limit of strong pinning. For example, such is the pinning at a solitary defect for an arbitrary parameter l given by Eq. (22). Another example of a system to which perturbation theory is never applicable is intersection of two Josephson junctions (a "cross"[12]).

3. PINNING ON JUNCTION INTERSECTION

The geometry of a cross in the xy plane is shown in Fig. 1. The four line segments comprising the cross are denoted by the index $i=1,2,3,4$. The equations for the phases ϑ_i on the segments have the form [cf. (17)]:

$$\vartheta_i''=\sin\vartheta_i-\sin\alpha_i, \tag{25}$$

where $\alpha_i=\alpha_x$ for $i=1,3$ and $\alpha_i=\alpha_y$ for $i=2,4$, while $\alpha_{x,y}$ is related to the force projections $F_{x,y}$ by (18). Far from the intersection point the phases satisfy the conditions

$$\vartheta_1(\infty)=2\pi+\alpha_x,$$
$$\vartheta_3(-\infty)=\alpha_x,\quad \vartheta_2(\infty)=\alpha_y,\quad \vartheta_4(-\infty)=\alpha_y$$

(it is assumed that the flux quantum is trapped on the segments 1 and 3). To give the problem more symmetric form we shift the phase ϑ_i by -2π, upon which $\vartheta_1(\infty)=\alpha_x$. At the intersection point the magnetic field is continuous, i.e.,

$$\vartheta_1'(0)=\vartheta_2'(0)=\vartheta_3'(0)=\vartheta_4'(0)=h, \tag{26}$$

and the phases have discontinuity, so that

$$\vartheta_x^-+\vartheta_y^-=-\pi, \tag{27}$$

where

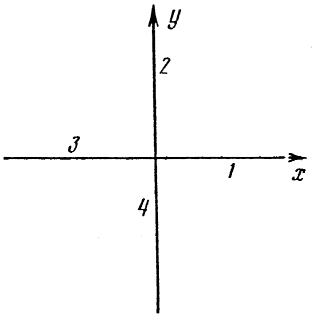

FIG. 1.

$$\vartheta_{x,y}^{\pm}={}^1/_2[\vartheta_{1,2}(0)\pm\vartheta_{3,4}(0)]. \tag{28}$$

Here $\vartheta_i(0)$ are the limiting phase values on the segments at the intersection point. Note that for the unshifted phase ϑ_i Eq. (27) has the form $\vartheta_x^-+\vartheta_y^-=0$.[12]

The first integrals of Eq. (25) give

$${}^1/_2\vartheta_i'^2=\cos\alpha_i-\cos\vartheta_i-(\vartheta_i-\alpha_i)\sin\alpha_i. \tag{29}$$

Hence, making use of the continuity condition (26), we get three equations

$$\begin{gathered}\sin\vartheta_{x,y}^+\sin\vartheta_{x,y}^-=\vartheta_{x,y}^-\sin\alpha_{x,y},\\ {}^1/_2h^2=\cos\alpha_{x,y}-\cos\vartheta_{x,y}^+\cos\vartheta_{x,y}^--(\vartheta_{x,y}^+-\alpha_{x,y})\sin\alpha_{x,y}.\end{gathered} \tag{30}$$

The relations (27) and (30) are four equations for the quantities $\vartheta_{x,y}^{\pm}$.

Upon finding $\vartheta_{x,y}^+$ from (27) and (30) as functions of α_x and α_y, we obtain the following expression for the energy $\widetilde{H}$ [see (4)]:

$$H=\frac{E_J}{2^{5/2}}\sum_{i=x,y}\int\limits_{\vartheta_i^+-\vartheta_i^-}^{\vartheta_i^+-\vartheta_i^-}d\vartheta\{\cos\alpha_i-\cos\vartheta-(\vartheta-\alpha_i)\sin\alpha_i\}^{1/2}. \tag{31}$$

To find an analytic solution we consider first the limiting case, when the force F is absent (i.e., $<\alpha_x=\alpha_y=0$). In this case we have five solutions: 1) $\vartheta_{x,y}^+=0$, $\vartheta_{x,y}^-=-\pi/2$. This corresponds to an equilibrium fluxon localized at the center (see Ref. 12), i.e. $\mathbf{R}=0$; 2) $\vartheta_x^{\pm}=-\pi$, $\vartheta_y^{\pm}=0$. This is an undeformed soliton on segment 1 whose center of gravity is infinitely far from the intersection point ($X=\infty$, $Y=0$); 3) $\vartheta_x^{\pm}=\pi$, $\vartheta_x^-=-\pi$, and $\vartheta_y^{\pm}=0$. This is a soliton on segment 3 ($X=-\infty$, $Y=0$); 4) $\vartheta_x^{\pm}=0$, $\vartheta_y^+=\pi$, and $\vartheta_y^-=-\pi$. This corresponds to a soliton on segment 2 ($X=0$, $Y=\infty$); and 5) $\vartheta_x^{\pm}=0$, $\vartheta_y^+=\pi$, and $\vartheta_y^-=-\pi$. This is a soliton on segment 4 ($X=0$, $Y=-\infty$).

