Inflationary solutions in homogeneous cosmological models with a scalar field

S. L. Parnovskiĭ

Astronomical Observatory of the T. G. Shevchenko Kiev State University, Ukraine
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The evolution of homogeneous cosmological models of the Bianchi types II, VI_0, and VII_0 in the presence of a scalar field is considered. In the initial stages the constructed dynamical systems are shown to reduce to a model of the Bianchi type I studied by Belinskiĭ and Khalatnikov [Sov. Phys. JETP **66**, 441 (1987)]. In view of this, for models of these types the ratio of the number of noninflationary solutions to the total number of solutions is also estimated at m/m_P, which is much smaller than unity.

1. INTRODUCTION

Belinskiĭ, Grishchuk, Zeldovich, and Khalatnikov[1,2] studied the inflationary stages in cosmological solutions with a scalar field and determined the degree to which these stages can be considered general. They demonstrated that for the flat and open Friedmann models and for a homogeneous space of the Bianchi type I, the ratio of the number of solutions without inflationary expansion to the total number of solutions is a small quantity of order m/m_P, where m is the mass of the field's quantum, and m_P is the Planck mass.

This paper shows that the results can easily be generalized to the case of homogeneous spaces of the Bianchi types II, VI_0, and VII_0 with a metric tensor of the form

$$ds^2 = dt^2 - \gamma_{\alpha\beta}dx^\alpha dx^\beta,$$
$$\gamma_{\alpha\beta} = a_1^2(t)l_\alpha l_\beta + a_2^2(t)m_\alpha m_\beta + a_3^2(t)n_\alpha n_\beta, \tag{1}$$

where **l**,**m**,**n** constitutes a set of basis vectors of the given space.

2. EVOLUTION OF A HOMOGENEOUS SPACE OF THE BIANCHI TYPE II WITH A SCALAR FIELD

Let us discuss in greater detail the case of a homogeneous Bianchi type-II space. To obtain a dynamical system of equations that links the scalar field φ, the "Hubble constant" $H = \frac{1}{3} d(\ln a_1a_2a_3)/dt$, the quantity $N = a_1/a_2a_3$, and their derivatives, we use the results of Ref. 3 by substituting the expressions for the effective energy density and the pressure of the scalar field into the system of equations derived in that paper. Setting the viscosity equal to zero, we obtain

$$\ddot{\varphi} = -3H\dot{\varphi} - m^2\varphi, \tag{2}$$

$$\dot{H} = -3H^2 + 4\pi\frac{m^2}{m_P^2}\varphi^2 + \frac{N^2}{6}, \tag{3}$$

$$\dot{N} = Np/2, \tag{4}$$

$$\dot{p} = -3N^2 - 8\pi\frac{m^2}{m_P^2}\varphi^2 - 3Hp. \tag{5}$$

The dots stand for time derivatives, m is the mass of the field's quantum, and m_P is the Planck mass, which appears in the Einstein equations because we use a system of units in which the speed of light, Planck's constant, and Boltzmann's constant are set to unity. To get rid of the factors m and m_P, we change the scale along each axis by introducing the new variables

$$\varphi = \sqrt{\frac{3}{4\pi}}\, m_P x, \quad \dot{\varphi} = \sqrt{\frac{3}{4\pi}}\, m_P m y,$$
$$H = mz, \quad t = \eta/m, \quad N = \sqrt{6}m\lambda, \quad p = m\alpha. \tag{6}$$

The system of equations assumes the form

$$x_\eta = y, \quad y_\eta = -3zy - x, \quad \lambda_\eta = \alpha\lambda/2,$$
$$z_\eta = 3x^2 - 3z^2 + \lambda^2, \quad \alpha_\eta = -18\lambda^2 - 6x^2 - 3z\alpha. \tag{7}$$

The auxiliary condition[3]

$$\frac{4\pi(m^2\varphi^2 + \dot{\varphi}^2)}{m_P^2} \leq 3H^2 - \frac{N^2}{4} \tag{8}$$

becomes

$$x^2 + y^2 + \lambda^2/2 \leq z^2. \tag{9}$$

In addition, both λ and z are positive (we consider only the inflationary expansion stage).

