Neutrino-pair synchrotron radiation of electrons and positrons in a hot plasma

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(Submitted 2 June 1992)
Zh. Eksp. Teor. Fiz. **103**, 438–454 (February 1993)

We calculate the neutrino energy loss rate Q for the neutrino-pair synchrotron radiation emitted by electrons and positrons in a nondegenerate (relativistic and nonrelativistic) plasma and a magnetic field of arbitrary strength. Synchrotron losses exhibit a disparate nature in seven regions of temperature (T) and magnetic field (B). The asymptotic behavior of Q in all seven regions is derived together with a general interpolation expression. We find, among other things, $Q \approx 7\times 10^{17}\rho_6 b^6$ erg cm^{-3} s^{-1} for $b \ll t \ll 1$ and $Q \approx 4\times 10^{19} t^5 b^2 \ln(t/t_B)$ erg cm^{-3} s^{-1} for $t \gg t_B$, where $t = T/(6\times 10^9\,\mathrm{K})$, $b = B/(4.4\times 10^{13}\,\mathrm{G})$, ρ_6 is the density in units of 10^6 g cm^{-3}, and $t_B \approx (1+b)^{3/2} b^{-1}$. For $\rho \approx 10^3$–10^6 g cm^{-3}, $T \approx 10^8$–10^{10} K, and $B \gtrsim 10^{12}$–10^{14} G synchrotron losses are comparable to, or exceed, other neutrino energy losses, a fact that may be important for the neutrino cooling of neutron star crusts.

1. INTRODUCTION

Neutrino radiation is a major cause of energy losses in the late stages of stellar evolution (in pre-supernovae and neutron stars, for instance). Neutrino losses result from a number of mechanisms (see, e.g., Refs. 1 and 2) and have been well studied for conditions where the effect of the magnetic field can be ignored.

On the other hand, neutrino losses are influenced by strong magnetic fields. As is well known from observations, there are strong fields ($B = 10^{11}$–10^{13} G) on surfaces of neutron stars. Inside stars magnetic fields can be even higher. Since neutrino radiation greatly cools neutron stars, it is important to study the effect of a magnetic field on various mechanisms of neutrino losses. More than that, a magnetic field brings to the fore a specific mechanism of neutrino losses: synchrotron emission of neutrino pairs by electrons e^- (or positrons e^+)

$$e^{\mp} \to e^{\mp} + \nu + \bar{\nu}. \tag{1}$$

For $B = 0$ this mechanism is forbidden by the law of conservation of energy and momentum.

The purpose of this paper is to study synchrotron neutrino cooling of a hot nondegenerate plasma. A general expression for the synchrotron neutrino energy loss rate Q has been derived in Ref. 3. There the authors also give a critical analysis of some of the previous work and derive computational formulas for the case of a nonrelativistic plasma at temperatures $T \ll 6\times 10^9$ K and in magnetic fields $B \ll 4.4\times 10^{13}$ G. For a degenerate relativistic plasma, calculations of Q have been performed in Ref. 4. Below we consider a nondegenerate (relativistic and nonrelativistic) electron-positron plasma in a magnetic field of arbitrary strength.

2. THE GENERAL FORMALISM

We use the formalism of relativistic electrons and positrons in a quantizing magnetic field in the Landau gauge (see, e.g., Refs. 5 and 6). In this picture a quantum state of an electron e^- (or a positron e^+) is determined by the projection p_z of the electron (positron) momentum in the direction of the field, the number $n = 0,1,2,\ldots$ of the Landau level, and the sign σ of the projection of the electron (positron) spin on the momentum. The levels $n > 0$ are degenerate in spin, $\sigma = \pm 1$. The ground level of $e^{\mp}$ is nondegenerate, and $\sigma = \mp\,\mathrm{sign}(p_z)$. The particle energy (here and in what follows, $c = \hbar = k = m_e = 1$, except in the final computational formulas) is

$$\varepsilon = (1 + p_z^2 + p_\perp^2)^{1/2}, \quad p_\perp = (2bn)^{1/2}, \quad b = B/B_c, \tag{2}$$

where $p_\perp$ has the meaning of particle momentum across the magnetic field, and $B_c = m_e^2 c^3/e\hbar \approx 4.414\times 10^{13}$ G.

The synchrotron radiation specified by Eq. (1) proceeds along two channels involving the charged and neutral weak intermediate bosons. We restrict our discussion to a plasma with a temperature $T \ll M_W$, with $M_W \sim 100$ GeV the boson mass. In this case the amplitude of process (1) is described by a single "four-leg" diagram. As a result the synchrotron loss rate Q (the amount of energy carried away by a neutrino from a unit plasma volume per unit time) is given by the following expression:[3]

$$Q = \frac{bQ_c}{3(2\pi)^6} \sum_{s=1}^{\infty} \sum_{n=1}^{\infty} \int_{-\infty}^{+\infty} dp_z \int d\mathbf{q}\, A\omega\, f(1-f'). \tag{3}$$

Here $\omega = \varepsilon - \varepsilon'$ and $\mathbf{q}$ are the energy and momentum carried away by a neutrino pair as the electron (or positron) passes from the initial state p_z, n, σ to the final state (p_z', n', σ') as the number of the Landau level changes by $s = n - n'$ (by analogy with electromagnetic synchrotron radiation, s can be called the cyclotron harmonic number). In Eq. (3), $f(\varepsilon)$ and $f'(\varepsilon') = f(\varepsilon - \omega)$ are the Fermi–Dirac distribution function for $e^{\mp}$ in the initial and final states:

$$f_{\mp} = \{\exp[(\varepsilon \mp \mu)/t] + 1\}^{-1}, \quad t = T/T_c, \tag{4}$$

where $T_c = m_e c^2/k \approx 5.930\times 10^9$ K, and μ is the chemical potential of the electrons. The quantity Q_c in Eq. (3) is equal to G^2, where G is the Fermi weak-coupling constant. In calculating Q we can take all quantities except Q_c on the right-hand side of the equations dimensionless, and for Q_c we can use the dimensional constant

$$Q_c = \frac{G^2}{\hbar}\left(\frac{m_e c}{\hbar}\right)^9 = 1{,}023\cdot 10^{23}\ \mathrm{erg\ cm^{-3}\ s^{-1}}, \tag{5}$$

 1063-7761/93/020229-07$10.00

which we call the Compton neutrino loss rate.