Consider now solutions in the vicinity of these points.

1) Small deviations of a fluxon from equilibrium at the pinning center ($R/\delta\ll 1$). In this region α_x, $\alpha_y\ll 1$, $\vartheta_{x,y}^-=-\pi/2+\delta_{x,y}$, $\delta_{x,y}\ll 1$, and $\vartheta_{x,y}^+\ll 1$. For these restrictions on the parameter values the solution of Eq. (30) has the form

$$\vartheta_{x,y}^{\pm}\simeq\frac{\pi}{2}\alpha_{x,y},\qquad \delta_x=-\delta_y\simeq\frac{\pi-1}{4}(\alpha_y^2-\alpha_x^2).$$

Now we can integrate in (31) to an accuracy of order α^2. As a result, after fairly cumbersome calculations, we get

$$H(\alpha)=E_J\left[2-2^{1/2}-\frac{A}{16}(\alpha_x^2+\alpha_y^2)\right], \tag{32}$$

where A is a numerical factor:

$$A=\frac{\pi^2}{2}\left(1+\frac{3}{2^{5/2}}\right)-2\pi\left(1+\frac{1}{2^{1/2}}\right)+3+\int\limits_0^{\pi/2}\frac{u^2\,du}{\sin u}=2{,}37232.$$

Differentiating now $\widetilde{H}$ given by (32) with respect to α and using Eqs. (5) and (18), we find

$$\mathbf{F}=-\frac{\pi^2}{2A\delta^2}\mathbf{R}. \tag{33}$$

The expression for the energy takes the form

$$H(\mathbf{R})=E_J\left(2-2^{1/2}+\frac{\pi^2R^2}{4A\delta^2}\right). \qquad (34)$$

For $\mathbf{R}=0$ Eq. (34) gives the energy of a localized fluxon, $E_J(2-2^{1/2})$, which is smaller than the energy of a free soliton E_J. This determines pinning at the center of the cross.[12] The corrections proportional to R^2 give small oscillations of a strongly deformed localized fluxon in the framework of the equation of motion (11). The external current $\mathbf{j}(t)$ plays the role of a driving force (e.g., in the problem of absorption of electromagnetic radiation).

2) The fluxon is far away from the pinning center on segment 1 ($X/\delta \gg 1$, $Y/\delta \ll 1$). In this case the fluxon shape differs from that of a free soliton at a junction only slightly. In this region $\alpha_x<0$, $|\alpha_x|$, $\alpha_y \ll 1$, $\vartheta_y^{\pm} \ll 1$, $\vartheta_x^{\pm} = -\pi+\delta_{\pm}$ and $\delta_{\pm} \ll 1$. For such restrictions the solution of Eq. (30) has the form

$$\vartheta_y^+\simeq\alpha_y,\quad \vartheta_y^-=\delta_-\simeq(-\pi\alpha_x/2)^{1/2},\quad \delta_+\simeq(-2\pi\alpha_x)^{1/2}. \qquad (35)$$

The Hamiltonian (31) for $\alpha_x \to 0$ is nonanalytic, so we consider first the case $\alpha_y=0$ ($\vartheta_y^+=0$). First of all, note that for $\alpha_x=0$, i.e. $\delta_{\pm}=\vartheta_y^-=0$, the value of $\widetilde{H}$ (31) is easily found and equals E_J. Then it is convenient to find the quantity $d\widetilde{H}/d\alpha_x$ for $\alpha_x\to 0$. Integrating over ϑ in (31), we find

$$\frac{d\widetilde{H}}{d\alpha_x}=-E_J\ln(-8/\pi\alpha_x).$$

Then, taking into account the conditions for $\widetilde{H}$, when $\alpha_x=0$, we get

$$\widetilde{H}|_{\alpha_y=0}=E_J\left\{1+\frac{\pi}{8}\alpha_x[\ln(-8/\pi\alpha_x)+1]\right\}.$$

Now we find corrections to $\widetilde{H}$ due to small α_y. The expansion of H in α_y has analytic character and begins with corrections of order α_y^2. After simple calculations we get

$$\widetilde{H}(\alpha)=\frac{1}{8}E_J\left\{8+\pi\alpha_x[\ln(-8/\pi\alpha_x)+1]+\frac{\pi}{8}\alpha_x\alpha_y^2\right\}.$$

