Bifurcations at the electrohydrodynamic effect threshold in a uniaxial nematic

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An infinite-dimensional dynamical system is constructed and its central manifold M_0 (of dimension $\leqslant 4$) is found within the linear theory of electrodynamic (EHD) instability in a uniaxial nematic liquid crystal in a low-frequency electric field. It is shown that the essential parameters of the problem are the strengths of the electric E and magnetic H fields, the thickness L of the liquid-crystal slab, and the anisotropy parameter $(\sigma_a \varepsilon_a / \sigma \varepsilon)$, where σ_a / σ and $\varepsilon_a / \varepsilon$ are, respectively, the relative anisotropies of the electric conductivity σ and the permittivity ε of the nematic. Two types of local bifurcations of codimension $\leqslant 4$, which are possible at the threshold of the EHD effect, are described taking into account the Fréedericksz effect and its influence of the EHD effect in a uniaxial nematic in the region where these two effects have the same threshold fields. For known nematics with planar orientation local bifurcations with codimension $\leqslant 3$ are allowed, while for nematics with homeotropic orientation local bifurcations with codimension $\leqslant 2$ are allowed. In an dielectrically isotropic nematic in the absence of an additional magnetic field only codimension-1 local bifurcations are allowed at the threshold of the EHD effect, irrrespective of the type of boundary orientation of the liquid crystal.

INTRODUCTION

The last ten years in the physics of liquid crystals (LCs) have been marked by intensive investigations of the nonlinear electrohydrodynamics (EHD) of uniaxial nematic LCs. The intense interest in this problem stems from the need to systematize diverse experimental nonlinear EHD results for NLCs,[1,2] which are beyond the scope of the linear theory of dynamical phenomena.[3] Moreover, experiments have shown that there is a deep analogy between the development of the nonlinear EHD effect in NLCs and transcritical Rayleigh–Bénard thermal convection in an isotropic liquid,[4] and the fact that NLCs have a wider variety of physical parameters than an isotropic liquid suggests that quite unusual types of instabilities, rarely or not at all encountered in other model problems, could be observed in uniaxial nematics. These include the oscillational EHD instability, predicted in Refs. 5 and 6 but not yet observed experimentally. This instability, together with the stationary EHD instability (Williams domains) and the orientational instability Fréederiksz effect) exhaust, from the standpoint of the theory of dynamical systems, the list of local codimension-1 bifurcations for the boundary-value problem of the theory of the EHD effect in a uniaxial nematic.[6] Bifurcations with codimension $\geqslant 2$ in the EHD effect, which correspond to multicritical singularities in the theory of phase transitions, have not been discussed in the literature, though in Ref. 7 it was pointed out that such bifurcations can exist in a phenomenon related to the EHD effect—thermal convection of an NLC in a magnetic field.

The linear part of a dynamical system is not sufficient for studying the stability of the new branched-off (bifurcated) motion, but with the help of the linear part it is not difficult to establish the limits on the existence of high-dimension bifurcations in model problems. The sparsity of the parameter set of the model precludes all local bifurcations with codimension $\geqslant 2$, which are engendered by the linear part of the dynamical system, in the existing three-dimensional models: Lorenz's thermal convection of an isotropic liquid,[8] Ricitake's for the magnetohydrodynamics of the earth's dynamo,[9] Rossler's for spiral chaos,[10] and Belousov–Zhabotinskiĭ's for chemical oscillatory reactions[11] with one essential parameter. However, this does not forbid the existence of local bifurcations with codimension $\geqslant 2$, engendered from codimension-1 bifurcations in these models by the linear part of the dynamical system due to degeneracy of some nonlinear terms. Bifurcations with codimension $\geqslant 2$ can be found by increasing the number of significant parameters in the model. In Ref. 12 Brand *et al.*, investigating thermal convection in a binary liquid slab, arrived at the conclusion that in an equivalent three-dimensional dynamical system with two essential parameters there exists a nonlocal codimension-2 bifurcation associated with the creation of a separatrix contour and a stable limit cycle.[1)]