The quantity A in Eq. (3) is proportional to the square of the absolute value of the matrix element summed over σ and σ'. The integration over $\mathbf{q}$ is limited to the domain of allowed values $\omega^2 \geqslant \mathbf{q}^2$. Analysis shows[3] that only transitions with $s > 0$ are allowed.

The general formula for A is[3]

$$\begin{aligned}
\varepsilon\varepsilon' A = C_+^2 \{ & [-2(\varepsilon\varepsilon' - p_z p_z')^2 + (\varepsilon\varepsilon' - p_z p_z')(2 + p_\perp^2 + p_\perp'^2 - 2q^2) \\
& - \frac{1}{2} q_\perp^2 (q_\perp^2 - p_\perp^2 - p_\perp'^2 - 3)]\Psi + [(\varepsilon\varepsilon' - p_z p_z')(1 + p_\perp^2 + p_\perp'^2) \\
& - \frac{1}{2}(p_\perp^2 + p_\perp'^2)(3 - q_\perp^2 + p_\perp^2 + p_\perp'^2)]\Phi\} \\
& + \frac{1}{2} C_-^2 [(-2\omega^2 + 2q_z^2 + q_\perp^2)\Psi + (-\omega^2 + q_z^2 + 2q_\perp^2)\Phi] \\
& - \langle C_V C_A \rangle (\varepsilon p_z' - \varepsilon' p_z)[(2\omega^2 - 2q_z^2 - 3q_\perp^2)\Psi_- + (p_\perp^2 - p_\perp'^2)\Phi_-].
\end{aligned} \tag{6}$$

Here $q_z = p_z - p_z'$ and $q_\perp$ are the components of the momentum of the neutrino pair along and across the magnetic field,

$$\Phi = F_{n'-1,n-1}^2 + F_{n',n}^2, \quad \Psi = F_{n'-1,n}^2 + F_{n',n-1}^2,$$

$$\Phi_- = F_{n'-1,n-1}^2 - F_{n',n}^2, \quad \Psi_- = F_{n'-1,n}^2 - F_{n',n-1}^2,$$

$$F_{n'n} = \left(\frac{n'!}{n!}\right)^{1/2} u^{(n-n')/2} e^{-u/2} L_{n'}^{n-n'}(u), \quad u = \frac{q_\perp^2}{2b}, \tag{7}$$

and $L_{n'}^s$ are the associated Laguerre polynomials. In cases where formally $n < 0$ or $n' < 0$ holds one should put $F_{n'n} = 0$. The quantities C_V and C_A are the transition constants for vector and axial vector interactions. For the process (1) in which electron neutrinos are produced (with allowance for charged and neutral currents), we have $C_V = 2\sin^2\theta_W + 0.5$ and $C_A = 0.5$, where θ_W is the Weinberg angle and $\sin^2\theta_W \approx 0.23$. For a process with muon-like and tau neutrino production (by neutral currents only), $C_V' = 2\sin^2\theta_W - 0.5$ and $C_A' = -0.5$. Summation in (6) involves neutrinos of all types, where the following notation is introduced:

$$C_\pm^2 = C_V^2 \pm C_A^2 + NC_V'^2 \pm NC_A'^2, \quad \langle C_V C_A \rangle = C_V C_A + NC_V' C_A', \tag{8}$$

where $N = 2$ is the number of types of non-electron neutrinos. To calculate the total synchrotron neutrino energy losses using Eq. (3), we must add the electron and positron losses.

Note that the electron and positron densities are given by the expressions:

$$n_\mp = \frac{b}{(2\pi)^2} \sum_{n=0}^{\infty} (2 - \delta_{n,0}) \int_{-\infty}^{+\infty} dp_z f_\mp, \quad n_- - n_+ = Zn_i, \tag{9}$$

where n_i is the density of the plasma ions, and Z the atomic number of the ions.

3. PHYSICAL CONDITIONS

We start by describing the range of temperatures T, densities ρ, and magnetic field strengths B considered below. It is convenient to introduce the following dimensionless parameter x_0:

$$x_0 = \frac{\hbar(3\pi^2 Zn_i)^{1/3}}{m_e c} \approx 1{,}009(\rho_6 Z/A)^{1/3}, \tag{10}$$

where ρ_6 is the density in units of 10^6 g cm^{-3}, and A is the mass number of the plasma ions. We restrict our discussion to a nondegenerate plasma, $t \gg t_F$, where t_F is the dimensionless temperature [see Eq. (4)] of electron degeneracy, $t_F = (1 + x^2)^{1/2} - 1$, $x = p_F/m_e c$, where p_F is the Fermi momentum of a degenerate gas with given b and ρ. For estimates we use the relations (see, e.g., Ref. 7) $x \approx x_0$ for $b \lesssim x_0^2$ and $x \approx \frac{2}{3}x_0^3/b$ for $b \gg x_0^2$, which readily follow from (9).

For $t \ll 1$ the electrons and positrons in the nondegenerate plasma are nonrelativistic, and for $t \gg 1$ they are relativistic. We will call a magnetic field with $b \ll 1$ nonrelativistic (and $b \gg 1$ relativistic) since for $b \ll 1$ the separation of adjacent Landau levels is much smaller than the particle rest mass. It is convenient to distinguish between the case of a nonquantizing magnetic field, in which the particles occupy many Landau levels, and the case of a quantizing field, in which the $n = 0$ level is primarily occupied. According to (2), for $t \ll 1$ a field with $b \ll t$ is nonquantizing and a field with $b \gtrsim t$ is quantizing. But for $t \gg 1$, a field with $b \ll t^2$ is nonquantizing and a field with $b \gtrsim t^2$ is quantizing.