Hence, taking into account (5) and the relation (18) between $\mathbf{F}$ and $\boldsymbol{\alpha}$ (for $\alpha \ll 1$), we find $\mathbf{R}$ as a function of α:

$$X=\frac{\delta}{2}\left\{\ln(-8/\pi\alpha_x)-\frac{1}{8}\alpha_y^2\right\},\qquad Y=\frac{1}{8}\delta\alpha_x\alpha_y, \qquad (36)$$

and, in accordance with (6),

$$H(\alpha)={}^1/_8E_J[8+\pi\alpha_x(1-{}^1/_4\alpha_y^2)].$$

Expressing α_x and α_y in terms of X and Y from (36) and substituting them into the found expression for $H(\alpha)$, we finally get, with the help of (6), the following expression

$$H(\mathbf{R})=E_J\left\{1-\exp\left(-\frac{2X}{\delta}\right)+\frac{2\pi^2Y^2}{\delta^2}\exp\left(\frac{2X}{\delta}\right)\right\}. \qquad (37)$$

Case 3) can be found from (37) by the change $X\to -X$, case 4) by the change $-X \rightleftarrows Y$, and case 5) by the change $X\to -Y$, $Y \to X$.

As seen from (37), the profile of potential energy $H(\mathbf{R})$, when the pinned fluxon flows from one site to another, has the form of narrow canyons. Their width is smaller the farther is the fluxon from the pinning center, if the fluxon size δ is smaller than the distance between the centers. The physical cause of formation of a canyon profile relief is a great loss in energy, when the fluxon stretches to the granule size in its attempt to go around the granule from both sides at once.

The largest pinning force F_p equals aE_p/δ, where $E_p=(2^{1/2}-1)E_J$ is the pinning energy, and a is a numerical coefficient which can be found only if the energy $H(X,0)$ is calculated exactly for all X. The critical current J_c for an ordered canyon structure in the form of a square lattice and for the current flowing along the [01] axis can be found by setting the right-hand side of Eq. (10) to zero and making the following change: $F(X)\to F_p$ and $j \to J_c$. As a result, we have

$$J_c=aE_pc/\delta\Phi=\frac{4a}{\pi}(2^{1/2}-1)j_c,$$

i.e., J_c is of the order of the critical current of a plane junction. As far as the dependence of J_c on the angle between $\mathbf{j}$ and the axes of the square lattice is concerned, it is the same as in the case of weak pinning [see the text following Eq. (15)].

Note also that for a canyon structure of the potential energy the fluxon velocity is directed along the canyon bottom rather than perpendicular to the current. Therefore, if there is a preferred direction in the canyon structure, the field arising upon vortex motion is not parallel to $\mathbf{j}$. As a result, the even Hall effect may arise, whose sigh is independent of the sign of the magnetic field. This effect has been observed in HTSC ceramics.[13]

The picture changes dramatically for disordered lattices. For homogeneous junctions the force $\mathbf{F}(\mathbf{R})$ is the same at all junction intersections, and disorder manifests itself only in random orientation of the current $\mathbf{j}$ with respect to the axes of the"cross." Since the fluxon moves along the bottom of canyons and chooses its path at canyon intersection in accordance with the condition of the steepest descent, the trajectory of fluxon motion from one site (intersection) to another has a one-dimensional determined character. Such a trajectory can be constructed in the following way. Let the current $\mathbf{j}$ be directed along the y axis and the fluxon move mainly along the x axis. Consider an initial site and draw rays from this site along the directions of canyons. If path segments between neighbors are not straight, the rays are directed along tangent lines. Let the angle between the rays and the x axis be ϑ. Since the force of the current is proportional to $\cos\vartheta$, the trajectory of the steepest descent corresponds to a ray with the smallest ϑ satisfying the condition $|\vartheta|<\pi/2$. Thus the nearest, along the trajectory, neighbor is chosen. Then the same construction is repeated at the neighboring site. There can be as many as possible trajectories entering each site on the left, but only one, which leaves it on its right. Such merging of trajectories leads to decrease in their density with x. For sufficiently large x (i.e., in sample depth) the trajectory segments between the neighboring points of their mergence become long and contain many sites, and the motion assumes a "one-dimensional" character. If the motion is one-dimensional, the fluxon will always encounter a site, for which $|\vartheta|\geqslant\pi/2$, so that, no matter how large the current, there is a barrier and the fluxon stops. To release it from this trap we need the pressure of other fluxons which have approached the trap, i.e., to describe fluxon kinetics it is necessary to introduce fluxon-fluxon interaction into the equation of motion (11). Strictly speaking, it is possible to operate in terms of "particles" (fluxons) only if their concentration is small. Taking into account only pair interaction, we can rewrite the equation of motion in the form

