Amplitude-phase multistabilty of multiatomic optical systems

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Dicke's model is used to discuss amplitude-phase multistability in a coherently driven cavity for the case of high Rabi frequencies and spontaneous decay rate that is low compared with the cavity mode decay rate. Quantum and quasiclassical interpretations, based on the Tavis–Cummings model, are presented and by regarding the system as a collection of damped harmonic oscillators is considered. An analytical solution of the problem is obtained for $N = 2$. A possible experimental arrangement for observing amplitude-phase multistability is examined.

1. INTRODUCTION

Collective effects in quantum optics, first studied by Dicke in his pioneering work,[1] continue to be a central problem. The initial formulation of the problem[1] of the decay of collectively radiating atoms contained in a volume small compared with the cube of the wavelength (Dicke's model) has been extended to collective decay (superradiation, superfluorescence) in extended systems (see, for example, Ref. 2).

Resonance fluorescence excited by strong optical radiation was studied in Dicke's model in Refs. 3–7. In particular, it was shown that for a certain value of the external field intensity a nonequilibrium phase transition, analogous to a second-order thermodynamic phase transition, can occur.[3,4] When the interatomic interaction between the particles of the system is taken into account, this model can exhibit a first-order nonequilibrium phase transition, associated with the fact that this system is equivalent to a quantum anharmonic oscillator in an external field.[7]

On the other hand, as shown in Ref. 8, a system of *independently* radiating atoms, placed in an optical cavity, manifests optical bistability of the absorptive type, for which the curve of the intracavity field versus the external pumping is analogous to a van-der-Waals isotherm describing a first-order liquid-vapor phase transition. It should be noted especially that optical bistability does not arise as a result of cooperative effects produced when the atoms interact with the cavity mode of the field, as was assumed in Ref. 8. This phenomenon is determined by single-atom nonlinear effects arising because the state of the atom depends on the intracavity field, which consists of the external field and the field reradiated by the atom and depends on the state of the atom. Thus it has been shown recently[9–11] that when a single atom is placed in a cavity and interacts strongly with the cavity mode, such a system can manifest bistable properties. In addition, besides the well-known absorption bistability (or bistability of the real part of the field amplitude),[9] a new effect appears: phase optical bistability.[10,11] As indicated in Ref. 10, phase bistability has a simple interpretation based on the representation of an atom interacting strongly with the intracavity field as a single quantum system in an external field.

In the present paper we investigate new effects produced by the combination of the collective character of the emission of a system of atoms and the nonlinearity of the atoms interacting strongly with the intracavity field. For this, we extend the single-atom model of quantum optics[10,11] to the case of a large number of two-level atoms in a volume which is small compared with the resonance wavelength.

This collection of atoms is located in a coherently driven high-Q optical cavity. The intracavity mode excited by the external optical field interacts strongly with the collection of atoms and decays because the mirrors are partially transmitting. The atomic states decay as a result of spontaneous emission into noncavity modes, which in this model are treated as free-space modes.

It is demonstrated below that for this model amplitude-phase multistability arises when the components of the resonance-fluorescence spectrum are widely separated (the Rabi frequencies are high)[5] and the rate of spontaneous transitions of the atoms is low compared with the rate of decay of the cavity mode. The number of stable points in the phase space of the model is analyzed as a function of the ratio of the numbers of the particles, and the spontaneous transition rates of the atoms and the boundary conditions for the appearance of multistability in a multiparticle system are determined.

2. THE MODEL

The system of two-level atoms with resonance frequency ω in a volume with linear dimensions much shorter than the wavelength is described with the help of the Dicke collective operators $J_+ = \Sigma_i \sigma_{i+}$, $J_- = \Sigma_i \sigma_{i-}$, $J_z = \Sigma_i \sigma_{iz}$ ($\sigma_{i\pm}$ and σ_{iz} are single-atom operators), which satisfy the commutation relations $[J_+, J_-] = 2J_z$ and $[J_z, J_\pm] = \pm J_\pm$. The action of the operators $J_\pm$ and J_z on the collective states $|j,m\rangle$, where j is the maximum value of the "energy spin" of the system ($j = N/2$, where N is the number of atoms, and m is the projection of the "energy spin" on the z axis, $-j \leqslant m \leqslant j$), is determined by the well-known relations:

$$J_+|j, m\rangle = \sqrt{(j + m + 1)(j - m)}\,|j, m + 1\rangle, \tag{1a}$$

$$J_-|j, m\rangle = \sqrt{(j + m)(j - m + 1)}\,|j, m - 1\rangle. \tag{1b}$$

The collective states $|j,m\rangle$ are eigenstates of the operator J_z and the total "energy spin" operator J^2:

$$J_z|j, m\rangle = m|j, m\rangle, \tag{2a}$$

$$J^2|j, m\rangle = j(j + 1)|j, m\rangle. \tag{2b}$$

The single-mode intracavity field is represented by a har-

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monic oscillator with frequency ω_0 and boson creation and annihilation operators a^+ and a, respectively. The excitation of a cavity mode by coherent pumping with amplitude E is described by the Hamiltonian $i\hbar E(a^+ - a)$. The interaction of the system of atoms and the field is represented by the collective Jaynes–Cummings Hamiltonian in the rotating-wave approximation, $\hbar g(a^+ J_- + aJ_+)$. Taking into account the decay of the field and the atoms, the corresponding quantum-mechanical description of this model in terms of the reduced density matrix in the interaction representation has the form

$$\dot{\rho} = E[a^+ - a, \rho] - i\Delta\omega[J_z, \rho] - ig[a^+ J_- + aJ_+, \rho] + k[a, \rho a^+] + [a\rho, a^+]) + (\gamma/2)([J_-, \rho J_+] + [J_-\rho, J_+]), \tag{3}$$

where $\Delta\omega = \omega - \omega_0$ is the detuning between the resonance transition of a two-level atom and the field, g is the interaction constant between the field and a separate atom, γ is the rate of spontaneous emission into field modes different from the cavity mode, $k = \pi/F\tau_c$ is the decay rate of the field in the cavity, $\tau_c = 2L/c$ is the round-trip passage time of the radiation in the cavity, L is the cavity length, and F is the sharpness of the cavity.

It is easy to show that for the model (3) the squared total "energy spin" is conserved:

$$\langle J^2 \rangle = \langle J_\pm J_\mp \rangle + \langle J_z^2 \rangle + \langle J_z \rangle = j(j+1). \tag{4}$$

We note that in the absence of atoms ($g = \gamma = 0$) in the cavity, the steady state of the intracavity field will be a coherent state with amplitude E/k. It is convenient to separate this state explicitly from the density matrix ρ by operating with the amplitude displacement operator $D(E/k) = \exp(E(a^+ - a)/k)$:

$$\bar{\rho} = D^{-1}(E/k)\rho D(E/k). \tag{5}$$

The resulting equation for the transformed matrix $\bar{\rho}$ assumes the form

$$\dot{\bar{\rho}} = -2i\vartheta[J_x, \bar{\rho}] - i\Delta\omega[J_z, \bar{\rho}] - ig[a^+ J_- + aJ_+, \bar{\rho}] + k([a, \bar{\rho}a^+] + [a\bar{\rho}, a^+]) + (\gamma/2)([J_-, \bar{\rho}J_+] + [J_-\bar{\rho}, J_+]), \tag{6}$$

where the first term describes the change in $\bar{\rho}$ under the action of the classical field E/k ($\vartheta = gE/k$ is the Rabi frequency), and the term $E[a^+ - a, \rho]$, corresponding to excitation of the intracavity field [see Eq. (3)], is absent.