An analysis of (3) shows that synchrotron losses behave differently in seven temperature (T) and magnetic field (B) regions (regions I–VII in Fig. 1). The curve $0AB$ in Fig. 1 separates the regions of nonquantizing and quantizing magnetic fields. An approximate equation describing curve AC is $b \approx 1/t$ ($t \gtrsim 1$). In the region lying above AC, relativistic electrons and positrons emit $\nu\bar{\nu}$ pairs and lose a considerable fraction of their energy (see Secs. 5.2 and 5.3).

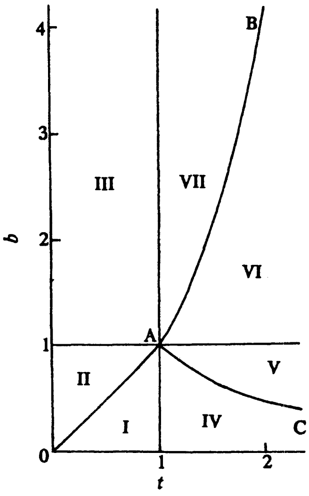

FIG. 1. Regions I–VII of the values of dimensionless temperature t and magnetic field strength b in which the neutrino synchrotron radiation from a nondegenerate plasma is of a disparate nature (for details see the text).

4. CYCLOTRON NEUTRINO RADIATION (REGIONS I AND II)

In a nonrelativistic gas with $t \ll 1$ and $b \ll 1$, the energy is of an emitted neutrino pair is approximately $\omega = sb$ (see Ref. 3), with the principal energy losses related to the first cyclotron radiation harmonic $s = 1$. As a result,[3]

$$Q = \frac{1}{120\pi^3} Q_c (C_+^2 - C_-^2)(n_- + n_+) b^6 \left[1 - \tanh\left(\frac{b}{2t}\right)\right]. \tag{11}$$

In a nonquantizing field ($b \ll t$, region I) we have $Q \propto (n_- + n_+) b^6$. In a strongly quantizing field ($b \gg t$, region II), Q is exponentially suppressed [$Q \propto \exp(-b/t)$]: the plasma particles are mainly on level $n = 0$ and are incapable of emitting neutrino pairs by the synchrotron mechanism. Only a small fraction of particles that remain on level $n = 1$ emit neutrino pairs (cf. the results of Sec. 6).

5. CYCLOTRON LOSSES IN A NONQUANTIZING FIELD (REGIONS IV, V, AND VI)

The electrons and positrons in regions IV–VI (Fig. 1) are relativistic, occupy many Landau levels, and emit a quasicontinuous spectrum of higher-order harmonics $s \gg 1$. In this case Eq. (3) can be simplified by a method used earlier in Refs. 4 and 8. Here, in contrast to those papers, we do not assume that in emitting a $\nu\bar{\nu}$ pair both e^- and e^+ lose only a small fraction of their energy.

For n and s fixed, the inequality $\omega^2 \geqslant q^2$, which limits the domain of integration over $\mathbf{q}$ in (3), can be written as[3]

$$(sb + p_z q_z + \frac{1}{2} q_\perp^2)^2 \geq \varepsilon^2 q^2, \quad q^2 = q_z^2 + q_\perp^2. \tag{12}$$

In the conditions chosen ($\varepsilon \gg 1, n \gg 1, s \gg 1$, and $p_\perp \gg 1$), only a small subdomain of (12) in which the function A in the integrand of (3) assumes its maximum value provides the principal contribution to (3). The peak value of A is primarily determined by the maximum of the functions $F^2_{n'n}(u)$. One can easily see that the domain specified by (12) corresponds to the values $u \lesssim (\sqrt{n} - \sqrt{n'})^2$, for which the $F^2_{n'n}(u)$ grow exponentially with u (see Ref. 6). Hence, the main contribution to the integral in (3) is provided by the subdomain of (12) near the point ($q_\perp \equiv \varkappa_\perp$, $q_z \equiv \varkappa_z$), where $\varkappa_\perp$ is the peak value of $q_\perp$. From (12) we find that

$$\varkappa_\perp = \sqrt{1 + p_\perp^2} - \sqrt{1 + p_\perp'^2}, \quad \varkappa_z = p_z \varkappa_\perp / \sqrt{1 + p_\perp^2}. \tag{13}$$

For $q_\perp \approx \varkappa_\perp$ and $q_z \approx \varkappa_z$ condition (12) can be replaced by the following inequality:

$$q_\perp^2 + \frac{(1 + p_\perp^2)^{3/2}}{\varepsilon^2 (1 + p_\perp'^2)^{1/2}} (q_z - \varkappa_z)^2 \leq \varkappa_\perp^2. \tag{14}$$

When integrating over the small subdomain (14), we can assume that the energies of the product particle and the emitted neutrino pair are constant: $\varepsilon = \varepsilon[(1 + p_\perp'^2)/(1 + p_\perp^2)]^{1/2}$ and $\omega = \varepsilon - \varepsilon'$. An analysis of (6) shows that it is sufficient to retain only the following terms in it:

$$2\varepsilon\varepsilon' A \approx C_+^2 \left\{ \frac{1}{2} (\omega^2 - q^2)(1 - \omega^2 + q^2)(\Phi + \Psi) \right.$$

$$\left. + [2(\omega^2 - q^2) \sqrt{1 + p_\perp^2} \sqrt{1 + p_\perp'^2} + \varkappa_\perp^2 (1 + \omega^2 - q^2)](\Psi - \Phi) \right\}$$

$$- C_-^2 \left\{ \frac{3}{2} (\omega^2 - q^2)(\Phi + \Psi) + \varkappa_\perp^2 (\Psi - \Phi) \right\}. \tag{15}$$