$$m\ddot{\mathbf{R}}_k+\eta\dot{\mathbf{R}}_k=-\frac{dH(\mathbf{R}_k)}{d\mathbf{R}_k}+\sum_{k'}{}' \mathbf{f}(\mathbf{R}_k,\mathbf{R}_{k'}), \tag{38}$$

where $\mathbf{f}(\mathbf{R}_k,\mathbf{R}_{k'})$ is the force of fluxon repulsion. If the pinning only slightly affects the fluxon shape, we have $\mathbf{f}(\mathbf{R},\mathbf{R}') = \mathbf{f}(\mathbf{R}-\mathbf{R}')$, i.e., the repulsion force is independent of location of a pair with respect to pinning centers.

As to the direct fluxon interaction with the external current j, entering into the right-hand side of (11), in (38) it is replaced by boundary conditions, and the distribution itself of the current in the sample is to be found from (38). This problem is similar to that of distribution of current over the sample cross section for currents smaller than critical in the presence of creep considered in Ref. 14.

The solution of the system of equations (38) is a very difficult problem and is not considered here. Note only that Eq. (38) also contains the problem of formation of a critical profile, when magnetic field penetrates into the sample. In the critical state $\ddot{\mathbf{R}}_k=\dot{\mathbf{R}}_k=0$, and $dH/d\mathbf{R}_k-\mathbf{F}$ is the critical pinning force independent of k. The equation

$$\mathbf{F}+\sum_{k'}\mathbf{f}(\mathbf{R}_k-\mathbf{R}_{k'})=0 \tag{39}$$

gives the value of soliton coordinates in the critical profile. An equation of this type has been considered in Ref. 15.

For a strongly compressed state of fluxons, it is necessary to consider many-soliton states (see Refs. 1 and 2) and make use of different procedures of perturbation theory based upon the inverse scattering problem.[4,5] Note also that fluxon activation from traps in disordered lattices can occur not only due to fluxon–fluxon interaction, but also due to interaction with random forces which can be inserted into Eq. (11).

4. FREQUENCY DEPENDENCE OF CONDUCTIVITY OF A DISORDERED JOSEPHSON LATTICE

Equation (10) allows to find the response to an alternating current

$$j(t)=j\exp(i\omega t)$$

for one-dimensionally and randomly arranged pinning defects. A random potential $H(R)$ has a set of l minima at points X_l defined by the relation $H(X_l)=0$. Expanding H in terms of $X-X_l$, we find in harmonic approximation:

$$\eta\dot{X}-H''(X_l)(X-X_l)=\frac{\Phi}{c}j(t). \tag{40}$$

It is technically more convenient to find the time-dependence of voltage, if a step-like current is applied: $j(t)=0$ for $t<0$ and $j(t)=j$ for $t>0$. The frequency dispersion of resistivity can be found with the help of usual relations (see below). The solution of Eq. (40), satisfying the initial condition $X|_{t=0}=X_l$, is

$$\dot{X}=\frac{\Phi j}{\eta}\exp\{-H''(X_l)t/\eta\}.$$

Hence a soliton in the lth well gives, with Eqs. (9a) and (8) taken into account, the following contribution to the voltage:

$$u=\rho j\exp\{-H''(X_l)t/\eta\}. \tag{41}$$

To find the total voltage we have to multiply (41) by the probability of finding a vortex in the lth well, sum the result over all wells, and average over all possible realization of random potential:

$$U=\rho jNg(t)/g(0),$$
$$g(t)=\langle\exp[-\beta dH(X_l)-H''(X_l)t/\eta]\rangle, \tag{42}$$

where N is the total number of vortices in the system, $\beta=1/kT$, and d is the vortex length. The "Boltzmann statistics" used here is applicable if the number of minima in the system is larger than N (small vortex density). If vortex density is large, "Fermi statistics" can be used provided the force of vortex repulsion in one well is infinitely large.

Note that in Refs. 16 and 17 a formula of the type (42) has been used to calculate the response to an alternating signal for Josephson structures and charge density waves. However the calculation has been based on averaging over the rigidity H'' with the Gaussian distribution function, so that the results obtained there have nothing in common with those found below.

Averaging in Eq. (42) reduces to calculation of the following expression

$$g(t)=\int_0^\infty K\,dK\varphi(K)\exp(-Kt/\eta), \tag{43}$$

where

$$\varphi(K)=\left\langle\int_{\mathscr{L}}dx\delta[H'(x)]\delta[K-H''(x)]\exp[-\beta dH(x)]\right\rangle. \tag{44}$$

Integration over only positive values of K corresponds to the choice of minima, $H''(x)>0$, among the extrema given by the equation $H'(x)=0$. To find (44) we have used the identity

$$\sum_l\delta(x-X_l)=H''(x)\delta[H'(x)].$$

Here and below we omit all constant factors in the expressions for $g(t)$ and $\varphi(t)$, since, according to (42), they do not enter into the result.