We note that operators describing the system of atoms and a single two-level system realize different irreducible representations of the same group $SU(2)$ and hence satisfy the same commutation relations. Therefore, the same canonical transformations as in Ref. 8 can be applied to the multiparticle model, and this yields transformations of the operators and equations for the density matrix ρ which are analogous to the single-particle model to within a substitution of the Pauli spin operators $\sigma_z, \sigma_\pm$ for the Dicke collective operators $J_z, J_\pm$.

3. HIGH RABI FREQUENCIES

The equation (6) derived above can be solved approximately in the case when the generalized Rabi frequency $\Omega = \sqrt{(2\vartheta)^2 + (\Delta\omega)^2}$ is sufficiently high. In order to obtain this solution we perform a rotation in the space of the vectors (J_x, J_y, J_z)

$$J_i' = \exp(-i\varphi J_y) J_i \exp(i\varphi J_y), \qquad i = x, y, z, \tag{7}$$

around the y axis through an angle φ such that $s = \sin\varphi = 2\vartheta/\Omega$ and $c = \cos\varphi = \Delta\omega/\Omega$. Then the eigenstates $|j,m'\rangle$ of the operator J_z' can be expressed as a linear superposition of the old basis states $|j,m\rangle$:

$$|j, m'\rangle = \sum_{m=-j}^{j} d^j_{mm'}(\varphi)|m\rangle, \tag{8}$$

where $d^j_{mm'}(\varphi) = \langle m|\exp(-i\varphi J_y)|m'\rangle$ is the Wigner d-function.[12] Assuming Ω^{-1} is the shortest characteristic time of the problem—shorter than the collective decay time $(N\gamma)^{-1}$ of the atoms, the decay time k^{-1} of the field in the cavity, and the period $(N/2)(g^2/k))^{-1}$ of the Rabi oscillations induced by the reradiated field,

$$\Omega^{-1} \ll (N\gamma)^{-1}, k^{-1}, \qquad \left(\frac{N}{2}\frac{g^2}{k}\right)^{-1}, \tag{9}$$

—we can transform to the "dressed state" representation for the density matrix ρ':

$$\rho = \exp(-i\Omega J_z' t)\rho' \exp(i\Omega J_z' t) \tag{10}$$

with the time dependence of the operators $J_i(t)$ given by

$$J_+(t) = sJ_z' + \frac{1+c}{2}J_+' e^{i\Omega t} - \frac{1-c}{2}J_-' e^{-i\Omega t}, \tag{11a}$$

$$J_z(t) = cJ_z' - \frac{s}{2}J_+' e^{i\Omega t} - \frac{s}{2}J_-' e^{-i\Omega t} \tag{11b}$$

and average the corresponding equation for the matrix ρ' over the fast Rabi oscillations. Like Eq. (3) above, the equation obtained as a result of such averaging,

$$\dot{\rho}' = -igs[(a^+ + a)J_z', \rho'] + k([a, \rho' a^+] + [a\rho', a^+]) + \frac{\gamma s^2}{2}([J_z', \rho' J_z'] + [J_z'\rho', J_z']) + \frac{d_{12}}{2}([J_+', \rho' J_-'] + [J_+'\rho', J_-']) + \frac{d_{21}}{2}([J_-', \rho' J_+'] + [J_-'\rho', J_+']), \tag{12}$$

where $d_{12} = \gamma(1-c)^2/4$, and $d_{21} = \gamma(1+c)^2/4$, satisfies the particle number conservation law (4).

According to the equations for the diagonal elements of the transformed and averaged density matrix ρ' (10)

$$\begin{cases} \dot{\rho}'_{-j'-j'} = \frac{igN}{2}[a^+ + a, \rho'_{-j'-j'}] + k([a, \rho'_{-j'-j'}a^+] + [a\rho'_{-j'-j'}, a^+]) + d_{21}N\rho'_{-j'+1,-j'+1} - d_{12}N\rho'_{-j'-j'}, \\ \rho'_{m'm'} = -igm'[a^+ + a, \rho'_{m'm'}] + k([a, \rho'_{m'm'}a^+] + [a\rho_{m'm'}a^+]) + d_{12}(j+m')(j-m'+1)\rho'_{m'-1,m'-1} + d_{21}(j+m'+1)(j-m')\rho'_{m'+1,m'+1} - (d_{12}(j+m'+1)(j-m') + d_{21}(j+m')(j-m'+1))\rho'_{m'm'}, \\ \dot{\rho}'_{j'j'} = -\frac{igN}{2}[a^+ + a, \rho_{j'j'}] + k([a, \rho_{j'j'}, a^+] + [a\rho_{j'j'}, a^+]) + d_{12}N\rho'_{j'-1,j'-1} - d_{21}N\rho_{j'j'}, \end{cases} \tag{13}$$

the atoms in different eigenstates $|j,m'\rangle$ (8) interact with the intracavity field independently if the spontaneous transition probabilities γ into noncavity modes are small (d_{ij} are small). In addition, the equations for the diagonal elements $\rho'_{m'm'}$ are equivalent to the equations for damped harmonic oscillators excited by coherent radiation with amplitude ($\sim gm'$), varying in intensity and phase for different states $|j,m'\rangle$. Therefore, for $d_{ij}=0$, the cavity field will have $2j+1$ possible stationary values of the amplitude

$$\alpha_{0m'} = E/k + igm'/k \qquad (m' = -j, \ldots, j). \tag{14}$$

The representation of the "field + atoms" quantum system as a collection of weakly coupled harmonic oscillators can also be interpreted conveniently using the Tavis–Cummings model.[13] For this, we now consider the excitation of eigenstates of such a model by external resonance radiation.

4. INTERPRETATION ON THE BASIS OF THE TAVIS–CUMMINGS MODEL

The eigenstates of the Tavis–Cummings Hamiltonian (see Appendix)

$$H = \hbar\omega a^+a + \hbar\omega J_z + \hbar g(a^+J_- + aJ_+)$$

for the case of exact resonance can be represented in the form (A2)

$$|2j, n, r\rangle = \sum_{m=-j}^{k} \alpha_m^n |n - j - m\rangle_f |m\rangle_a, \tag{15}$$

where the indices f and a denote the eigenstates of the intracavity field and the system of atoms. The number n is the number of the excitations in the "field + atoms" system and the number r enumerates the eigenstates belonging to a fixed number of excitations n. For $n<2j$ the number $k=n-j$ and r varies from $-n/2$ to $n/2$ in steps of 1. For $n\geqslant 2j$ the number $k=j$ and r ranges from $-j/2$ to $j/2$. The eigenvalues of the Hamiltonian H

$$\mathcal{E}_{2j,n,r} = \hbar\omega(n-j) + E_{2j,n,r}, \tag{16}$$

where $E_{2j,n,r}$ are the eigenvalues of the interaction Hamiltonian $\hbar g(a^+J_- + aJ_+)$, determine the energy spectrum of the system. The spectrum consists of the levels of the noninteracting atoms and the field, each level being split by the interaction into $n+1$ sublevels for $n<2j$ and $2j+1$ sublevels for $n\geqslant 2j$ (Fig. 1).