The many-parameter model of the EHD effect in a uniaxial nematic differs advantageously from the models listed above in that it allows the existence of several types of bifurcations with codimension $\geqslant 2$. Separation of such bifurcations means that the infinite-dimensional dynamical system obtained with the help of Galerkin approximations in the boundary-value problem of the theory of the EHD effect is reduced to a finite-dimensional central manifold M_0. In nonlinear dynamical systems the manifold M_0 is, generally speaking, a very complicated, sometimes multisheet (in connection with the strange-attractor structure[14]) hypersurface in phase space. However, even the nondegenerate linear part of the dynamical system can give the dimension of the smooth central manifold and in addition the possible types of bifurcations of the phase portrait of the dynamical system on M_0.[15]

The object of this work is to find the central manifold of the dynamical system for the EHD effect and to describe some possible bifurcations of high codimension at the threshold of the EHD effect.

LINEAR DYNAMICAL SYSTEM

In this section we give a brief exposition of the linear theory of the low-frequency EHD effect in an NLC with

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planar (p) and homeotropic (h) orientation. We neglect second- and higher-order infinitesimals in the variations of the electro- and hydrodynamic variables and we confine our attention to the two-dimensional model of the EHD effect (a more detailed discussion is given in Refs. 6 and 16).

Consider an infinite plane-parallel slab of an NLC with thickness L and free boundaries. Nonstationary flow of the liquid-crystal medium is described by a system of four differential equations:

1) equation of continuity for an incompressible liquid;
2) equation of conservation of volume charge;
3) Navier–Stokes equation in the NLC; and,
4) equation of motion of the director $\mathbf{n}$, neglecting the small specific moment of inertia of the nematic liquid.

These equations are supplemented by free boundary conditions at the surface of the liquid crystal layer. The functions sought are: the components v_x and v_z of the velocity vector $\mathbf{v}$ of points of the liquid-crystal medium; the tilt angle $\theta = \mathbf{n}_0\mathbf{n}$ of the director $\mathbf{n}$ away from its unperturbed position $\mathbf{n}_0$; the deviation ψ of the potential of the electric field in the NLC from the uniform distribution Ez. The electric $\mathbf{E}$ and magnetic $\mathbf{H}$ field vectors lie in the same plane xz as $\mathbf{n}_0$; the z axis is directed along the electric field and the x axis is parallel to the plane of the liquid-crystal slab; the origin of the coordinate system is placed at the center of the plane-parallel slab. There are four types of possible mutual orientations of the vectors $\mathbf{n}_0$, $\mathbf{E}$, and $\mathbf{h}$ for which the breakdown of the undisturbed structure of the liquid crystal exhibits a threshold:[17] in p-NLC ($\varphi = 0$) and h-NLC ($\varphi = \pi/2$)—$\mathbf{E}\|\mathbf{H}$ ($\lambda = \pi/2$) and $\mathbf{E}\perp\mathbf{H}$ ($\lambda = 0$), where φ and λ are the tilt angles of the director $\mathbf{n}_0$ and the magnetic field with respct to the plane of the cell, respectively.

Solving the system of equations with the help of the Fourier transform leads to separation of the spatial modes. We choose for the basis functions for the electro- and hydrodynamic variables $u = (v_z, \theta, \Psi)$ the planar harmonics

$$u_0^{(k)}\exp[ikq_z z], \qquad u_1^{(k)}\exp[i(q_x x + kq_z z)], \qquad q_z = \pi/L, \tag{1}$$

where $k = \pm 1, \pm 2, \ldots$; q_x is the wave number of the surface-modulated structure of the NLC, described by the spatial mode $u_1^{(k)}$; the unmodulated structure in the Fréedericksz effect is described by the mode $u_0^{(k)}$. Eliminating the spatial modes for v_x from the system with the help of the equation of continuity, we obtain an infinite-dimensional linear dynamical system that decomposes into independent sets of three equations for the modes $u_1^{(k)}$ and independent equations for the components of the modes $u_0^{(k)}$:

$$\gamma_1 \frac{d\theta_1^{(k)}}{dt} = i\Gamma_3^{(k)} q_x^{-1} v_{z1}^{(k)} + (\varkappa E^2 - \Gamma_2^{(k)})\theta_1^{(k)} - i\varkappa E q_x \psi_1^{(k)},$$