Further calculations require a model for the potential $H(x)$. We consider pinning centers randomly arranged at the points $X=X_n$ of a junction of length $\mathscr{L}$. We restrict the discussion to the weak pinning model, when the potential at an isolated center is given by (24). If the center density is not very large so that $lL/\delta\ll1$ ($L=\mathscr{L}/\nu$, where ν is the total number of pinning centers), the potential $H(x)$ can be written as

$$H(X)=E_s\left\{1-\frac{l}{2}\sum_n f[(X-X_n)/\delta]\right\},\quad f(x)=\mathrm{ch}^{-2}x. \tag{45}$$

In such a model averaing in (44), after we have used the

Fourier transformation for the δ-function, takes the form

$$\varphi(K)=\int_{(\mathscr{L}/\delta)} dx\int_{-\infty}^{\infty} ds\, d\sigma \exp(iK\sigma)$$

$$\times\left\{\frac{\delta}{\mathscr{L}}\int_{(\mathscr{L}/\delta)} dx_n \exp\left[\frac{1}{2}E_J l\left(\beta d f(x-x_n)+\frac{is}{\delta}f'(x-x_n)+\frac{i\sigma}{\delta}f''(x-x_n)\right)\right]\right\}^{\nu}$$

$$=\int_{-\infty}^{\infty} d\sigma\, ds \exp(iK\sigma)$$

$$\times\left\{\frac{\delta}{\mathscr{L}}\int_{(\mathscr{L}/\delta)} dx \exp[\gamma f(x)+isf'(x)+i\sigma f''(x)]\right\}^{\nu}.$$

where $k=2\delta^2K/lE_J$ is the dimensionless rigidity, and $\gamma=lE_J\beta d/2$. Turning to an infinitely long chain $(\mathscr{L},\nu\to\infty,\ \mathscr{L}/\nu=L)$, we get a much simpler expression

$$\varphi(k)=\int_{-\infty}^{\infty} ds\, d\sigma \exp[ik\sigma+G(is,i\sigma)], \tag{46}$$

$$G(a,b)=\frac{\delta}{L}\int_{-\infty}^{\infty} dx\{\exp[\gamma f(x)+af'(x)+bf''(x)]-1\}.$$

Then Eq. (43) acquires the form

$$g(t)=\int_0^{\infty} k\,dk\int_{-\infty}^{\infty} ds\, d\sigma \exp\left\{-k\left(\frac{t}{2\tau}-i\sigma\right)+G(is,i\sigma)\right\}, \tag{47}$$

where $\tau=\delta^2\eta/lE_J$ is the characteristic time of the problem.

The case of small density of defects ($\delta/L\ll 1$) is trivial. Expanding the exponent in (47) in G to an accuracy of terms in G, we integrate over k, s, σ and x. As a result, we get

$$U(t)=\rho jN\exp(-t/\tau), \tag{48}$$

which corresponds to the usual exponential relaxation. This result is evident, if we recall that in a sparse lattice a vortex is pinned at a solitary center, and the fluxons localized at neighboring centers overlap only slightly. However, the criterion of validity of Eq. (48) turns out to be more rigid than $\delta/L\ll 1$. Analysis shows that Eq. (48) is valid for $\delta\exp(\gamma/L)\ll 1$, i.e., for very sparse lattices and not very low temperatures. The fact is that there are always regions of center condensation, where pinning is anomalously strong, and therefore fluxons are localized just near them.

The lattices and temperature ranges, for which $\delta\exp(\gamma/L)\gg 1$, are of significantly greater interest. The parameter $\delta l/L$ remains small, so that it is assumed that Eq. (45) holds. In this case the integrals over s and σ can be taken by the saddle-point method. The saddle point is $s=0$ and $\sigma=-i\eta$, where η is given by the equation

$$k+G'(\eta)=0, \tag{49}$$

where $G(\eta)\equiv G(0,\eta)$ and

$$G'(\eta)=\frac{\delta}{L}\int_{-\infty}^{\infty} dxf''\exp(\gamma f+\eta f'').$$

Upon integrating over s and σ near the saddle point, we find, omitting insignificant constant factors:

$$g(t)=\int_0^{\infty} k\,dk[G_2(\eta)G''(\eta)]^{-1/2}\exp\left\{-k\left(\frac{t}{2\tau}-\eta\right)+G(\eta)\right\}. \tag{50}$$

One has to bear in mind that η is a function of k in accordance with (49). Now it is convenient to go over in (50) from integration over k to integration over η. As a result, we have

$$g(t)=\int_{-\infty}^{\eta_0} d\eta G'(\eta)\{G''(\eta)/G_2(\eta)\}^{1/2}$$

$$\times\exp\left\{G'(\eta)\left(\frac{t}{2\tau}-\eta\right)+G(\eta)\right\}. \tag{51}$$

Here η_0 is that value of η for which $k=0$, i.e.,

$$G'(\eta_0)=0,\quad G_2(\eta)=\frac{d^2}{da^2}G(a,\eta)|_{a=0}. \tag{52}$$

It is easy to show that $\eta_0>0$.