The external field, exciting the "atoms + intracavity field" system (the interaction Hamiltonian is $-iE(a-a^+)$), induces transitions between the eigenstates $|2j,n,r\rangle$. The different paths of excitation of the numerically computed energy levels of the Tavis–Cummings system are displayed in Fig. 1a ($N=2$) and Fig. 1b ($N=3$). The first two paths start from the ground state and excite the states $|2j,1,1/2\rangle$ and $|2j,1,-1/2\rangle$ with the same probability (since the frequency mismatches $|\hbar\omega - \mathcal{E}_{2j,1,1/2}|$ and $|\hbar\omega - \mathcal{E}_{2j,1,-1/2}|$ are equal):

$$|2j,0,0\rangle \begin{cases} \nearrow |2j,1,1/2\rangle \\ \searrow |2j,1,-1/2\rangle \end{cases}.$$

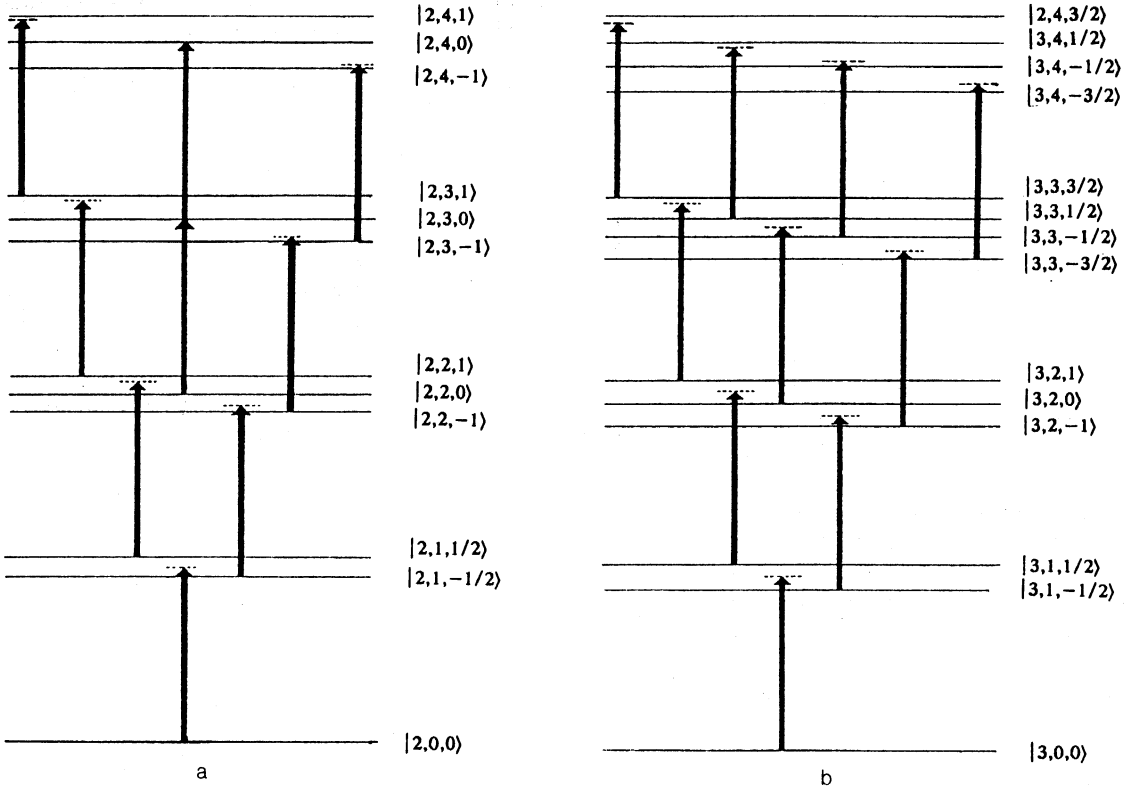

FIG. 1. Formation of independent excitation paths in the system of energy levels of the Tavis–Cummings model[13] under the action of external classical resonance radiation. The numerical calculation was performed using the formulas from the appendix for even $N=2$ (a) and odd $N=3$ (b).

The excitation probability of the next states now depends on the state from which the system starts. The most likely transitions (due to the small frequency detuning) are

$$|2j, 1, 1/2\rangle \to |2j, 2, 1\rangle,$$
$$|2j, 1, -1/2\rangle \to |2j, 2, -1\rangle.$$

As n increases the situation repeats. For $n \geqslant 2j$ transitions without a change in r are the most likely of these two paths:

$$|2j, n, j\rangle \to |2j, n+1, j\rangle,$$
$$|2j, n, -j\rangle \to |2j, n+1, -j\rangle.$$

As a result, the first two excitation paths can be represented as

$$|2j, 0, 0\rangle \begin{cases} \to |2j, 1, 1/2\rangle \to |2j, 2, 1\rangle \to \ldots |2j, n, j\rangle \to |2j, n+1, j\rangle \\ \to |2j, 1, -1/2\rangle \to |2j, 2, -1\rangle \to \ldots |2j, n, -j\rangle \to |2j, n+1, -j\rangle \end{cases} \quad (17)$$

The next series starts from the state $|2j,2,0\rangle$ corresponding to two excitations in the system ($n=2$) and $r=0$. One [for $2j=2$ (Fig. 1a)] or two [for $2j>2$ (Fig. 1b)] new excitation paths starting from the state $|2j,2,0\rangle$ are formed similarly to the paths (17). Thus each state $|2j,2l,0\rangle$, where l is an integer and $2l<2j$, gives rise to new excitation paths. The total number of excitation paths distinguished in this manner, is equal to the maximum number of split energy levels with the same value of n and is equal to $2j+1$.

As shown in the appendix, for large numbers of excitations $n \gg 2j$ the eigenstates of the Tavis–Cummings Hamiltonian can be represented in the factored form [see Eq. (A17)]

$$|2j, n, r\rangle \approx |n\rangle_f |r\rangle_a, \quad (18)$$

where the states $|r\rangle_a = \Sigma_{m=-j}^{j} d^j_{mr}(-\pi/2)|m\rangle_a$ are identical to the "dressed" states (8) with $\varphi = -\pi/2$. In addition, according to Eqs. (16) and (A12), for $n \gg 2j$ the energy eigenvalues are

$$\mathcal{E}_{2j,n,r} = \hbar\omega(n-j) + 2\hbar g\sqrt{n}r, \quad (19)$$

i.e., the detuning of the frequency of the transition

$$|2j, n, r\rangle \to |2j, n+1, r\rangle$$

for each separated rth excitation path decreases as n increases.

Thus, according to Eq. (18), the eigenstates (8) of the operator J_x are dressed atomic states which determine the quasiclassical ($n \gg 2j$) eigenstates of the Tavis–Cummings Hamiltonian. Using these states as a basis for expansion, we obtained the equations (13) which represent the excitation of the "atoms + intracavity field" system as a motion of $2j+1$ weakly coupled oscillators; for large n this is equivalent to separation of $2j+1$ independent excitation paths (of the form (17)) in the Tavis–Cummings model.

Taking decay into account in this model will lead to transitions from one distinct path to another. In Eqs. (13) these transitions are described by terms containing the spontaneous transition constants d_{21} and d_{12} and characterizing random quantum jumps between states of neighboring harmonic oscillators with probability proportional to $d_{12}(j+m)(j-m+1)$ and $d_{21}(j+m+1)(j-m)$. As the transition probabilities decrease, the intracavity field will have for most of the time a definite amplitude (14) and the system of atoms will be in the corresponding state $|j,m'\rangle$. This makes it possible to talk about multistability of the system under consideration.