In the subdomain (14) it is enough to put $\omega^2 - q^2 \approx 2\varkappa_\perp \delta q_\perp - (1 + p_\perp^2)(q_z - \varkappa_z)^2 \, (\varepsilon\varepsilon')^{-1}$, where $\delta q_\perp = \varkappa_\perp - q_\perp$. Since both Φ and Ψ are independent of q_z, we can easily integrate A with respect to q_z within the limits (14):

$$\varepsilon\varepsilon' \int A \, dq_z = C_+^2 \delta q_z \left[(\Psi + \Phi) \left(\frac{2}{3} \varkappa_\perp \delta q_\perp - \frac{16}{15} \varkappa_\perp^2 \delta q_\perp^2 \right) \right.$$

$$\left. + (\Psi - \Phi) \left(\frac{8}{3} \varkappa_\perp \delta q_\perp \sqrt{1 + p_\perp^2} \sqrt{1 + p_\perp'^2} + \varkappa_\perp^2 + \frac{4}{3} \varkappa_\perp^3 \delta q_\perp \right) \right.$$

$$- C_-^2 \delta q_z [2(\Psi + \Phi) \varkappa_\perp \delta q_\perp + (\Psi - \Phi) \varkappa_\perp^2], \tag{16}$$

where $\delta q_z^2 = 2\varkappa_\perp \varepsilon\varepsilon' \delta q_\perp (1 + p_\perp^2)$. Next we assume $p_\perp \gg 1$ and $p_\perp' \gg 1$. Under these conditions we have[9]

$$\Psi + \Phi = \frac{4}{3\pi^2 p_\perp p_\perp'} (1 + \xi^2) K_{1/3}^2(z),$$

$$\Psi - \Phi = \frac{2}{3\pi^2 p_\perp^2 p_\perp'^2} (1 + \xi^2)^2 [K_{1/3}^2(z) + K_{2/3}^2(z)], \tag{17}$$

where $K_{1/3}(z)$ and $K_{2/3}(z)$ are modified Bessel functions of the second kind with argument $z = 0.5y(1 + \xi^2)^{3/2}$, with $y = 2\varkappa_\perp / 3p_\perp p_\perp' b$ and $\xi^2 = 2p_\perp p_\perp' \delta q_\perp / \varkappa_\perp$. The functions (17) decrease exponentially as $\delta q_\perp$ grows, which justifies the above procedure. We substitute (17) into (16) and integrate with respect to $q_\perp$ in accordance with (3). Going on to integration with respect to ξ and sending the upper limit of integration to infinity, we obtain

$$I \equiv \int q_\perp dq_\perp \int dq_z A = \frac{\varkappa_\perp}{3\pi^2 \varepsilon} \left(\frac{\varkappa_\perp}{p_\perp p_\perp'} \right)^4 \left\{ C_+^2 \left[\frac{4}{3} R_2 + \frac{8}{3} R_0 \right. \right.$$

$$\left. \left. + 2R_1 + \frac{8}{9} \frac{\varkappa_\perp^2}{p_\perp p_\perp'} (R_0 + R_2) \right] - C_-^2 (4R_2 + 2R_1) \right\}. \tag{18}$$

Here

$$R_0(y) = \int_0^\infty d\xi \, \xi^4 (1 + \xi^2)^2 (K_{1/3}^2(z) + K_{2/3}^2(z))$$

$$= \frac{2\pi\sqrt{3}}{9y^3} \int_y^\infty K_{1/3}(x) dx,$$

$$R_1(y) = \int_0^\infty d\xi \, \xi^2 (1 + \xi^2)^2 (K_{1/3}^2(z) + K_{2/3}^2(z)) = \frac{2\pi\sqrt{3}}{9y^2} K_{1/3}(y),$$

$$R_2(y) = \int_0^\infty d\xi \, \xi^4 (1 + \xi^2) K_{1/3}^2(z) = \frac{\pi\sqrt{3}}{12y^2} K_{1/3}(y)$$

$$+ \frac{\pi\sqrt{3}}{8y} \left(\int_y^\infty K_{1/3}(x) dx - K_{2/3}(y) \right). \tag{19}$$

In (18) we have allowed for the identity

$$\int_0^\infty d\xi\,\xi^6(1+\xi^2)K_{1/3}^2(z)=\frac{5}{12}(R_0-2R_2), \tag{20}$$

which can easily be derived from the theory of Bessel functions.[9] The integrals R_0, R_1, and R_2 are expressed in terms of linear combinations of modified Bessel functions.[4,9]

In the approximation adopted here the characteristic numbers of the Landau levels of the particles before and after neutrino emission are high ($n\gg 1$ and $n'\gg 1$). Hence in (3) we can replace the sums over n and s (i.e., n') by integrals with respect to $dp_\perp$ and $dp'_\perp$ ($dn=p_\perp dp_\perp/b$, and similarly for n'). As a result we obtain

$$Q=\frac{2Q_c}{3b(2\pi)^5}\int_{-\infty}^{+\infty}dp_z\int_0^\infty p_\perp dp_\perp\int_0^{p_\perp}p'_\perp dp'_\perp f(1-f')\omega I, \tag{21}$$

where I is given by Eq. (18). In (21) we have allowed for the fact that in a nondegenerate relativistic gas the chemical potential of electrons and positrons vanishes. For this reason the distribution functions (4) for e^- and e^+ are the same, and summing the contributions of e^- and e^+ to (21) reduces to introducing a factor 2.

Equations (18) and (21) generalize Eqs. (5), (11), and (12) of Ref. 4 to the case where the radiating particles may lose a sizable fraction of their energy and momentum (the magnetic field, however, remains nonquantizing). The difference in the formulas derived in this paper lies in the new term in (18) that contains the sum R_0+R_2. If the particles lose a small fraction of their momentum, then $\varkappa_\perp^2/p_\perp p'_\perp\ll 1$ holds and this term is negligible.

By the very meaning of our derivation, formula (21) is valid in regions IV–VI (Fig. 1). Neutrino losses in these regions do not depend on the density of matter (they are functions of only temperature and magnetic field strength). Below we consider regions IV–VI separately.