The exponent in (51) has a maximum at $\eta=t/2\tau$. Therefore the function $g(t)$ behaves differently for $\eta_0>t/2\tau$ and $\eta_0<t/2\tau$. For $\eta_0>t/2\tau$ the maximum is within the integration interval. Expanding the exponent in a series near $\eta=t/2\tau$, we get

$$g(t)=G'(t/2\tau)\left\{\frac{2\pi}{G_2(t/2\tau)}\right\}^{1/2}\exp[G(t/2\tau)].$$

Here, going over from (51), we keep the constant factor in order to determine $g(0)$ properly:

$$g(0)=G'(0)\left\{\frac{2\pi}{G_2(0)}\right\}^{1/2}\exp[G(0)]. \tag{53}$$

Finally, according to (41), for $\eta_0>t/2\tau$ we have

$$U(t)=\rho jN\left[G'\left(\frac{t}{2\tau}\right)\Big/G'(0)\right]$$

$$\times\left[G_2(0)/G_2\left(\frac{t}{2\tau}\right)\right]^{1/2}\exp\left[G\left(\frac{t}{2\tau}\right)-G(0)\right]. \tag{54}$$

Now we turn to long relaxation times $\eta_0<t/2\tau$. In this range the main contribution to the integral in (51) comes from the vicinity of the point $\eta\simeq\eta_0$. Hence, allowing for (52), we have

$$g(t)=-\left(\frac{t}{2\tau}-\eta_0\right)^{-2}[G_2(\eta_0)G''(\eta_0)]^{-1/2}\exp[G(\eta_0)].$$

Now we get $U(t)$ for $\eta_0<t/2\tau$:

$$U(t)=-\rho jN\left(\frac{t}{2\tau}-\eta_0\right)^{-2}\frac{1}{G'(0)}$$

$$\times\left\{\frac{G_2(0)}{2\pi G_2(\eta_0)G''(\eta_0)}\right\}^{1/2}\exp\{G(\eta_0)-G(0)\}. \tag{55}$$

Note that $G'(0)<0$, and therefore $U(t)>0$ for $j>0$.

To find the actual time and temperature dependences of U from (54) and (55), it is necessary to carry out the integration in the expressions for G and G_2. Using the relations

$$f'=-2f(1-f)^{1/2}\,\mathrm{sign}x,\quad f''=4f-6f^2,$$

we can write the expression for $G(\eta)$ in the form

$$G(\eta)=\frac{\delta}{L}\int_0^1\frac{du}{(1-u)u^{1/2}}\exp\{\gamma-2\eta+(8\eta-\gamma)u-6\eta u^2\}.$$

Consider now the most interesting limit $\gamma\gg 1$, when the vortex energy is larger than kT. The exponent in the integrand has its maximum at $u=u_0=(8\eta-\gamma)/12\eta$, and the asymptote for G has different forms for $8\eta>\gamma$ (i.e., for $u_0>0$) and $8\eta<\gamma$ ($u_0<0$). For $8\eta<\gamma$ the main contribution to the integral comes from the vicinity of the point $u=0$, and

$$G(\eta)=\frac{\delta}{L}\left(\frac{\pi}{\gamma-8\eta}\right)^{1/2}\exp(\gamma-2\eta).$$

For $8\eta>\gamma$ the main contribution comes from the point $u=u_0$, and

$$G(\eta)=\frac{\delta}{L}\frac{\eta 12(2\pi)^{1/2}}{(\gamma+4\eta)(8\eta-\gamma)^{1/2}}\exp\left\{\frac{1}{3}\left(\gamma+2\eta+\frac{\gamma^2}{8\eta}\right)\right\}.$$

In a similar way we find $G_2(\eta)$:

$$G_2(\eta)=\frac{\delta\pi^{1/2}}{L}\times\begin{cases}2(\gamma-8\eta)^{-3/2}\exp(\gamma-2\eta), & 8\eta<\gamma,\\ \dfrac{2^{1/2}}{36\eta^2}(4\eta+\gamma)(8\eta-\gamma)^{1/2}\exp\left\{\dfrac{1}{3}\left(\gamma+2\eta+\dfrac{\gamma^2}{8\eta}\right)\right\}, & 8\eta>\gamma.\end{cases}$$