5. NUMERICAL SOLUTION

In order to solve the system (13) we make use of the definition of the characteristic matrix

$$\Im_{m'n'} = \mathrm{Sp}(\exp(\lambda a^+)\exp(-\lambda^* a)D(E/k)\rho'_{m'n'}D^{-1}(E/k)). \quad (20)$$

Note that in the basis of dressed states $|j,m'\rangle\langle j,n'|$ the system of equations for $\mathcal{F}_{m'n'}$, which is obtained on the basis of Eq. (13), splits into two subsystems for the diagonal and off-diagonal elements. The diagonal part of the system has the form

$$\begin{aligned}\dot{\Im}_{m'm'} = {} & igm'(\lambda+\lambda^*)\Im_{m'm'} - k\lambda^*\frac{\partial\Im_{m'm'}}{\partial\lambda^*} - k\lambda\frac{\partial\Im_{m'm'}}{\partial\lambda} \\ & + E(\lambda-\lambda^*)\Im_{m'm'} \\ & + d_{12}(j+m')(j-m'+1)\Im_{m'-1,m'-1} \\ & + d_{21}(j+m'+1)(j-m')\Im_{m'+1,m'+1} \\ & - d_{12}(j+m'+1)(j-m') \\ & + d_{21}(j+m')(j-m'+1))\Im_{m'm'}. \end{aligned} \quad (21)$$

We have solved the system (21) numerically for four and eleven particles and found the probability distribution of the field $\mathcal{P}(\alpha) = \Sigma_{m'}\mathcal{P}_{m'm'}(\alpha)$, where $\mathcal{P}_{m'm'}(\alpha) = (1/\pi^2)\int d^2\lambda \exp(\lambda^*\alpha - \lambda\alpha^*)\mathcal{F}_{m'm'}(\lambda,t)$ is the probability density for finding the field in a coherent state with amplitude α and the atom in the superposition state $|j,m'\rangle$ (8). Since in the case at hand $\mathcal{P}_{m'm'}$ as a function of the real part of the amplitude of the field can be found explicitly in the form $\mathcal{P}_{m'm'}(\alpha) = \mathcal{P}_{m'm'}(y)\delta(x - E/k)$, where $\alpha = x + iy$, we investigated the distribution function as a function of the imaginary part of the amplitude. The results of calculating this dependence for different values of the parameters g/k and d/k are displayed in Fig. 2 ($N=4$) and Fig. 3 ($N=11$). For fixed values of g/k and d/k the number of peaks in the distribution function is equal to the number of independent paths of excitation of the "molecule," and the positions of the peaks correspond to the amplitudes of the fields that would be established in a cavity pumped by a classical field with amplitudes $E + igm'$: $\alpha = E/k + igm'/k$. Since the imaginary part of the field amplitude cannot exceed the value $gN/2k$, the distribution function is confined to the segment $[\alpha_{\min}, \alpha_{\max}]$, where $\alpha_{\min} = E - igN/2k$ and $\alpha_{\max} = E + igN/2k$. The qualitative nature of the probability distribution is not changed by increasing or decreasing the parameter g/k; the actual value of g/k determines only the scale of the change in the imagi-

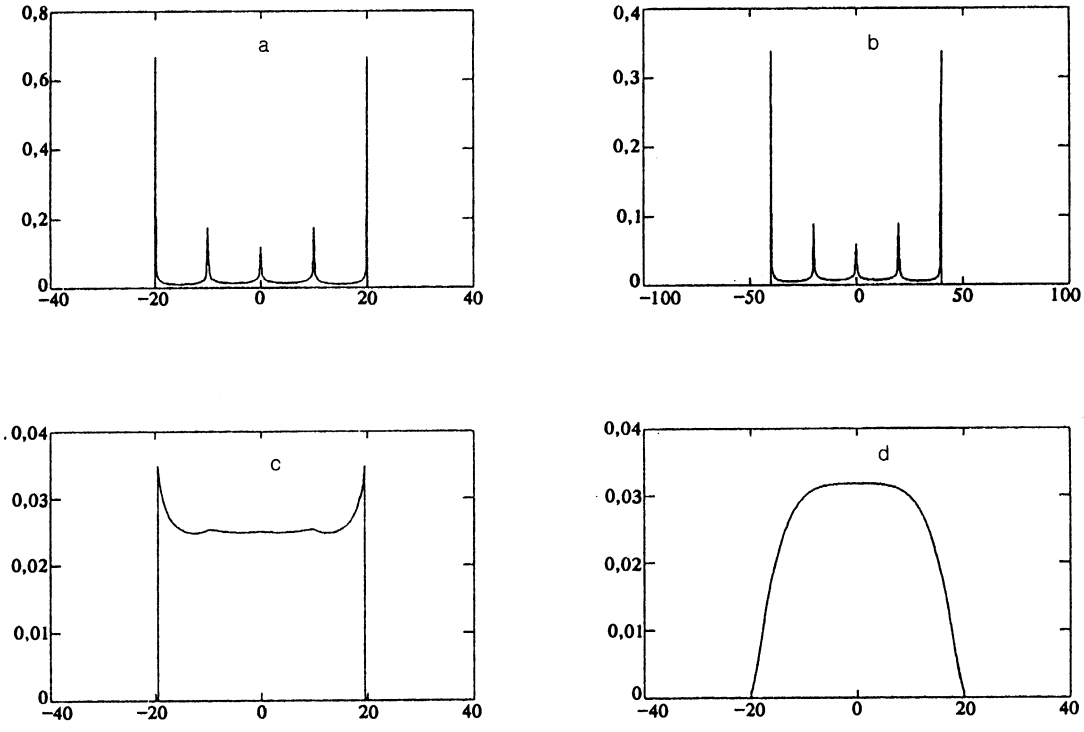

FIG. 2. Stationary distribution function $\mathscr{P}$ of the field inside the cavity for $N = 4$ with a) $g/k = 10$, $d/k = 0.05$, b) $g/k = 20$, $d/k = 0.05$, c) $g/k = 10$, $d/k = 0.2$, and d) $g/k = 10$, $d/k = 0.6$.

nary part of the amplitude of the field (compare Figs. 2a and 2b).

The sharpness of the peaks in the distribution function increases symmetrically from the center to the limits of the segment (Fig. 2a and Fig. 3a), and for fixed d/k it increases as the number of particles in the system decreases. As the ratio d/k increases [Fig. 2a(3a) and Fig. 2b(3b)] the distribution of the peaks becomes more diffuse and above some critical limit, which depends on N, the probability density function assumes the form shown in Figs. 2d and 3d with a single stable state at the center of the segment.

In order to find the critical value of d/k let us assume

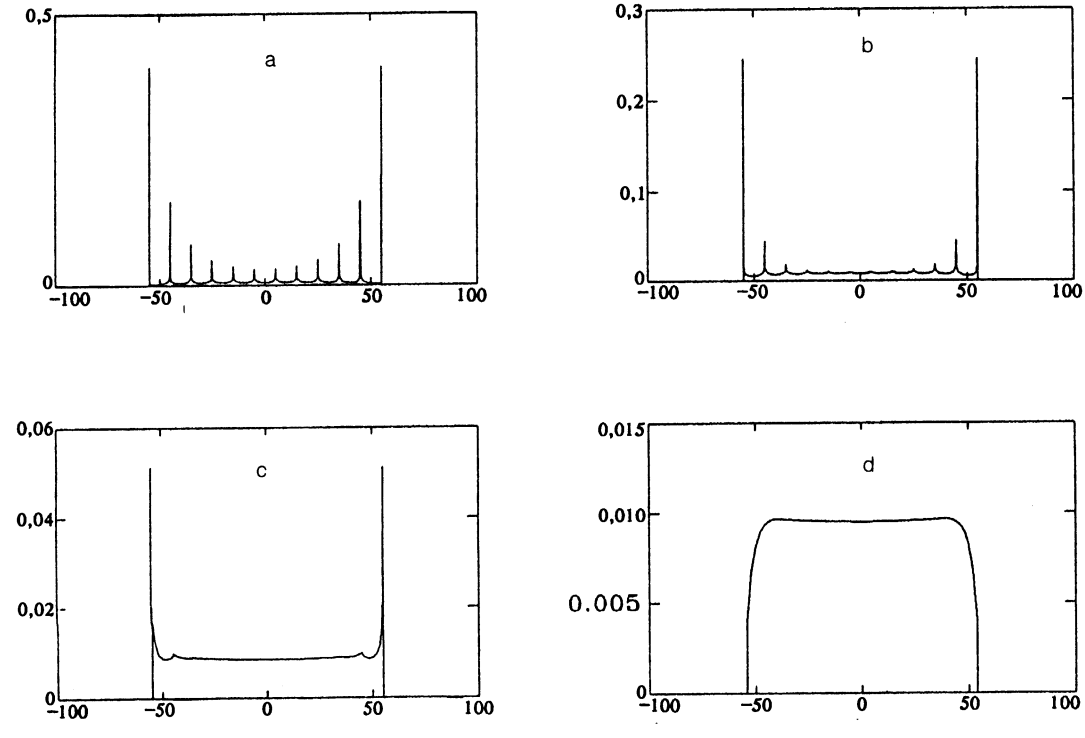

FIG. 3. Stationary distribution function $\mathscr{P}$ of the field inside the cavity with $N = 11$, $g/k = 10$, and $d/k = 0.01$ (a), 0.05 (b), 0.1 (c), and 0.15 (d).