$$\Gamma_\varepsilon^{(k)} \frac{d\psi_1^{(k)}}{dt} = \varkappa\gamma_1^{-1} E\Gamma_3^{(k)} v_{z1}^{(k)} - iq_x E[\sigma_a \cos 2\varphi + \varkappa\gamma^{-1}(\varkappa E^2 - \Gamma_2^{(k)})]\theta_1^{(k)} - (\Gamma_\sigma^{(k)} + \varkappa^2\gamma_1^{-1}E^2 q_x^2)\psi_1^{(k)},$$

$$\rho(q_x^2 + q_z^2 k^2)\frac{dv_{z1}^{(k)}}{dt} = (\gamma_1^{-1}\Gamma_3^{(k)2} - \Gamma_1^{(k)})v_{z1}^{(k)} + iq_x[\varkappa E^2(q_x^2 - \gamma_1^{-1}\Gamma_3^{(k)}) + \gamma_1^{-1}\Gamma_2^{(k)}\Gamma_3^{(k)}]\theta_1^{(k)} + q_x^2 E(\Gamma_\varepsilon^{(k)} - \varkappa\gamma_1^{-1}\Gamma_3^{(k)})\psi_1^{(k)}, \tag{2}$$

$$\gamma_1 \frac{d\theta_0^{(k)}}{dt} = (\varkappa E^2 - \Gamma_2^{(k)})\theta_0^{(k)},$$

$$\Gamma_\varepsilon^{(k)} \frac{d\psi_0^{(k)}}{dt} = -\Gamma_\sigma^{(k)}\psi_0^{(k)},$$

$$v_{z0}^{(k)} = 0. \tag{3}$$

In Eqs. (2) and (3) we introduce the following notation:

$$\Gamma_1^{(k)} = B_2 k^4 q_z^4 + (B_1 + B_4)k^2 q_x^2 q_z^2 + B_3 q_x^4, \qquad \varkappa = \frac{1}{4\pi}\varepsilon_a \cos 2\varphi,$$

$$\Gamma_2^{(k)} = (K_1\cos^2\varphi + K_3\sin^2\varphi)k^2 q_z^2 + (K_1\sin^2\varphi + K_3\cos^2\varphi)q_x^2 + \chi_a H^2\cos 2(\varphi - \lambda),$$

$$\Gamma_3^{(k)} = \frac{1}{2}(\gamma_1 + \gamma_2\cos 2\varphi)k^2 q_z^2 + \frac{1}{2}(\gamma_1 - \gamma_2\cos 2\varphi)q_x^2,$$

$$\Gamma_\sigma^{(k)} = \sigma_c q_x^2 + \sigma_s k^2 q_z^2, \qquad \Gamma_\varepsilon^{(k)} = \frac{1}{4\pi}(\varepsilon_c q_x^2 + \varepsilon_s k^2 q_z^2),$$

$$\gamma_1 = \alpha_3 - \alpha_2, \qquad \gamma_2 = \alpha_3 + \alpha_2,$$

$$B_1 = \alpha_1\cos^2\varphi(\cos^2\varphi - 2\sin^2\varphi) + \frac{1}{2}[\alpha_4 + (\alpha_3 + \alpha_6 + 2\alpha_5)\cos^2\varphi - (\alpha_2 + \alpha_5)\sin^2\varphi],$$

$$B_2 = \frac{1}{2}[\alpha_4 + (\alpha_3 + \alpha_6)\cos^2\varphi + (\alpha_5 - \alpha_2)\sin^2\varphi],$$

$$B_3 = \frac{1}{2}[\alpha_4 + (\alpha_3 + \alpha_6)\sin^2\varphi + (\alpha_5 - \alpha_2)\cos^2\varphi],$$

$$B_4 = \alpha_1\sin^2\varphi(\sin^2\varphi - 2\cos^2\varphi) + \frac{1}{2}[\alpha_4 + (\alpha_3 + \alpha_6 + 2\alpha_5)\sin^2\varphi - (\alpha_2 + \alpha_5)\cos^2\varphi],$$

$$\varepsilon_a = \varepsilon_\| - \varepsilon_\perp, \qquad \chi_a = \chi_\| - \chi_\perp, \qquad \sigma_a = \sigma_\| - \sigma_\perp,$$

$$\varepsilon_s = \varepsilon_\perp + \varepsilon_a\sin^2\varphi, \qquad \varepsilon_c = \varepsilon_\perp + \varepsilon_a\cos^2\varphi,$$

$$\sigma_s = \sigma_\perp + \sigma_a\sin^2\varphi, \qquad \sigma_c = \sigma_\perp + \sigma_a\cos^2\varphi, \tag{4}$$