Now we can find η_0 from (52). For this purpose we differentiate $G(\eta)$ with respect to η:

$$G'(\eta)=\frac{\delta\pi^{1/2}}{L}\times\begin{cases}2(\gamma-8\eta)^{-1/2}\exp(\gamma-2\eta), & 8\eta<\gamma,\\ \dfrac{\gamma-4\eta}{\eta[2(8\eta-\gamma)]^{3/2}}\exp\left\{\dfrac{1}{3}\left(\gamma+2\eta+\dfrac{\gamma^2}{8\eta}\right)\right\}, & 8\eta>\gamma.\end{cases}$$

Hence one can see that, in particular, $\eta_0=\gamma/4$.

Thus, the function $U(t)$ has three characteristic regions. In the first one, $0<t/\tau<\gamma/4$, according to (54),

$$U(t)=\rho jN\left(1-\frac{4t}{\tau\gamma}\right)^{1/4}\times\exp\left\{-\frac{\delta}{L}\left(\frac{\pi}{\gamma}\right)^{1/2}e^{\gamma}\left[1-\left(1-\frac{4t}{\tau\gamma}\right)^{-1/2}\right]e^{-t/\tau}\right\}. \tag{56}$$

Hence, for $t/\tau<1$ exponential relaxation occurs:

$$U(t)\simeq\rho jN\exp\left\{-\frac{\delta}{L}\left(\frac{\pi}{\gamma}\right)^{1/2}e^{\gamma}\frac{t}{\tau}\right\}. \tag{56a}$$

In the intermediate region, $\gamma/4<t/\tau<\gamma/2$,

$$U(t)=\rho jN\frac{3(\gamma-2t/\tau)}{(2\gamma)^{1/4}(4t/\tau-\gamma)^{3/4}(2t/\tau+\gamma)^{1/2}}\times\exp\left\{\frac{\delta\pi^{1/2}}{L\gamma^{3/2}}e^{\gamma}\left\{1-\frac{6(2\gamma)^{1/2}t/\tau}{(4t/\tau-\gamma)^{1/2}(2t/\tau+\gamma)}\times\exp\left[-\frac{1}{3}\left(2\gamma-\frac{t}{\tau}-\frac{\gamma^2\tau}{8t}\right)\right]\right\}\right\}. \tag{57}$$

The long-time asymptote (55) has a power-law character:

$$U(t)=\rho jN\left(\frac{t}{\tau}-\frac{\gamma}{2}\right)^{-2}\left(\frac{\delta}{L}\right)^{3/2}\frac{\gamma^{5/4}}{2^{5/2}\pi^{5/4}}\times\exp\left\{-\frac{7}{6}\gamma-\frac{\delta\pi^{1/2}}{L\gamma^{1/2}}e^{\gamma}\left(1-\frac{3}{2}e^{-\gamma/3}\right)\right\}, \tag{58}$$

where we have used the relation

$$G_2(\eta_0)=\frac{\delta}{L}\frac{36(2\pi)^{1/2}}{\gamma^{3/2}}\exp\left(\frac{2}{3}\gamma\right).$$

Thus, at short times the relaxation has a rapid exponential character given by (56), the relaxation rate being the greatest at very short times t/τ. In this region the relaxation is similar to that at an isolated defect [see (48)]. However, the voltage relaxation on the lattice occurs much faster within times of order

$$\tau^*=\frac{\tau}{\delta}\frac{L\gamma^{1/2}}{\pi^{1/2}}e^{-\gamma},$$

where $\gamma=E_p/kT$, and $E_p=lE_Jd/2$ is the energy of vortex pinning at an isolated defect. The relaxation time τ^* drops exponentially with temperature. At this stage the deepest pinning centers, which correspond to defect condensation and have the largest rigidity, undergo relaxation. Since for small vortex density (for the Boltzmann statistics) the largest part of vortices are localized just at these centers, the largest part of drop of the initial voltage ρjN down to the magnitude of order $\rho jN\exp(-\delta e^{\gamma}/L)$ corresponds to the exponential region of relaxation. The parameter $\delta e^{\gamma}/L$ is assumed to be large. However, if we want the voltage to be still noticeable upon its exponential drop, we must have a sparse lattice, $\delta/L\ll 1$, and not a very large parameter E_p/kT.

Here we must note the following. First, the pinning energy being proportional to the vortex length, the sample thickness should not be very large. Furthermore, for a large vortex length it is necessary to allow for its deformation (bending). Second, for not too small vortex density (for the Fermi statistics) the fraction of vortices at ultradeep centers decreases, therefore the voltage drop in the exponential region should decrease as well. In the intermediate time region the relaxation assumes a slower character (57).