that as the transition probabilities between neighboring functional states of the "molecule" increase, all stable states in the system vanish except two states corresponding to the extreme excitation paths. In this case the system (21) can be averaged over the fast intermediate motions and we arrive at equations for the probability densities similar to the single-atom variant:[10]

$$\begin{cases} \dfrac{\partial \mathcal{P}_{-j'-j'}}{\partial t} = \dfrac{\partial}{\partial x}\left[(kx - E)\mathcal{P}_{-j'-j'}\right] \\ + \dfrac{\partial}{\partial y}\left[\left(ky + \dfrac{gN}{2}\right)\mathcal{P}_{-j'-j'}\right] + d_{21}N\mathcal{P}_{j'j'} - d_{12}N\mathcal{P}_{-j'-j'}, \\ \dfrac{\partial \mathcal{P}_{j'j'}}{\partial t} = \dfrac{\partial}{\partial x}\left[(kx - E)\mathcal{P}_{j'j'}\right] + \\ + \dfrac{\partial}{\partial y}\left[\left(ky + \dfrac{gN}{2}\right)\mathcal{P}_{j'j'}\right] - d_{21}N\mathcal{P}_{j'j'} + d_{12}N\mathcal{P}_{-j'-j'}. \end{cases} \tag{22}$$

Then the condition for the appearance at least two stable states in the system (21) is given by the inequality

$$\gamma/4k < 1/N. \tag{23}$$

Compared with the single-atom phase optical bistability ($\gamma/4k < 1$) the collective interaction of the particles weakens the phase stability of the states in proportion to the number of atoms. The corresponding solution of Eq. (22) for the function $\mathcal{P}$ is, to within the substitution $k \to k/N$, the beta distribution,[10] and it describes a curve averaged over all intermediate states.

6. ANALYTICAL SOLUTION FOR $N=2$

We have solved the problem (21) analytically for a three-level Dicke system, corresponding to two atoms interacting with the field in a cavity. In this case the stationary system of equations for the probability densities $\mathcal{P}_{mm'}$, where $m = 0$ and ± 1 is

$$\begin{cases} \dfrac{\partial}{\partial y}\left[(ky - g)\mathcal{P}_{-1-1}\right] + 2d\mathcal{P}_{00} - 2d\mathcal{P}_{-1-1} = 0, \\ \dfrac{\partial}{\partial y}\left[ky\mathcal{P}_{00}\right] + 2d(\mathcal{P}_{-1-1} + \mathcal{P}_{11}) - 4d\mathcal{P}_{00} = 0, \\ \dfrac{\partial}{\partial y}\left[(ky + g)\mathcal{P}_{11}\right] + 2d\mathcal{P}_{00} - 2\mathcal{P}_{11} = 0. \end{cases} \tag{24}$$

In solving this system, it is convenient to normalize the imaginary part of the field amplitude to the ratio g/k : $z = y/(g/k)$. This change of variables "cleans up" the explicit dependence on the constant g/k in the system (24). The role of the parameter g/k as a scale factor of the imaginary part of the field amplitude can also be demonstrated in the case of an arbitrary number of particles [see Eq. (28) in Sec. 7]. The numerical solution given in Sec. 5 confirms that the form of the distribution function does not depend on the ratio g/k.

In order to derive the equation satisfied by the distribution function it is necessary to take into consideration the relation

$$(ky - g)\mathcal{P}_{-1-1} + ky\mathcal{P}_{00} + (ky + g)\mathcal{P}_{11} = 0,$$

which follows directly from the system (21) for the characteristic functions $\mathcal{F}_{ii}(\lambda,t)$. Then the normalized distribution function of the two-particle problem is found from the hypergeometric equation

$$u(1-u)\frac{d^2\mathcal{P}}{du^2} + [c - (a+b+1)u]\frac{d\mathcal{P}}{du} - ab\mathcal{P} = 0,$$

$$u = z^2$$

and can be expressed as follows:

$$\mathcal{P}(x,z) = \mathcal{P}_0^{-1}\delta\left(x - \frac{E}{k}\right)\theta(1-z^2)z^{2(1-c)}$$
$$\times(1-z^2)^{c-a-b}{}_2F_1(1-a, 1-b, c-a-b+1, 1-z^2), \tag{25}$$

where the constants a, b, c, and $\mathcal{P}_0$ are given by the relations

$$a = \frac{3}{2}\left(1 - 2\frac{d}{k}\right), \tag{26a}$$

$$b = 1 - \frac{d}{k}, \tag{26b}$$

$$c = \frac{1}{2}\left(3 - 4\frac{d}{k}\right), \tag{26c}$$

$$\mathcal{P}_0 = \frac{g}{2k}B(c-a-b+1, \frac{3}{2}-c)_2$$
$$\times F_1(1-a, 1-b, \frac{5}{2}-a-b, 1). \tag{26d}$$

Here ${}_2F_1$ is the hypergeometric function, θ is the Heaviside step function, and B is the beta function.

The distribution function (25) is quite easy to analyze qualitatively, since the behavior of the function $\mathcal{P}(x,y)$ is determined mainly by the poles $z = 0, \pm 1$ and not by the hypergeometric function ${}_2F_1$. For this reason, the critical point of the transition for the extreme poles from unstable to stable states is $d/k = 1/2$, which corresponds to $c - a - b = 0$. The relation $1 - c = 0$ or the value $d/k = 1/4$ gives the critical ratio of the parameters d and k for the appearance of an interior stability pole. For $d/k < 1/4$ the distribution has three pronounced peaks at points with amplitude $\alpha = E/k \pm ig/k$ and $\alpha = E/k$ (Fig. 4a).

The solution in the form (25) contains the hypergeometric series, which converges well only for $d/k < 1/4$ (for $d/k > 1/4$ it converges conditionally). For $d/k > 1/4$ the distribution function can be represented in the following form equivalent to Eq. (25):

$$\mathcal{P}(x,z) = \mathcal{P}_0^{-1}\delta\left(x - \frac{E}{k}\right)\theta(1-z^2)$$
$$\times(1-z^2)^{c-a-b}{}_2F_1(c-a, c-b, c-a-b+1, 1-z^2), \tag{27}$$

whence it follows directly that for $d/k > 1/N$ [see Eq. (23)] the function $\mathcal{P}(x,z)$ has a stable distribution with a maximum at the center of the segment $[-1,1]$ and vanishing at the ends of the segment.

Figure 4 displays for different values of d/k the form of the function $\mathcal{P}(y)$ computed numerically from the formulas (25)–(27).