The constructed dynamical system of equations describes the evolution of a homogeneous model of the Bianchi type II in the five-dimensional space with coordinates x, y, z, λ, and α. Two subspaces can be isolated in this space where the evolution has already been studied. By setting $\lambda = \alpha = 0$ we reduce the system (7) to that for the Bianchi type-I model studied by Belinskiĭ and Khalatnikov.[2] Naturally, the case $\lambda = 0$, which corresponds to $a_1 = 0$, is nonphysical, but many singular points of (7) lie in this hyperplane, as we will shortly see.

The second subspace studied, $x = y = 0$, describes the evolution of a homogeneous cosmological Bianchi type-II model in the absence of a scalar field. The solution of (7) in this case is a single change of epochs in the oscillatory solution of Belinskiĭ, Lifshitz, and Khalatnikov,[4] which exhibits, as $\eta \to 0$ and $\eta \to \infty$, the Kasner asymptotic behavior

$$a_1 \propto t^{q_1}, \quad a_2 \propto t^{q_2}, \quad a_3 \propto t^{q_3}, \tag{10}$$

where q_1, q_2, and q_3 are numbers obeying the condition

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$$q_1 + q_2 + q_3 = q_1^2 + q_2^2 + q_3^2 = 1. \tag{11}$$

One of these numbers must be negative. Near the initial point in the evolution, $t = 0$, in the subspace $x = y = 0$ we have the asymptotic solution (10), (11) with $q_1 > 0$. In response to the perturbation terms related to spatial curvature the solution is transformed into a similar solution with $q_1 < 0$, which is the asymptotic solution for $t \to \infty$. As a result we find that $a_1 \to 0$ and $\lambda \to 0$ as $\eta \to 0$ and $\eta \to \infty$.

Let us now study the system of equations (7) and (9). One can easily see that the only singular point for finite values of x, y, z, and λ is $x = y = z = \lambda = 0$, which lies at the intersection of these two subspaces and corresponds to the final stages in the expansion. Near this point the following solutions are admissible ($\eta \to \infty$):

$$z \to \frac{2}{3\eta}, \quad \lambda \to \frac{1}{\sqrt{24}\eta}, \quad x \to \sqrt{\frac{5}{12}}\frac{1}{\eta}\sin(\eta - \eta_0),$$
$$y \to \sqrt{\frac{5}{12}}\frac{1}{\eta}\cos(\eta - \eta_0). \tag{12}$$

Here

$$a_1 \propto t^{1/2}, \quad a_2 \propto t^{3/4}, \quad a_3 \propto t^{3/4}. \tag{13}$$

In addition, there are two other types of asymptotic behavior established in Ref. 2: the Kasner asymptotic behavior (in our case it must satisfy the condition $a_1 \to 0$) unstable in the presence of an arbitrarily low field φ, and

$$z \to \frac{2}{3\eta}, \quad x \to \frac{2}{3\eta}\sin(\eta - \eta_0), \quad y \to \frac{2}{3\eta}\cos(\eta - \eta_0), \tag{14}$$

which is unstable for $\lambda \neq 0$. These solutions correspond to separatrices lying in the planes $x = y = 0$ and $\lambda = 0$, respectively. Thus, in the initial stages of the evolution, "isotropization" does not occur in the case of the Bianchi type-II space, in contrast to the Bianchi type I. However, the average pressure also vanishes and the scalar field imitates a dusty medium.

But we are most interested in the singular points for $z \to \infty$. If condition (9) is met, these points coincide with the singular points of the dynamical system for a homogeneous Bianchi type I space studied in Ref. 2. Three solutions are admissible near these singular points. One is the vacuum Kasner solution (10), (11) for $t \to 0$. Another is the generalization of this solution with a scalar field. This was found by Belinskiĭ and Khalatnikov[5] and has the form of (10) with

$$q_1 + q_2 + q_3 = 1, \quad q_1^2 + q_2^2 + q_3^2 = 1 - 8\pi m_P^{-2} C^2,$$
$$\varphi = C\ln(t/t_0), \quad C = \text{const}. \tag{15}$$

As noted earlier, for the Bianchi type-II space these solutions must have $q_1 > 0$ and a_1 must tend to 0 as $\eta \to 0$. The third admissible solution is the inflationary solution for $\eta \to -\infty$:

$$z \to -\eta/3, \quad x \to \pm\eta/3, \quad \lambda \ll z. \tag{16}$$

For this the functions in the metric (1) have the form

$$a_i \propto \exp\left[-\frac{\eta^2}{6} + C_i\int_0^\eta \exp\left(\frac{\xi^2}{2}\right) d\xi\right],$$
$$C_i = \text{const}, \quad \sum_{i=1}^{3} C_i = 0. \tag{17}$$