5.1. Region IV ($1\ll t\ll b^{-1}$ and $b\ll 1$)

In this region the typical values $p_\perp\sim\varepsilon\sim t$ satisfy the condition $p_\perp b\ll 1$. Relativistic electrons and positrons emit cyclotron harmonics with moderate numbers ($1\ll s\ll n$) and lose a small fraction of their energy and momentum when emitting $\nu\bar\nu$ pairs ($\omega\ll\varepsilon$, $|q_z|\ll|p_z|$, and $q_\perp\ll p_\perp$). Equations (13) then yield

$$\varkappa_\perp\approx\frac{bs}{p_\perp},\quad \omega\approx\frac{\varepsilon sb}{p_\perp^2},\quad y\approx\frac{2s}{3p_\perp^3}. \tag{22}$$

According to (19), for $y\gg 1$ the functions $R_i(y)$ decrease exponentially as y grows. The main contribution to the neutrino losses is provided by harmonics with $s\sim p_\perp^3$, which correspond to $y\sim 1$. The same is true of ordinary electromagnetic synchrotron losses (see, e.g., Ref. 9). Using Eqs. (22), we can easily show that nondegenerate relativistic electrons and positrons do indeed lose a small fraction of their energy and momentum if $1\ll t\ll b^{-1}$ holds. This is possible only for $b\ll 1$ (region IV in Fig. 1).

Here we can ignore the term in (18) that contains the sum R_0+R_1. It is convenient in Eq. (21) to go from integration with respect to $p'_\perp$ to integration with respect to y [see Eq. (22)] by using $p'_\perp dp'_\perp=b\,dn=b\,ds=1.5p_\perp^3 b\,dy$. We let the upper limit of integration go to infinity since the contribution of large values of y is exponentially small. The most convenient way to perform integration with respect to p_z and $p_\perp$ is to transfrom to spherical coordinates and introduce the classical particle momentum p and the pitch angle ϑ: $p_z=p\cos\vartheta$ and $p_\perp=p\sin\vartheta$. The result is

$$Q=\frac{3^5Q_c}{2^{11}\pi^7}b^6\int_0^\infty dp\,p^8\frac{e^{p/t}}{(1+e^{p/t})^2}\int_0^\pi d\vartheta\sin^7\vartheta$$

$$\times\int_0^\infty dy\,y^6\left[C_+^2\left(\frac{8}{3}R_0+2R_1+\frac{4}{3}R_2\right)-C_A^2(4R_2+2R_1)\right]. \tag{23}$$

All the integrals can be evaluated, which yields

$$Q=\frac{5\cdot 2^9}{3\pi^5}Q_c b^6 t^9\zeta(8)\left(1-\frac{1}{2^7}\right)(25C_+^2-21C_-^2), \tag{24}$$

where $\zeta(x)$ is the Riemann zeta function.

5.2. Region V ($t^{-1}\ll b\ll 1$)

Here the characteristic values $p_\perp\sim\varepsilon\sim t$ satisfy the condition $p_\perp b\gg 1$, the main contribution to the neutrino losses is provided by harmonics with $s\sim\varepsilon^2/b$, and electrons and positrons may lose a sizable fraction of their energy and momentum when emitting $\nu\bar\nu$ pairs. We partition the domain of integration with respect to $p'_\perp$ [in (Eq. 21)] into two domains, one from 0 to p_* and the other from p_* to $p_\perp$, with p_* subject to the condition $b^{-1}\ll p_*\ll t$.

For $p'_\perp\gg p_*$ the characteristic values of the arguments of the $R_i(y)$ are small, $y\ll 1$. The main contribution to (18) is provided by the function $R_0\approx 2\pi^2/9y^3$, which grows faster than other functions as $y\to 0$. This yields

$$I\approx\frac{2}{3}C_+^2\frac{b^3\varkappa_\perp^2}{\varepsilon p_\perp p'_\perp}\left(1+\frac{\varkappa_\perp^2}{3p_\perp p'_\perp}\right), \tag{25}$$

$$\int_{p_*}^{p_\perp}p'_\perp dp'_\perp\omega(1-f')I$$

$$=\frac{2}{3}C_+^2b^3p_\perp^2\int_{x_*}^1\frac{dx(1-x)^3}{1+\exp(-\varepsilon x/t)}\left[1+\frac{(1-x)^2}{3x}\right]$$

$$\approx\frac{2}{3}C_+^2b^3p_\perp^2\left\{\int_0^1 dx\left[\frac{(1-x)^3}{1+\exp(-\varepsilon x/t)}\left(1+\frac{(1-x)^2}{3x}\right)\right.\right.$$

$$\left.\left.-\frac{1}{6x}\right]-\frac{1}{6}\ln x_*\right\}, \tag{26}$$

where $x_*=p_*/p_\perp\ll 1$. The logarithmic term appears because x_* is small.

For $p'_\perp < p_*$ the main contribution to (18) is provided by the term containing the sum $R_0 + R_1$. When $p'_\perp$ is small, the energy of the final electron or positron is low, $\varepsilon' \ll t$, that is, $f' \approx 0.5$. As a result,

$$\int_0^{p_*} p'_\perp dp'_\perp \omega(1-f')I = \frac{p_\perp^2}{2\pi^2} b^3 C_+^2 \int_{y_*}^{\infty} dy\, y^2 (R_0 + R_2)$$

$$\approx \frac{2}{3} C_+^2 b^3 p_\perp^2 \left[\frac{3}{4\pi^2} \int_0^\infty dy\, y^2 R_2 + \frac{1}{2\pi\sqrt{3}} \int_0^\infty dy \ln(y) K_{1/3}(y) - \frac{1}{6} \ln y_* \right], \qquad (27)$$

where $y_* = 2/(3p_\perp b x_*) \ll 1$. The integral containing $R_2(y)$ converges as $y_* \to 0$, which allows us to replace the lower limit of integration by zero. The integral containing $R_0(y)$ possesses a logarithmic singularity when $y_* \to 0$. This is written out explicitly.

After we add (26) to (27), the dependence on the artificially introduced parameter p'_* vanishes, which justifies the above procedure. We substitute this sum into (21). The integration in (26) can be done numerically. The other integrals in (27) and (21) can be evaluated analytically. The result is

$$Q = \frac{20\zeta(5)}{9(2\pi)^5} C_+^2 t^5 b^2 Q_c [\ln(tb) - 0{,}346]. \qquad (28)$$

In the case considered here $tb \gg 1$, that is, the leading term in the square brackets is the large logarithm. Note that the given mode of synchrotron losses is similar to the one occurring in a degenerate relativistic electron gas with a nonquantizing magnetic field for $t \ll bx_0^2$, where x_0 is given by (10). In the latter case $Q = (2/9)\zeta(5)\pi^{-5} C_+^2 t^5 b^2 Q_c$ (see Ref. 4).