7. DISCUSSION

In order to determine the conditions under which the internal states of the "molecule" are stable, we consider the system of equations for the probability densities $\mathcal{P}_{m'm'}$ in the case of exact resonance $\Delta\omega = 0$:

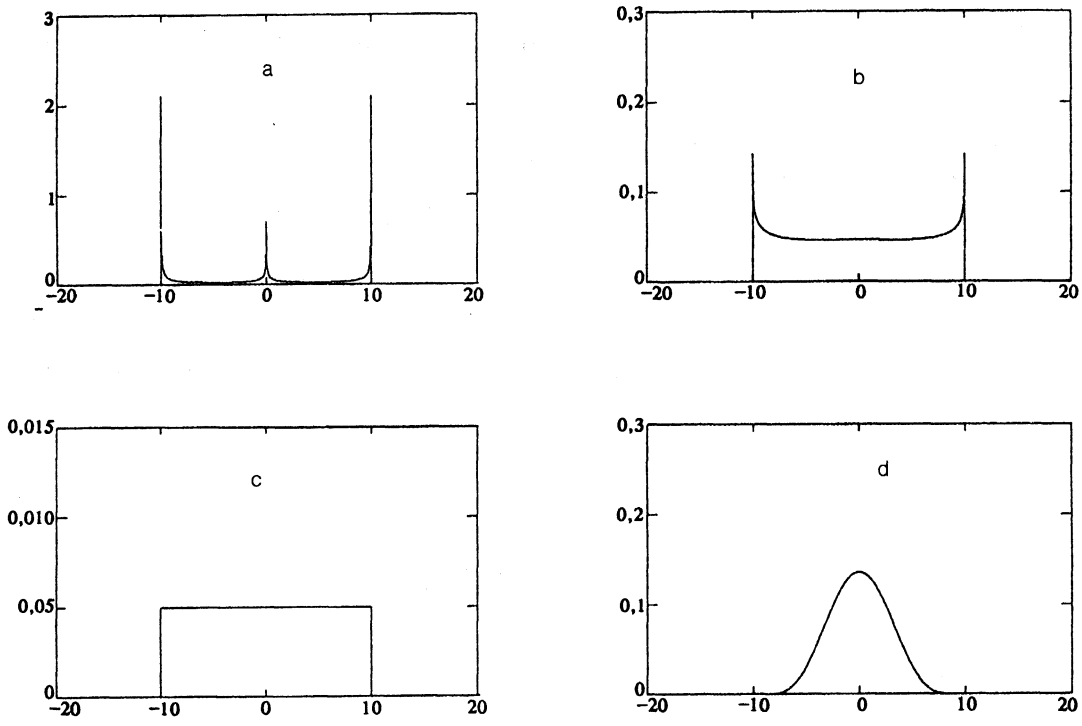

FIG. 4. Analytically computed stationary distribution function $\mathscr{P}$ of the field in the cavity for $N = 2, g/k = 10$, and $d/k = 0.1$ (a), 0.4 (b), 0.5 (c), and 4 (d).

$$\frac{\partial \mathscr{P}_{m'm'}}{\partial t} = \frac{\partial}{\partial x}[(kx - E)\mathscr{P}_{m'm'}] + \frac{\partial}{\partial y}[(ky - gm')\mathscr{P}_{m'm'}] - 2d(j + j^2 - m'^2)\mathscr{P}_{m'm'} + d(j + m')(j - m' + 1)\mathscr{P}_{m'-1,m'-1} + d(j + m' + 1)(j - m')\mathscr{P}_{m'+1,m'+1}, \quad d = \frac{\gamma}{4}. \tag{28}$$

The phase space of the model is represented by a collection of $N + 1$ phase planes, corresponding to "dressed" states of the "molecule" (Fig. 5). The system (28) describes a discrete-continuous Markov stochastic process, consisting of phase-plane motion along the trajectories $y + gm'/k = \text{const}$ $(x - E/k)$ (for the m'th phase plane) in the direction toward the pole with the coordinates $x_{0m} = E/k$, $y_{0m} = gm'/k$ and random hops from one phase plane to another in accordance with the transition probabilities. The continuous part of the Markov process is determined by the terms of Eq. (28) which contain partial derivatives of the probability densities and the discrete part is determined by terms with the constant probability d of a spontaneous transition.

The conditional probability density $c_{m'm',nn}(t + \tau|t)$ of a transition at the time $t + \tau$ into the state m' can be calculated by means of the theory of semi-Markov processes, using the Kolmogorov–Feller equations under the assumption that at time t the system was in the state n:

$$c_{m'm',nn}(t + \tau|t) = c_{m'n}(\tau) = d_{nm'}\exp(-\gamma_{nn}\tau). \tag{29}$$

Here $d_{nm'}$ is the probability of a transition from the plane n into the plane m', and in our case it is different from zero only for $m' = n \pm 1$:

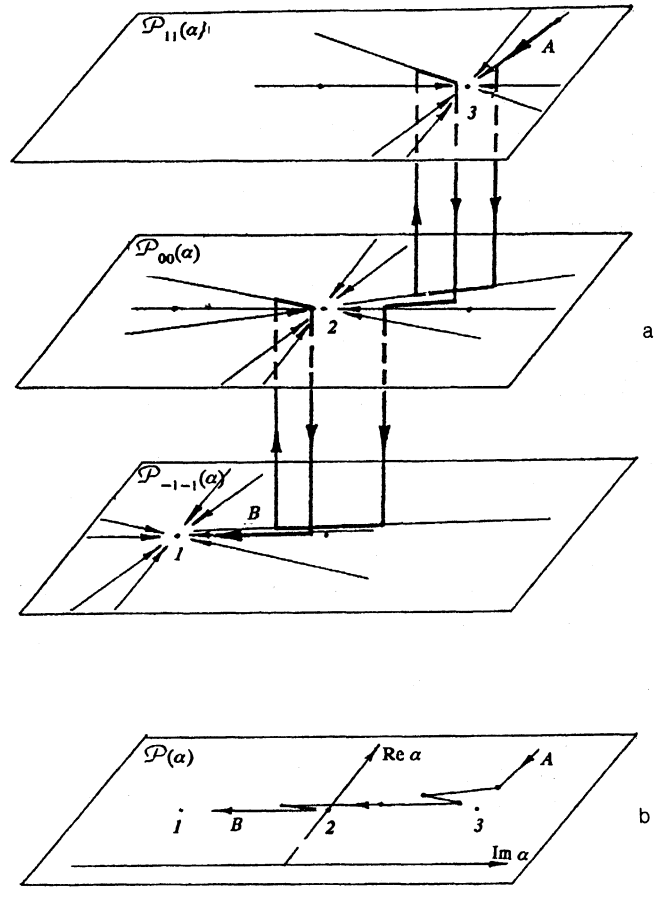

FIG. 5. Possible motion of the system in phase space for $N = 2$. a) Motion in a definite plane m' corresponds to the choice of the state of the system of atoms $|j,m'\rangle$ (and the choice of the mth excitation path in Fig. 1). The amplitude of the intracavity field is determined by fixing a point in the plane. b) Stochastic character of the variation of the intracavity field.

$$d_{n,n\pm1} = d(j \mp n)(j \pm n + 1). \tag{30}$$

The quantity γ_{nn} determines the probability of leaving the plane n ($\gamma_{nn} = \Sigma_l d_{nl}$) and is equal to $2d(j + j^2 - n^2)$ because the balance of the populations in this system is closed. The inverse of γ_{nn} is the average time between transitions of the system from one phase plane into another.