Finally, in the region of the longest times, $t>\tau E_p/2kT$, the relaxation drop assumes a power character (58), for which $U\propto t^{-2}$. In this region the relaxation is governed by centers with very small rigidity, so that the effective relaxation time here grows with time, $\tau\propto t\ln t$. Note that in the power region of relaxation the current-voltage characteristic

$U(j)$ probably becomes nonlinear already for a very small j, since the centers with the small rigidity are separated from the neighboring wells by low barriers. This means that even a small current leads to above-the-barrier transition, and the linearized Eq. (40) becomes invalid.

The relation between the time dependence of relaxation, $U(t)$, and the system response to the alternating current $j\exp(i\omega t)$ is

$$U(\omega)=i\omega\int_0^\infty dt\,U(t)\exp(i\omega t).$$

According to (56a), the function $U(\omega)$ in the high frequency limit has a character of the Debye losses, $U(\omega)\propto i\omega/(1+i\omega\tau^*)$. In the low frequency range $U(\omega)\propto\omega^2\ln\omega$.

Note in conclusion that the adiabatic approximation used above is valid only in the case of sufficiently slow fluxon motion, since the effects of fluxon radiation excitation related to the inner degrees of freedom of a vortex state are ignored, as are the processes of fluxon creation and annihilation. For a weak interaction with defects the radiation effects arise in the second order of perturbation theory and are sufficiently well studied.[2,18-20] For a strong interaction the radiation effects can, in principle, be studied by allowing for nonadiabatic corrections to the technique exposed in the present paper. These effects can be completely ignored in the case when the time of interaction of a solitary wave with a defect is larger than the period of intrafluxon structure oscillations, i.e., if $v/\delta\ll\omega_0$, where v is the fluxon velocity, δ is its size, and ω_0 is the characteristic frequency of the radiation modes.

[1] K. Longren and A. Scott, *Soliton in Action*, Academic Press, New York, 1978.
[2] Yu. S. Kivshar and B. A. Malomed, Rev. Mod. Phys. **61**, 703 (1989).
[3] V. U. Zakharov, S. V. Manakov, S. P. Novikov, and L. P. Pitaevskiĭ, *Teoriya solitonov (Theory of Solitons),* Moscow, 1980.
[4] D. J. Kaup and A. C. Newell, Proc. R. Soc. London, Ser. A **361**, 413 (1978).
[5] V. I. Karpman and E. M. Maslov, Phys. Lett. A **60**, 307 (1977).
[6] M. B. Vogel, S. E. Trullinger, A. R. Bishop, and J. F. Krumhansl, Phys. Rev. B **15**, 1578 (1977).
[7] K. K. Likharev and A. B. Zorin, Low Temp. Phys. **59**, 347 (1985).
[8] R. O. Starobogatov, M. M. Trunov, and G. Yu. Yashin, Fiz. Tverd. Tela **29**, 56 (1987) [Sov. Phys. Solid State **29**, 31 (1987)].
[9] K. Yoshihiro, J. Kinoshita, K. Inagaki, and C. Yamanouchi in *Proceedings of the 2nd International Symposium on Foundation of Quantum Mechanics, Tokyo, 1987.*
[10] A. Barone and G. Paterno, *Physics and Applications of the Josephson Effect*, Wiley, New York, 1982.
[11] V. I. Karpman and E. M. Maslov, Zh. Eksp. Teor. Fiz. **75**, 504 (1978) [JETP **48**, 252 (1978)].
[12] V. V. Bryksin, A. V. Goltsev, and S. N. Dorogovtsev, Physica C **160**, 103 (1989).
[13] Ya. V. Kopelevich, V. V. Lemanov, E. B. Sonin, and A. L. Holkin, Pis'ma Zh. Eksp. Teor. Fiz. **50**, 188 (1989) [JETP Lett. **50**, 212 (1989)].
[14] V. M. Vinokur, M. V. Feigelman, and V. B. Geshkenbein, Phys. Rev. Lett. **67**, 915 (1991).
[15] V. V. Bryksin, A. V. Goltsev, S. N. Dorogovtsev and A. N. Samukhin, J. Mod. Phys. B, in print.
[16] B. A. Malomed, J. Phys. C **21**, 5163 (1988).
[17] B. A. Malomed, Phys. Lett. A **136**, 395 (1989).
[18] D. J. Kaup and P. J. Hansen, Physica D **18**, 77 (1986).
[19] O. H. Olsen and M. R. Samuelsen, Phys. Rev. B **33**, 595 (1986).
[20] B. A. Malomed, Physica D **27**, 113 (1987).

Translated by E. Khmelnitski