5.3. Region VI ($1 \ll b \ll t^2$)

The calculation of synchrotron losses in region VI is similar to the one done for region V. We again partition the domain of integration with respect to $p'_\perp$ in (21) into two domains (one from 0 to p_* and the other from p_* to $p_\perp$). We require p_* to satisfy the condition $\sqrt{b} \ll p_* \ll t$. The integral over the second domain leads to (26), as it did in Sec. 5.2. The difference lies in the first domain. While in Sec. 5.2 the main contribution in this domain was provided by values of $p'_\perp$ much larger than unity, corresponding to $n' \gg 1$, now the values $n' \sim 1$ provide a sizable contribution. Hence, from integrating with respect to $p'_\perp$ from 0 to p_* we go back to summation over n' from $n' = 0$ to $n_* = p_*/\sqrt{2b} \gg 1$.

Equations (17), which were used in deriving (18), become invalid for small values of n'. Let us use the exact formulas for $F_{n'n}(u)$. These can be made simpler by accounting for the fact that we are interested only in values $n \gg 1$ and in the domain of arguments $u \lesssim n$, which provides the largest contribution to the integrals with respect to $q_\perp$ in (18). As a result we find that it is sufficient to use the following formulas in (15):

$$\Psi + \Phi = 2\vartheta, \quad \Psi - \Phi = \frac{v}{n}\vartheta \quad (n = 0),$$

$$\Psi + \Phi = \frac{2}{n}(v^2 + n)\vartheta, \quad \Psi - \Phi = \frac{v}{n^2}(v^2 - 3n)\vartheta \quad (n = 1),$$

$$\Psi + \Phi = \frac{1}{n^2}(v^4 + n^2)\vartheta,$$

$$\Psi - \Phi = \frac{v}{2n^3}[(v^2 - n)^2 - 6nv^2 + 8n^2]\vartheta \quad (n = 2),$$

$$\Psi + \Phi = \frac{1}{3n^3}(v^6 - 3v^4 n + 3v^2 n^2 + 3n^3)\vartheta,$$

$$\Psi - \Phi = \frac{v}{6n^4}(v^6 - 15nv^4 + 51n^2v^2 - 33n^3)\vartheta \quad (n = 3),$$

$$\vartheta = (2\pi n)^{-1/2} \exp\left(-\frac{v^2}{2n}\right), \quad v = n - u. \qquad (29)$$

Clearly, Eq. (16) remains valid up to values $n' \gg 1$. Substituting (29) into (16) and integrating with respect to $q_\perp$, we obtain

$$I = \frac{2}{9} C_+^2 \frac{b^3 n}{\varepsilon n'} a_{n'}, \qquad (30)$$

where $a_{n'} = 0.697$, 0.790, and 0.833 for $n' = 1$, 2, and 3, respectively. These values can be approximated by the formula $a_{n'}/n' \approx (n')^{-1} - 0.3(n')^{-1.5}$. Assuming that the same formula is valid for $n' > 3$ yields

$$\sum_{n'=1}^{n_*} I = \frac{2}{9} C_+^2 b^3 n \varepsilon^{-1} (0{,}577 + \ln(n_*) - 0{,}3\zeta(1{,}5)). \qquad (31)$$

The case $n' = 0$ requires special treatment since the domain of integration with respect to $q_\perp$ and q_z is described then by Eq. (14) only for $|p_z - q_z| \ll 1$. For $|p_z - q_z| \gg 1$ but in the neighborhood of the point ($q_\perp = \varkappa_\perp$, $q_z = \varkappa_z$), the domain of integration is found from Eq. (12). For $q_z > 0$ (for the sake of definiteness) this equation assumes the form $(\varkappa_\perp - q_\perp) = (\varepsilon \pm p_z)(q_z - \varkappa_z)/\varkappa_\perp$, where $\varkappa_\perp = p_\perp$, $\varkappa_z = p_z$, the "plus" corresponds to $q_z > \varkappa_z$, and the "minus" to $q_z < \varkappa_z$. Employing (15) and (29), we get $I = C_+^2 b^3 n \varepsilon^{-1}$. We add this to Eq. (31), multiply the product by b (to adjust summation over n' and integration with respect to $p'_\perp$), and add the result to (26). As in Sec. 5.2, all dependence on p_* (or on n_*) vanishes. The remaining transformations are similar to those performed in Sec. 5.2. The result is

$$Q = \frac{20\zeta(5)}{9(2\pi)^5} C_+^2 t^5 b^2 Q_c \left[\ln\left(\frac{t}{\sqrt{b}}\right) + 2{,}33\right]. \qquad (32)$$

This is similar to (28). The difference lies in the arguments of the logarithm and the coefficients of the logarithms. In region VI the quantity $t/\sqrt{b}$ under the logarithm sign is large, that is, the leading term in the square brackets is still the large logarithm. For this reason the synchrotron loss modes in regions V and VI are similar.

6. CYCLOTRON LOSSES IN A RELATIVISTIC QUANTIZING MAGNETIC FIELD (REGIONS III and VII)

In regions III ($t \ll 1 \ll b$) and VII ($1 \ll t \ll \sqrt{b}$), as in region II, the majority of the plasma electrons and positrons are on the ground Landau level and are incapable of emitting synchrotron $\nu\bar{\nu}$ pairs. The main contribution to the radiation is provided by a small fraction of particles on level $n = 1$. When summing in (3), it is sufficient to leave only one term, with $n = 1$, $n' = 0$, and $s = 1$. The processes of emission of a $\nu\bar{\nu}$ pair by particles in regions III and VII are similar. Prior to emitting, a particle is highly nonthermal ($\varepsilon \gg t$). In both regions we have $b \gg 1$. Hence in calculating A in (6) we can put $\varepsilon \approx \sqrt{2b}$, with $p_z \sim t^{1/2} b^{1/4} \ll \varepsilon$. After emitting, the particle retains its high energy but finds itself on level $n' = 0$. It is sufficient to assume that $\varepsilon' \approx |p_z| \gg t$ and $\varepsilon' \gg 1$, with the result that $1 - f' \approx 1$ in (3). Equation (6) yields