If at time t_n the system passed into the state m', then the amplitude of the field on the given plane evolves toward its stationary value at the pole according to the law

$$\alpha(t) - \alpha_{0m'} = \exp(-k(t - t_n))(\alpha(t_n) - \alpha_{0m'}), \tag{31}$$

where $\alpha_{0m} = E/k + igm'/k$ and k^{-1} establishes the time scale for reaching the stationary point $\alpha_{0m'}$. If

$$k^{-1} \ll \Delta t < (2d(j + j^2 - m'^2))^{-1} \tag{32}$$

the system will be found in one of the stationary states during the time Δt. It is obvious from the inequalities (32) that for $m' = j$ the condition for the appearance of stable states on the limiting plane is Eq. (23) (maximally stable states). As m' decreases, the states become less stable, and states with $m' = \pm 1/2$ for odd number of particles and $m' = 0$ for an even number of particles in the system are least stable. The inequality (32) indicates the ratio of the decay rates of a separate atom γ and the field k for which stable states can be observed. At the critical value, as Q of the cavity decreases continuously the successive appearance of 1 to $N + 1$ stable states in the collective model of $N + 1$ particles will be observed experimentally. The single-particle problem solved analytically in Ref. 10 (critical point $d/k = 1$) and the problem for two particles interacting with the radiation inside the cavity solved in Sec. 6 (critical points $d/k = 1/2$ for $m' = \pm 1$ and $d/k = 1/4$ for $m' = 0$), make it possible to indicate exactly, by comparing with the condition (21), the value of the ratio of the parameters γ and k from which the next stable state will arise as the ratio γ/k decreases:

$$(2d/k)(j + j^2 - m'^2) = 1. \tag{32a}$$

The phase trajectory of the motion after the first few transitions, as one can see from Eq. (31), will be limited to the segment $[\alpha_{\min}, \alpha_{\max}]$ (see Fig. 6). In the stationary limit of unstable states the system moves rapidly between the poles lying closest to the center of the segment $[\alpha_{\min}, \alpha_{\max}]$, so that it can be observed mainly at the central point of the trajectory. In the case of multistability, however, the system spends most of its time near one of the poles. The difference in the values of the amplitude of the field at the transition from one pole to another is displayed in Fig. 6. On all points of the phase trajectory the real part of the field amplitude E/k is the same and the imaginary part increases in an arithmetic progression with the difference g/k, starting with the minimum value mg/k. Besides the phase difference between the field at the entrance and a value of the field at the exit

$$\varphi_{m'} = \operatorname{arctg}(gm'/E), \tag{33}$$

a difference in the absolute values of the amplitudes also appears:

$$\Delta|\alpha_{m'}| = (E/k)[(1 + (m'g/E)^2)^{1/2} - 1], \tag{34}$$

which makes it possible to interpret the effect observed in the system as amplitude-phase multistability. In the case when the number of particles increases without bound, the maximum phase difference approaches $\pi/2$, and the maximum difference of the amplitudes increases as N.

Thus multistability of the states can be observed experimentally by comparing the amplitude and phase of the incident field and the field exiting the cavity, for example, by investigating their interference. It should be kept in

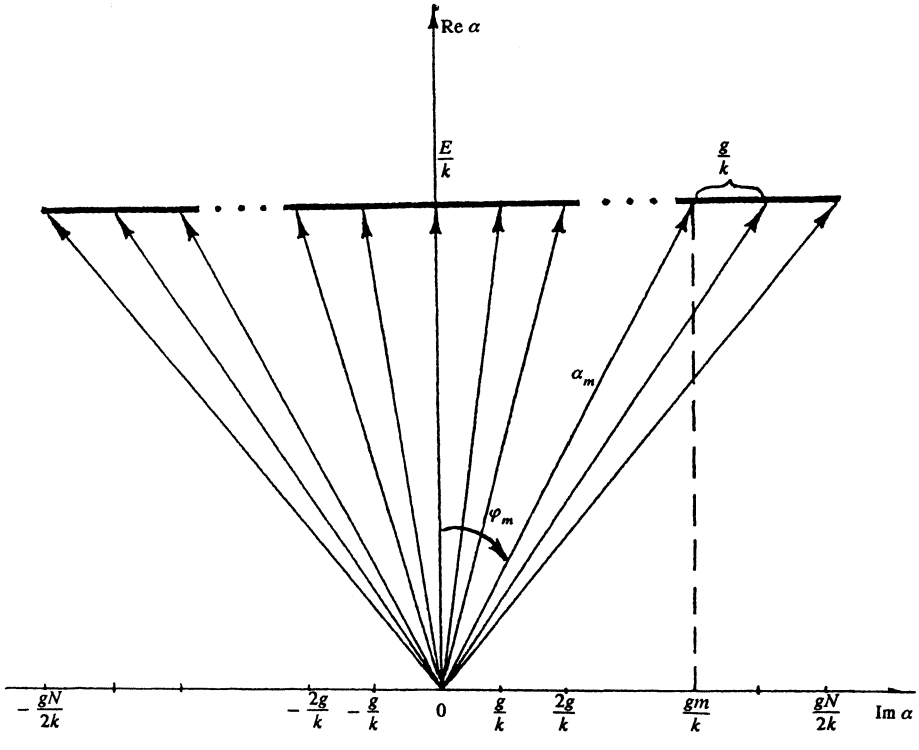

FIG. 6. Diagram of stationary values of the complex amplitude of the intracavity field for an even number of particles.

mind, however, that in order to obtain all possible amplitude shifts the system must be observed not over the time interval Δt but rather over the interval $\Delta T \sim \Sigma_{m'=-j}^{j} \times [2d(j+j^2-m'^2)]^{-1}$.

8. POSSIBLE EXPERIMENT

The amplitude-phase multistability is more easily observed experimentally than single-atom phase bistability.[10] The cavity Q in the multiparticle experiment can be made N times smaller than in the single-atom case without destabilizing the extreme states if the pump power is high and the interaction is strong in order to satisfy the conditions (9). The observed phase difference will be $\tan^{-1}(gN/2E)/\tan^{-1}(g/2E)$ times greater than the difference $\Delta\varphi$ for the single-particle problem.

The multistability effect predicted here can be observed under conditions achievable in existing experimental facilities. Thus in Ref. 14 bistability of the absorption type was investigated for the case of strong interaction of a beam of Cs atoms ($\lambda_{D_2} = 852$ nm) with the field in a high-Q cavity with transmission coefficient $T_0 = 4\cdot 10^{-5}$ and $k = cT_0/2L = 2\pi\,(0.9 \pm 0.1)$ MHz, where $L = 1$ mm is the cavity length. The interaction constant g_0 was of the same order of magnitude as the spontaneous emission probability γ: $[g_0,\gamma] = [2\pi(3.2 \pm 0.2), 5 \pm 0.4]$ MHz. The normalized input pumping intensity $Y = I_{\text{in}}/(I_{\text{sat}} T_0 \approx 10^4$ (titanium-sapphire laser with beam waist $\omega_0 = 50\,\mu$m), where $I_{\text{sat}} = 1$ MW/cm^2, $I_{\text{in}}/T_0 = (c\hbar\omega/4\pi\omega_0^2 L)(E/k)^2$, makes it possible to estimate the Rabi frequency of atomic oscillations in the experiment of Rempe *et al.*[14] as $\Omega \sim 10^2 g$, for which the conditions (9) are satisfied. In this experiment, however, the relation (32) does not hold because of the large value of the ratio γ/k. As the coefficient k increases with Ω^{-1} remaining the shortest characteristic time in the problem, it becomes possible to observe the multistability effect in the system of atoms in the cavity.

In conclusion it should be noted that the predicted multistability of a system of two-level atoms in a cavity reflects the quantum nature of the interaction of the atoms and the radiation to a greater degree than the phase bistability of a single atom in a cavity studied in Refs. 10 and 11. The phase bistability effect can be explained by means of the quasiclassical approach, treating the formation of two stable states as a manifestation of stationary points on the Bloch sphere,[10] whereas such a quasiclassical approach, based on equations for the averages $\langle J_i \rangle$, cannot be used to explain the presence of stable internal states in the case of a system of atoms. The quasiclassical approach can only be used to explain the existence of two extreme stable states. A systematic quantum approach, not limited only by the lowest-order averages $\langle J_i \rangle$, is required in order to explain the appearance of stable internal states.