$\varepsilon_\|$, $\varepsilon_\perp$, $\chi_\|$, $\chi_\perp$, $\sigma_\|$, and $\sigma_\perp$ are the principal values of the diagonalized tensors of permittivity ε, diamagnetic susceptibility χ, and electric conductivity σ; K_i are Frank's elastic constants; α_i are the Leslie coefficients of viscosity; and, ρ is the density of the NLC.

The equations (2) contain complex coefficients. This is because the real basis functions constructed with the help of Eq. (1) have different symmetry. For example, the single-mode approximation has the form[18] (taking into account the boundary conditions)

$$v_x \propto \sin q_z z \sin q_x x, \qquad \theta \propto \cos q_z z \sin q_x x\,,$$
$$v_z \propto \cos q_z z \cos q_x x, \qquad \psi \propto \cos q_z z \cos q_x x\,. \tag{5}$$

It is no accident that the equations (3) of the Fréedericksz effect are included in the general dynamical system of the EHD effect in the NLC when studying the degenerate states of the system which are associated with the existence of high-dimension bifurcations. As a rule, the Fréedericksz effect in NLCs is considered as an independent model effect irrespective of competing electrohydrodynamic processes. This is valid when the threshold fields (bifurcation values of the parameters) of the Fréedericksz effect E_F and the EHD effect (E_s for the stationary effect or E_0 for the oscillation effect) are significantly different:

$$g = \frac{|E_F - E_{s,o}|}{E_F} \gtrsim 1\,.$$

In the region where these fields are close ($g \ll 1$), however, this approach may turn out to be incorrect because of the nonlinear character of both of these effects. Indeed, it can be shown[18] that when the nonlinear terms are retained in Eqs. (2) and (3) the modes $u_0^{(k)}$ and $u_1^{(k)}$ start to mix with one another, and the systems of equations (2) and (3) are no longer independent. The situation is similar to including, in the basis of the solutions of Lorenz's nonlinear three-dimensional model of thermal convection of an isotropic liquid, a Z mode not modulated by the surface,[14] and with which it is impossible to describe the evolution of the dynamical system through a cascade of bifurcations to a strange attractor.

The possibility cannot be excluded that the system of equations (3) will be partially independent of the modes $u_1^{(k)}$, namely, nonlinear terms associated only with $\theta_0^{(k)}$ will enter into the equation of dynamics of the Fréedericksz effect [the first equation in the system (3)]. Then the Fréedericksz effect can indeed be regarded an an independent effect, even in the region $g \ll 1$.

The system of equations (3) describes the relaxation of the director $\mathbf{n}$ in the Fréedericksz effect and Maxwellian relaxation of the volume charge in the absence of macroscopic hydrodynamic flows. This system engenders in the complex plane of the decrement μ of small disturbances [$u^{(k)} \propto \exp(\mu^{(k)} \cdot t)$] an infinite ordered sequence of real roots

$$\mu_F^{(k)} = \gamma_1^{-1}(\varkappa E^2 - \Gamma_2^{(k)}) \tag{6}$$

and one infinitely degenerate real root

$$\mu_m = -\frac{4\pi\sigma_s}{\varepsilon_s}\,. \tag{7}$$

Each mode $u_1^{(k)}$ engenders in the μ plane three roots of the characteristic equation of the dynamical system (2)