$$A = 2C_+^2 e^{-u}(b - bu - \sqrt{2b}|q_z|). \tag{33}$$

Here $\omega = \sqrt{2b} - |q_z|$, and the domain (12) of admissible values of $q_\perp$ (or u) and q_z can be expressed as follows: $u < 1$, $2q_z^2 \leqslant b(1-u)^2$. Integration over this domain yields

$$Q = \frac{4}{9} Q_c C_+^2 \frac{b^3}{(2\pi)^3}\left(1 - \frac{9}{4e}\right)(n_+^{(1)} + n_-^{(1)}), \tag{34}$$

where $e = \exp(1)$, and $n_\pm^{(1)}$ are the concentrations of particles (e^+ and e^-) on level $n = 1$ [see Eq. (9)]. The above formula is valid in both region III and region VII.

Calculating $n_\pm^{(1)}$ in region III, we get

$$Q = \frac{8}{9} Q_c C_+^2 \frac{b^3}{(2\pi)^3}\left(1 - \frac{9}{4e}\right)(2b)^{1/4}\exp\left(-\frac{\sqrt{2b}}{t}\right)(n_- + n_+), \tag{35}$$

where $n_\pm$ are the total particle densities.

A similar calculation for region VII yields

$$Q = \frac{16}{9} Q_c C_+^2 \frac{b^4}{(2\pi)^5}\left(1 - \frac{9}{4e}\right)(2\pi t\sqrt{2b})^{1/2}\exp\left(-\frac{\sqrt{2b}}{t}\right). \tag{36}$$

The main feature of Eqs. (34)–(36) is the presence of the exponentially small factor $\exp(-\sqrt{2b}/t)$, which reflects the fact that only a small fraction of particles are capable of emitting neutrino pairs by the synchrotron mechanism.

7. AN INTERPOLATION FORMULA

After deriving the asymptotic formulas (11), (24), (28), (32), (35), and (36) describing the neutrino synchrotron losses in regions I–VII, it is well to derive a general interpolation formula. Allowing for the special features of the synchrotron mechanism of the emission of $\nu\bar{\nu}$ pairs (Secs. 3–6), we can write the interpolation formula as

$$Q = \frac{Q_c}{120\pi^3}(n_- + n_+)j(C_+^2 F_+ - C_-^2 F_-), \tag{37}$$

where $n_- + n_+$ is the total concentration of e^- and e^+, j is the fraction of electrons on Landau levels with $n > 0$, and $F_\pm$ are functions of temperature and magnetic field strength (t and b). To use (37), we must have computational formulas for $n_- + n_+$, j, and $F_\pm$.

The total concentration of e^- and e^+ is given by the following formulas:

$$n_- + n_+ = Zn_i\sqrt{1+u}, \quad u = 4n_-n_+/(Zn_i)^2, \tag{38}$$

which follow from (9). For the product n_-n_+ we can use the interpolation formula

$$n_-n_+ = \frac{b^2 t(1+\alpha t)}{(2\pi)^3}\exp\left(-\frac{2}{t}\right)\coth^2\left[\frac{b}{2t(1+\beta t)}\right], \tag{39}$$

where $\alpha = (2/\pi)(\ln 2)^2 = 0.3059$, and $\beta = 1.5\zeta(3)/\ln 2 = 2.601$. For $t \ll 1$ formula (39) describes the nonrelativistic asymptotic behavior, which can be obtained from (9) and is valid for all values of b. The above choice of the parameter α enables (39) to be transformed into its asymptotic form for $1 \ll t \ll \sqrt{b}$. Finally, the choice of the parameter β makes it possible to reproduce the proper asymptotic behavior when $t \gg 1$ in a nonquantizing field. Using (39), we arrive at the following expression for the dimensionless parameter u in (38):

$$u = 13,41\left(\frac{A}{Z\rho_6}\right)^2 b^2 t(1+\alpha t)\exp\left(-\frac{2}{t}\right)\coth^2\left[\frac{b}{2t(1+\beta t)}\right], \tag{40}$$

where A is the mass number of the plasma ions, and ρ_6 is the plasma density in units of 10^6 g cm^{-3}.

For the fraction j of electrons occupying the excited Landau levels we can employ the following interpolation formula:

$$j = \left[1 - \tanh\left(\frac{\sqrt{1+2b}-1}{2t}\right)\right]\left(1 + \frac{\sqrt{1+2b}-1}{\alpha t+1}\right)^{1/2}. \tag{41}$$

Here α is the same parameter as in (39). Formula (41) reproduces the nonrelativistic asymptotic behavior for $t \ll 1$ and $b \ll 1$ and the asymptotic behavior for $1 \ll t \ll \sqrt{b}$ and yields the correct value $j = 1$ in the limit $t \gg 1$ for nonquantizing fields.

Finally, employing Eqs. (37)–(41), we arrive at the following expressions for the functions F_+ and F_-:

$$F_+ = \frac{b^6}{(1+\gamma b)^3}\left(1 + \frac{t\delta}{\sqrt{1+\gamma b}}\right)^6\left[1 + \frac{\lambda t b}{(1+\gamma b)^{1,5}}\right]^{-4} \times\left\{1 + \ln\left[1 + \frac{\chi t b}{(1+\eta b)^{1,5}}\right]\right\},$$

$$F_- = \frac{b^6}{(1+\gamma b)^6}(1+t\delta_-)^6\left[1 + \frac{\lambda t b}{(1+\gamma b)^{1,5}}\right]^{-5}, \tag{42}$$

with the constants

$$\gamma = \left[\frac{20}{3}\left(1 - \frac{9}{4e}\right)\right]^{-1/3} = 0,955;$$

$$\delta = \left[\frac{5^4 2^{12}\zeta(8)}{3\zeta(3)}\left(1 - \frac{1}{2^7}\right)\right]^{1/6} = 9,439,$$