If we want the condition $\lambda \ll z$ to hold, we must assume $C_1 < 0$. This solution describes an inflationary separatrix that is unstable as $\eta \to -\infty$. Thus, in the inflationary stage the universe rapidly becomes isotropic at a rate considerably greater than the rate of inflationary expansion. The first to obtain such a solution for the inflationary regime created by a massive scalar field and the probability of such a regime setting in was apparently Starobinskiĭ.[6]

Note that the system of equations has two more possible asymptotic states for $z \to \infty$ and $\eta \to 0$,

$$x \to \frac{\sqrt{2}}{3\eta}, \quad z \to \frac{2}{3\eta}, \quad \lambda \propto \eta^{-2/3}\exp\left(-\frac{C^2}{\eta}\right), \quad C = \text{const} \tag{18}$$

and

$$x \to \sqrt{\frac{5}{6}}\frac{1}{2\eta}, \quad \lambda \to \frac{1}{\sqrt{24}\eta}, \quad \alpha \to -\frac{1}{\eta} \tag{19}$$

with $a_1 \propto t^{3/4}$, $a_2 \propto t^{5/4}$, and $a_3 \propto t^{5/4}$. But neither solution meets condition (9).

As a result we see that for $z \to \infty$ the singular points lie in the subspace $\lambda = \alpha = 0$. More than that, for large z we can ignore the term λ^2 in the equation for z_η, the only equation among those for x_η, y_η, and z_η that distinguishes a Bianchi type-I space from a Bianchi type-II space. Hence, the question of finding the probability of the inflationary stage in the two cases is resolved in the same manner and we can simply employ the results of Ref. 2. The probability of noninflationary evolution constitutes an extremely small quantity of the order of m/m_P for the Bianchi type-II space, too.

3. EVOLUTION OF HOMOGENEOUS SPACES OF THE BIANCHI TYPES VI_0 AND VII_0 WITH A SCALAR FIELD AT HIGH VALUES OF H

Let us now show that the above result can be applied to homogeneous cosmological models of the Bianchi types VI_0 and VII_0 with a scalar field. We set up a system of equations for this case by introducing auxiliary quantities $M = a_2/a_1a_3$ and $q = 2d(\ln M)/dt$. Equations (2) and (4) retain their form, but instead of (3) and (5) we get

$$\dot{H} = -3H^2 + 4\pi\frac{m^2}{m_P^2}\varphi^2 + \frac{(N - kM)^2}{6}, \tag{20}$$

$$\dot{p} = -3N^2 + M^2 + 2kNM - 8\pi(m^2/m_P^2)\varphi^2 - 3Hp. \tag{21}$$

Here $k = -1$ for the Bianchi type-VI space and $k = 1$ for the Bianchi type-VII space. Two more equations must be added: for the variable M and for its derivative:

$$\dot{M} = Mq/2, \tag{22}$$

$$\dot{q} = -3M^2 + N^2 + 2kNM - 8\pi(m^2/m_P^2)\varphi^2 - 3Hq. \tag{23}$$

After we have introduced the variables x, y, z, η, λ, and α and the quantities $\beta = q/m$ and $\mu = M/\sqrt{6m}$, the system of equations (2), (4), and (20)–(23) becomes

$$z_\eta = 3x^2 - 3z^2 + (\lambda - k\mu)^2, \tag{24}$$

$$x_\eta = y, \quad y_\eta = -3zy - x, \quad \lambda_\eta = \alpha\lambda/2, \quad \mu_\eta = \beta\mu/2, \tag{25}$$

$$\alpha_\eta = -18\lambda^2 + 6\mu^2 + 12k\lambda\mu - 6x^2 - 3z\alpha, \tag{26}$$

$$\beta_\eta = -18\mu^2 + 6\lambda^2 + 12k\lambda\mu - 6x^2 - 3z\beta. \tag{27}$$

The auxiliary condition takes the form

$$x^2 + y^2 + (\lambda - k\mu)^2/2 \leq z^2. \tag{28}$$

In addition, we must assume $z \geqslant 0$, $\lambda > 0$, and $\mu > 0$. Ignoring all the details concerning the singular points of this system for z finite, we turn to the form of the solution in the limit $z \to \infty$. A simple analysis reveals that the Kasner solution (10), (11), its generalization (10), (15), with $q_1 > 0$ and $q_2 > 0$, and the inflationary solution (16), (17), with $C_1 < 0$ and $C_2 < 0$, are still admissible. The constants $C_1 = C_2$ may be positive only in the special case where for the Bianchi type-VII space we also assume $a_1 = a_2$. But then we have $\lambda - k\mu = 0$, and the system of equations for x, y, and z is reduced to the system for the Bianchi type-I space.