9. ACKNOWLEDGMENTS

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APPENDIX

In this section we study the structure of the energy levels of the many-body analog of the Jaynes–Cummings Hamiltonian[13] for the case of exact resonance $\omega = \omega_0$:

$$H = \hbar\omega a^+ a + \hbar\omega J_z + \hbar g(a^+ J_- + a J_+), \tag{A1}$$

for which there exists a system of eigenvectors constructed from a superposition of eigenstates of the free Hamiltonian (A1) with $g = 0$: $|n-j-m\rangle_f |m\rangle_a$ (where the indices f and a designate the eigenstates of the intracavity field and the system of atoms) with a vacuum state corresponding to no field quanta $n = 0$ and the lowest Dicke state $m = -j$: $|0\rangle_f |-j\rangle_a$. As shown in Ref. 13, the eigenstates of the interaction Hamiltonian $\hbar g(a^+ J_- + a J_+)$ are

$$|2j, n, r\rangle = \sum_{m=-j}^{k} \alpha_m^n |n-j-m\rangle_f |m\rangle_a, \tag{A2}$$

where the number $n = 0, 1, 2, \ldots$ corresponds to the number of excitations in the "field + atoms" system,

$$k = \begin{cases} j, & \text{for} \quad n \geq 2j, \\ n-j, & \text{for} \quad n < 2j, \end{cases}$$

and the number r, ranging from $-n/2$ to $n/2$ for $n < 2j$ and from $-j$ to j for $n \geqslant 2j$, labels the eigenstates of the Hamiltonian $\hbar g(a^+ J_- + a J_+)$ which belong to a fixed number of excitations n. The states $|2j,n,r\rangle$ are eigenstates of the Hamiltonian H with energy eigenvalues

$$\mathcal{E}_{2j,n,r} = \hbar\omega(n-j) + E_{2j,n,r}, \tag{A3}$$

where $E_{2j,n,r}$ are the eigenvalues of the interaction Hamiltonian

$$\hbar g(a^+ J_- + a J_+)|2j, n, r\rangle = E_{2j,n,r}|2j, n, r\rangle. \tag{A4}$$

According to Eqs. (A2) and (A4), the coefficients α_m^n satisfy the recurrence relation

$$\alpha_{m+1}^n S_m^n - E_n \alpha_m^n + \alpha_{m-1}^n S_{m-1}^n = 0 \tag{A5a}$$

with the boundary conditions

$$\alpha_{-j-1}^n = \alpha_{j+1}^n = 0, \quad n \geq 2j, \tag{A5b}$$

$$\alpha_{-j-1}^n = \alpha_{n-j+1}^n = 0, \quad n < 2j,$$

and the energies E_n are found from the equation

$$\det{}_{-j}^{n} = \begin{vmatrix} -E & S_{-j}^n & 0 & \ldots & 0 & 0 & 0 \\ S_{-j}^n & -E & S_{-j+1}^n & \ldots & 0 & 0 & 0 \\ 0 & S_{-j+1}^n & -E & \ldots & 0 & 0 & 0 \\ 0 & 0 & 0 & & S_{k-1}^n & -E & S_k^n \\ 0 & 0 & 0 & & 0 & S_k^n & -E \end{vmatrix} = 0, \tag{A6}$$

where $S_m^n = \hbar g\sqrt{(n-j-m)(j+m+1)(j-m)}$ and $E \equiv E_{2j,n,r}$. The dimension of the matrix (A6) fixes the number of separate sublevels into which the energy level of the free Hamiltonian $E_{0n} = \hbar\omega(n-j)$ is split for fixed values of n and j. Starting with the state $n = 0$, which corresponds to the unsplit energy level of the single "intracavity field + atoms" system with $E_{2j,0,0} = 0$, as the number of excitations increases, the number of sublevels corresponding to the level E_{0n} increases as n (Fig. 1). The maximum amount of splitting is limited by the maximum dimension of the matrix (A6) and is equal to $2j+1$. As the number of excitations increases further, only the splitting of the sublevels increases.

From the recurrence relation for $\det^n_{-j}$, obtained by expanding the determinate (A6) with respect to the elements in first row,

$$\det^n_{-j} = E\det^n_{-j+1} - (S^n_{-j})^2\det^n_{-j+2}, \tag{A7}$$

where the lower index coincides with the index of the coefficient S^n_{-j} in the first row of the determinant, it follows that in the case when the dimension of the matrix (A6) is even the splitting into sublevels E_n is symmetric with respect to E_{0n}, and in the case of odd dimension, besides the symmetric sublevels, there is also the level with $E_{2j,n,0} = 0$. The corresponding energy level diagrams for an even ($2j \equiv N = 2$) or odd ($2j \equiv N = 3$) number of particles are displayed in Fig. 1a (1b).

In the case of a large number of excitations in the "atoms + intracavity field" system (for example, for a large number of field quanta), such that

$$n \gg 2j, \tag{A8}$$

the equations (A5) and (A6) can be solved approximately. Indeed, setting in Eqs. (A5a) $S_m \approx \hbar g\sqrt{n}\times\sqrt{(j+m+1)(j-m)}$, we find that in this approximation the equations (A5a) are equivalent to the following eigenvalue problem:

$$\hbar g n^{1/2}(J_+ + J_-)|\psi_E\rangle = E|\psi_E\rangle, \tag{A9}$$

where

$$|\psi_E\rangle = \sum_{m=-j}^{j} \alpha^n_m |m\rangle_a, \tag{A10}$$

and $|m\rangle_a$ are the eigenstates of the operator J_z. The solution of Eq. (A9) is obvious: applying the rotation transformation

$$J'_z = \exp(-i\frac{\pi}{2}J_y)J_x\exp(i\frac{\pi}{2}J_y), \tag{A11}$$

we find that in this approximation the eigenvalues E are, to within a factor, the same as the eigenvalues of the angular momentum operator J'_z:

$$E_{2j,n,r} = 2\hbar g n^{1/2} r, \qquad r = -j/2, \ldots, j/2, \tag{A12}$$

and the eigenvectors $|\psi_E\rangle$ are related to the eigenvectors $|r\rangle \equiv |m'\rangle$ of the operator J'_z by the relation

$$\exp(-i\frac{\pi}{2}J_y)|\psi_E\rangle = |r\rangle. \tag{A13}$$

According to the last relation the coefficients α^n_m are determined by the Wigner d-functions and do not depend (for $n \gg 2j$) on the number n of excitations:

$$\alpha^n_m = d^j_{mr}(-\pi/2). \tag{A14}$$

Therefore the approximate eigenstates of the interaction Hamiltonian with a large number of excitations have the form

$$|2j, n, r\rangle \approx \sum_{m=-j}^{j} d^j_{mr}(-\pi/2)|n-j-m\rangle_f |m\rangle_a. \tag{A15}$$

Under the action of the external field with interaction Hamiltonian $iE(a^+ - a)$ the "atoms + intracavity field" system can make transitions between the states $|2j,n,r\rangle$. The transition matrix elements are

$$\begin{aligned}\langle 2j, n', r'|a|2j, n, r\rangle &= \langle 2j, n', r'|\sum_{m=-j}^{j} d^j_{mr}(-\pi/2) \\ &\times (n-j-m)^{1/2}|n-j-m-1\rangle_f |m\rangle_a \\ &\approx n^{1/2}\delta_{n',n-1}\delta_{r,r'}.\end{aligned} \tag{A16}$$

i.e., transitions between states with the same quantum number r are allowed for $n \gg 2j$, and this is what explains the existence of $2j+1$ independent paths of excitation of the "atoms + intracavity field" system by an external field. Moreover, since the approximate matrix elements (A16) can be used to calculate averages of field operators for $n \gg 2j$, the eigenstates (A15) can be represented in the factor form

$$|2j, n, r\rangle \approx |n\rangle_f |r\rangle_a, \tag{A17}$$

where the states $|r\rangle_a = \Sigma^j_{m=-j} d^j_{mr}(-\pi/2)|m\rangle_a$ are identical to the "dressed" states (8) with $\varphi = -\pi/2$.

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