$$\delta_- = \left(\frac{21}{25}\right)^{1/6}; \quad \delta = 9,169; \quad \chi = 0,260;$$

$$\eta = 0,168; \quad \lambda = 23,31 \tag{43}$$

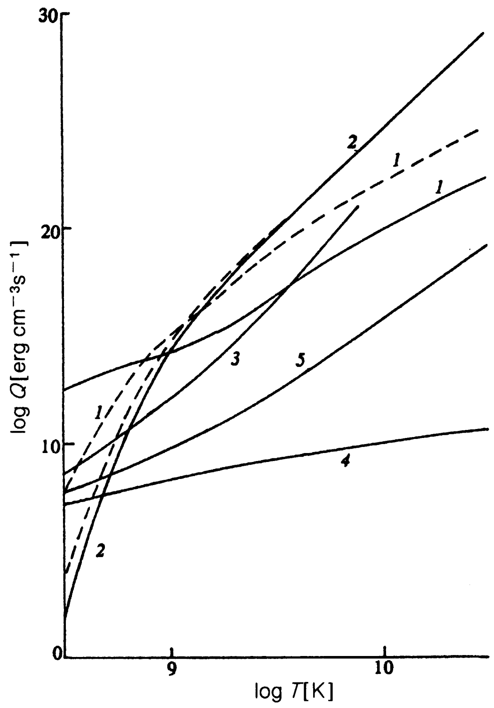

FIG. 2. Neutrino energy loss rates versus the plasma temperature at a density $\rho = 2\times10^5$ g cm^{-3}. Curves *1* and *2* correspond to synchrotron and annihilation losses for $B=10^{13}$ G (solid curves) and $B=10^{14}$ G (dashed curves); curves *3*, *4*, and *5* correspond to photon decay,[2] plasmon decay,[2] and electron bremsstrahlung on nuclei ($Z=6$ and $A=12$; Ref. 11) at $B=0$. Curve *3* has been plotted using the data of Table I in Ref. 2, where the values of Q are listed only for $T\lesssim10^{9.9}$ K. For higher temperatures the authors of Ref. 2 suggest using the interpolation formulas (5)–(16) of Ref. 2. The interpolation curves, however, depart strongly from the tabulated data and are not shown here.

chosen in such a way so as to satisfy the asymptotic forms of Eqs. (11), (24), (28), (32), (35), and (36).

Finally, substituting (38) into (37), we get

$$Q = 9{,}53\cdot10^{17}\,\frac{Z\rho_6}{A}\,\sqrt{1+u}\,j(C_+^2F_+ - C_-^2F_-)\ \mathrm{erg\,cm^{-3}\,s^{-1}}, \tag{44}$$

where $C^2_+ = 1.675$, $C^2_- = 0.175$ [see Eq. (8), $\sin^2\theta_W = 0.23$, the number N of types of non-electron neutrino is two], and the quantities u, j, and $F_\pm$ are specified in (40), (41), and (42).

8. DISCUSSION AND CONCLUSIONS

Formula (44) provides an easy way to calculate the synchrotron neutrino loss rate in a nondegenerate plasma with an arbitrary magnetic field. To illustrate this fact, Fig. 2 depicts the temperature dependence of the loss rate for $\rho Z/A = 10^5$ g cm^{-3} in magnetic fields $B=10^{13}$ and 10^{14} G. Synchrotron losses are compared with neutrino losses caused by other mechanisms of neutrino production, such as the annihilation of e^-e^+ pairs ($e^- + e^+ \to \nu + \bar\nu$), plasmon decay ($\hbar\omega_p \to \nu + \bar\nu$), photon decay in a plasma ($\gamma + e^- \to e^- + \nu + \bar\nu$), and electron bremsstrahlung on nuclei ($e^- + Z \to e^- + Z + \nu + \bar\nu$). Annihilation losses in a magnetic field have been calculated in Ref. 10. Up to now the effect of a magnetic field on other processes has not been discussed. The respective curves in Fig. 2 are depicted for the case of a zero field for illustration (with the data taken from Refs. 2 and 11).

According to Fig. 2, synchrotron radiation is the major cause of neutrino losses in a plasma that is not too hot ($T \lesssim 10^9$ K) and is placed in a strong magnetic field. Such conditions are realized in neutron star crusts. The above discussion shows that neutrino synchrotron radiation is an important cause of the cooling of nondegenerate plasma in neutron star crusts. According to Ref. 4, neutrino synchrotron radiation may also be one of the main reasons for cooling in the deeper, degenerate, layers of neutron stars. Our results augment the data of Ref. 4 and make it possible to calculate neutrino synchrotron losses in all the sections of neutron star crusts (degenerate and nondegenerate, relativistic and nonrelativistic) in magnetic fields of arbitrary strength. This is especially important in studying the cooling of young neutron stars (the first 10–100 years), where neutrino losses in the crust have a strong effect on the lowering of the temperature of the star's surface (see, e.g., Ref. 12).

A more complete analysis of the effect of a magnetic field on the cooling of a neutron star must include the effect of the field on the neutrino losses caused by all the mechanisms of neutrino production.

In conclusion we note that a simple scaling criterion for neutrino and electromagnetic synchrotron losses has been derived in Ref. 4. It makes it easy to estimate neutrino synchrotron losses if electromagnetic synchrotron losses are known. In Ref. 4 the criterion has been applied to the case of a degenerate electron gas. Using the results of Secs. 4–6 and the well-known formulas for electromagnetic synchrotron losses, one can easily verify that the scaling criterion also makes it possible to estimate the neutrino synchrotron losses in all seven regions of variation of the magnetic field strength and temperature in a nondegenerate plasma. What this approach does not achieve is an estimate of the large logarithms in Eqs. (28) and (32). Note that electromagnetic synchrotron losses of plasma energy are much larger than neutrino synchrotron losses. However, electromagnetic radiation cannot penetrate an optically thick medium and is not as important a cause of cooling of hot, dense matter as neutrino losses.

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Translated by Eugene Yankovsky