To prove the above statements we consider the three terms on the right-hand side of Eq. (24). According to (28), the order of the term with z^2 cannot be lower than that of the other terms. Hence, if the leading terms in the right-hand side of (24) do not cancel out, we have $z_\eta \propto z^2$ and $z \propto \eta^{-1}$. If $x \propto \eta^{-1}$, then $y \propto \eta^{-2}$ and condition (28) is not met. Hence, $x \ll z$ as $\eta \to 0$, and the scalar field in Eqs. (24), (27), and (28) can be discarded. In the absence of this field at $x = 0$ we have the Kasner asymptotic behavior (10), (11). In the presence of the field incorporated into the vacuum metric, we have $x \propto \ln \eta$ and $y \propto \eta^{-1}$, and we obtain the solution (10), (15) for $\eta \to 0$.

But if the leading terms on the right-hand side of (24) do cancel out, for $z^2 \approx x^2 \gg (\lambda - k\mu)^2$ we arrive at the inflationary solution (16), (17). The variant with $z^2 \approx (\lambda - k\mu)^2/3 \gg x^2$ is not suitable. Proof of this can be obtained from considering the latter case more closely and also from the fact that in the absence of a scalar field only the Kasner asymptotic behavior is possible. If we assume $x \propto z \propto \lambda - k\mu$, we find $z \propto \eta$ as $\eta \to -\infty$ and, hence, α and β tend to $2\eta^{-1}$ as $\eta \to \infty$. Subtracting Eq. (27) from Eq. (26), we find that $\lambda - \mu \to 0$ as $\eta \to \infty$. Hence, at $k = 1$ (type VII) the system is reduced to the system of equations for the Bianchi type-I space. On the same assumptions the system of equations for the Bianchi type-VI space has no solution.

In the case of the Bianchi type-VI space we must consider the variant where $\lambda \approx \mu \gg \lambda - \mu \propto z$ holds. This has no solution for $\lambda \neq \mu$, and this special case reduces the system of equations (24)–(27) to the system for the Bianchi type-I space.

Generally, the system of equations (24)–(27) has additional singular points and related asymptotic solutions different from type I, but all such points lie outside the cone specified by condition (28). As in the Bianchi type-II case considered earlier, we can ignore the exponentially small term $(\lambda - k\mu)^2$ in Eq. (24), which reduces the system to that for the Bianchi type-I space. This means that if the universe was indeed born as a result of quantum creation in a highly anisotropic state with $N \approx H$, this terms falls off much faster than the scalar field, and the universe passed through the inflationary stage. An exception is the extremely unlikely case of the birth of the universe with an initially small scalar field. The probability of this is a quantity of the order of m/m_P and determines the probability of noninflationary evolution for the homogeneous cosmological models of the Bianchi types I, II, VI_0, and VII_0.

4. CONCLUSION

We have found that the investigation of the initial stages of the expansion of a homogeneous cosmological model of the Bianchi type I conducted in Ref. 2 is more general and can also be applied to the cases of types II, VII_0, and $VIII_0$. In this respect it is also true of models such that the ratio of the number of noninflationary solutions to the total number of solutions is a small quantity of order m/m_P and, hence, the inflationary stage is certain to occur.

A massive scalar field acts as a cosmological constant. Wald[7] has shown that for all homogeneous cosmological models except the Bianchi type IX a cosmological constant $\Lambda > 0$ ensures that the inflationary asymptotics is reached in a time period of the order of $(3/\Lambda)^{1/2}$. Hence, our result can be expected to be valid for other types of homogeneous cosmological models, except for type IX. This statement supports the investigation, reported in Refs. 1 and 2, of the evolution of a particular model of the Bianchi type IX, a closed Friedmann model with a massive scalar field. The probability of noninflationary evolution obtained proved not to be a negligible small quantity, in contrast to all other cases considered.

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