$$D_3^{(k)}\mu^3 + D_2^{(k)}\mu^2 + D_1^{(k)}\mu + D_0^{(k)} = 0\,, \tag{8}$$

where

$$D_0^{(k)} = E^2(q_x^2 + k^2q_z^2)(\varkappa\sigma_s\Gamma_1^{(k)} - q_x\Gamma_3^{(k)}\Gamma_4^{(k)}) - \Gamma_1^{(k)}\Gamma_2^{(k)}\Gamma_\sigma^{(k)}\,,$$
$$D_1^{(k)} = \varkappa E^2(q_x^2 + k^2q_z^2)\left[\frac{\varepsilon_s}{4\pi}\Gamma_1^{(k)} + \rho\sigma_s(q_x^2 + k^2q_z^2)\right] + d_1^{(k)}\,,$$
$$D_2^{(k)} = \varkappa E^2\frac{\varepsilon_s}{4\pi}\rho(q_x^2 + k^2q_z^2)^2 + d_2^{(k)}\,, \tag{9}$$
$$D_3^{(k)} = -\gamma_1\rho\Gamma_\varepsilon^{(k)}(q_x^2 + k^2q_z^2),$$
$$\Gamma_4^{(k)} = \frac{1}{4\pi}(\varepsilon_a\sigma_\perp - \varepsilon_\perp\sigma_a)q_x\cos 2\varphi\,,$$
$$d_1^{(k)} = \Gamma_\sigma^{(k)}(\Gamma_3^{(k)^2} - \gamma_1\Gamma_1^{(k)}) - \Gamma_2^{(k)}[\Gamma_1^{(k)}\Gamma_\varepsilon^{(k)} + \rho\Gamma_\sigma^{(k)}(q_x^2 + k^2q_z^2)],$$
$$d_2^{(k)} = \Gamma_\varepsilon^{(k)}(\Gamma_3^{(k)^2} - \gamma_1\Gamma_1^{(k)}) - \rho(q_x^2 + k^2q_z^2)(\Gamma_2^{(k)}\Gamma_\varepsilon^{(k)} + \gamma_1\Gamma_\sigma^{(k)})\,.$$

We note that all coefficients $D_j^{(k)}$ of the cubic equation (8) have the even $D_j(k^2)$ on k, i.e., the spatial modes $u_1^{(k)}$ can be enumerated with positive integers.

Depending on the sign of the discriminant Δ_k of the characteristic polynomial from Eq. (8)

$$\Delta_k = 27(D_0^{(k)}D_3^{(k)})^2 + 4D_0^{(k)}D_2^{(k)^3} + 4D_3^{(k)}D_1^{(k)^3} - (D_1^{(k)}D_2^{(k)})^2 - 18D_0^{(k)}D_1^{(k)}D_2^{(k)}D_3^{(k)} \tag{10}$$

the roots $\mu_j^{(k)}, j = 1, 2, 3$, of the equation lie in the complex plane:

$\Delta_k < 0$—three real roots;

$\Delta_k = 0$—two real roots, one of which is doubly degenerate;

$\Delta_k > 0$—one real root and two complex-conjugate roots.

The further approach in studying threshold phenomena of the EHD effect consists of applying the general methods of the theory of dynamical systems. From this standpoint the trivial state $u = 0$ of the system is replaced by one of the nontrivial types of motion when the essential parameters of the problem[2)] reach bifurcation values. In the absence of $\mathbf{E}$ and $\mathbf{H}$ fields, whose values are significant parameters, all roots of the infinite-dimensional dynamical system, including $\mu_F^{(k)}$ (6), μ_m (7), and $\mu_j^{(k)}$ from Eq. (8), lie in the left-hand half-plane of the decrement μ. This is easily verified, keeping in mind the facts that $\gamma_1 > 0$, $D_3^{(k)} < 0$, and $\Gamma_2^{(k)} > 0$ with $H = 0$ and also the inequality

$$\Gamma_3^{(k)^2} - \gamma_1\Gamma_1^{(k)} < 0\,,$$

proved in Refs. 6 and 18 for p- and h-NLCs. As the electric and magnetic fields increase, the root μ_m remains stationary while the other roots move in the complex plane. For some values of the fields $E = E_*$ and $H = H_*$ one of the roots $\mu_F^{(k)}$ or $\mu_j^{(k)}, j = 1, 2, 3$, from the infinite set of real roots or a pair of complex-conjugate roots $\mu_j^{(k)}, \bar{\mu}_j^{(k)}$ from an infinite set of pairs will fall on the imaginary axis $\operatorname{Im}\mu$, while among all other roots of the dynamical system n_- roots will lie in the left-hand half-plane ($\operatorname{Re}\mu < 0$) and n_+ roots will lie in the right-hand half-plane ($\operatorname{Re}\mu > 0$). If the parametric space of the problem is sufficiently rich, then in fields $E = E_*$ and

$H = H_*$ for some set of functionally related parameters (a so-called system of nongeneral position) n_r real roots and n_c complex-conjugate pairs of roots will lie at the same time on the Im μ axis. It turns out that in this case Shoshitaĭshvili's "theorem about the suspension of a saddle[3] over a real central manifold of dimension $n_r + 2n_c$" holds:[15,20] Near the bifurcation values E_*, H_* of the parameters E, H the initial nonlinear dynamical system is split into three independent dynamical systems with the help of a continuous nonlinear transformation: 1) n_--dimensional system, describing damped modes; 2) n_+-dimensional system, describing growing modes; and, 3) $n_r + 2n_c$-dimensional system, describing the nontrivial behavior of phase trajectories, including new stationary states and limit cycles. In other words, the infinite-dimensional phase space of the dynamical system reduces to a real finite-dimensional central manifold of dimension $n_r + 2n_c$.

In the next section we investigate the central manifold M_0 of an infinite-dimensional dynamical system describing the transition from the trivial state $u = 0$, corresponding to $n_+ = 0, n_- = \infty$. For this reason, in the present situation it is topologically more appropriate to talk about the suspension of not a "saddle" but rather an infinite-dimensional "cap" over the manifold M_0.

CENTRAL MANIFOLD OF A DYNAMICAL SYSTEM

From the technical standpoint, the application of the theorem about the central manifold consists of two steps. First, the central manifold M_0 itself is found with the required accuracy. Next, the system of differential equations which describes the behavior of phase trajectories on M_0 is put into normal form. The second step is important for nonlinear dynamical problems. In the present paper we focus our attention on the first step of the problem. In order to investigate this step it is sufficient to consider the linear part of the dynamical system.

In the present section we show that with increasing electric and magnetic fields the first roots to appear on the imaginary axis of the complex μ plane are $\mu_F^{(1)}$ (6) and $\mu_j^{(1)}$, $j = 1, 2, 3$, from Eq. (8), corresponding to the single-mode approximation. No other roots $\mu_F^{(k)}$ and $\mu_j^{(k)}$ with $k > 1$ can appear together with $\mu_F^{(1)}$ and $\mu_j^{(1)}$ on the Im μ axis. Since the degeneracy of the three roots is unrestricted, on the basis of what was said above the dimension of the central manifold M_0 of the dynamical system of the boundary-value problem of the theory of the EHD effect will not exceed four. We note that for a dielectrically isotropic ($\varkappa = 0$) NLC in the absence of a magnetic field ($H = 0$) the root $\mu_F^{(k)}$ becomes stationary as a function of the electric field **E**; this reduces the upper limit of dim M_0 to three, though, in reality, as will be shown below, even this number overestimates $n_r + 2n_c$ under the chosen conditions.

We construct a proof of the assertion formulated above in several steps, with no restrictions on the parameters of the NLC and the electric and magnetic fields. First, it is easy to show that the infinite sequence of roots $\mu_F^{(k)}$ is ordered as follows:

$$\ldots < \mu_F^{(3)} < \mu_F^{(2)} < \mu_F^{(1)} \tag{11}$$

so that no two roots from this sequence can coincide, and the root $\mu_F^{(1)}$ is the first one to land on the Im μ axis.

We now consider Eq. (8) and show that if the real root $\mu_j^{(1)}$ of the first mode lies on the Im μ axis, then all real roots of the other modes with $k > 1$ lie in the region $\mu < 0$. The position of the real root $\mu_*^{(k)}$ on the imaginary axis is determined by the condition

$$D_0^{(k)} = 0, \tag{12}$$

which gives a spectrum of the threshold field $E_s^{(k)}(q_x, q_z)$ of the stationary instability for the corresponding mode $\mu_1^{(k)}$. Since the functions $D_j^{(k)}$ from Eq. (9) and the initial position of all roots $\mu_j^{(k)}$ in the region Re $\mu < 0$ are continuous, the real root corresponding to the mode $\mu_1^{(k)}$ for which the condition (12) is satisfied first, in other words, the mode for which the threshold field $E_S^{(k)}$ is minimum, will be the first root to land on the imaginary axis as the electric and magnetic fields increase. Using the results of Refs. 6 and 16 for the threshold fields $E_S^{(k)}$ of the corresponding modulated EHD structures it is easy to show for p- and h-NLCs (see Appendix) that

$$E_s^{(k)} = C_1(k^2 + C_2)^{1/2}, \tag{13}$$

where C_i are constants ($C_1 > 0$) which depend only on the properties of the NLC. Minimizing $E_S^{(k)}$ with respect to k gives $k = 1$, i.e., if $\mu_*^{(1)} = 0$, then $\mu_j^{(k)} < 0$ for $k > 1$.

We now consider the complex-conjugate pair of roots of Eq. (8). We shall show that if the pair of complex-conjugate roots $\mu_*^{(1)}, \bar{\mu}_*^{(1)}$ corresponding to the first mode lies on the Im μ axis, then all complex-conjugate roots of the higher-order modes with $k > 1$ lie in the region Re $\mu < 0$. The position of the imaginary pair of roots $\mu_*^{(k)}$ and $\mu_*^{(k)}$ on the Im μ axis is determined by the condition

$$D_1^{(k)} D_2^{(k)} - D_0^{(k)} D_3^{(k)} = 0, \tag{14}$$

which gives a spectrum of two threshold fields $E_0^{(k)}(q_x, q_z)$ of the oscillatory EHD instability for the corresponding mode $u_1^{(k)}$. In Ref. 16 it is shown that only the lower branch of the oscillatory EHD effect can be observed in practice. Repeating the arguments for the real roots, we obtain that as the electric and magnetic fields increase, the pair of complex-conjugate roots of the mode $\mu_1^{(k)}$ for which the condition (14) is satisfied first, i.e., for which the threshold field $E_0^{(k)}$ is lowest, will arrive first on the imaginary axis. Using the results for the threshold fields $E_0^{(k)}$ obtained in Refs. 6 and 16 for the corresponding modulated EHD structures it is easy to show for p- and h-NLCs (see Appendix) that

$$E_o^{(k)} = C_3 k(k^2 + C_4)^{1/2}, \tag{15}$$

where C_i are constants ($C_3 > 0$) which depend on the properties of the NLC. Minimizing $E_0^{(k)}$ with respect to k gives $k = 1$, i.e., if Re $\mu_*^{(1)} = 1$, then Re $\mu_j^{(k)} < 0$ for $k > 1$.

In order to complete the proof of our assertion, it remains to show that the roots of the first- ($k = 1$) and higher-order ($k > 1$) modes cannot arrive first on the imaginary Im μ axis simultaneously in the following combinations: 1) $\mu_F^{(1)}$, $\mu_j^{(k)}$; 2) $\mu_F^{(1)}, \mu_j^{(k)}, \bar{\mu}_j^{(k)}$; 3) $\mu_j^{(1)}, \mu_F^{(k)}$; 4) $\mu_j^{(1)}, \mu_j^{(k)}, \bar{\mu}_j^{(k)}$; 5) $\mu_j^{(1)}, \bar{\mu}_j^{(1)}, \mu_F^{(k)}$; 6) $\mu_j^{(1)}, \bar{\mu}_j^{(1)}, \mu_j^{(k)}$. Indeed, as shown above, the first roots from the set of real roots $\mu_F^{(k)}$ and $\mu_j^{(k)}$ to arrive on the Im μ axis as the electric and magnetic fields increase are the roots $\mu_F^{(1)}$ and $\mu_j^{(1)}$, and the first roots to arrive on the Im μ axis from the set of complex-conjugate pairs of roots

$\mu_j^{(k)}$ and $\bar{\mu}_j^{(k)}$ are also the roots corresponding to the first mode. For this reason, the roots corresponding to the first- and higher-order modes cannot arrive first on the imaginary axis simultaneously in the combinations enumerated above.

Thus, recognizing that the infinitely degenerate root μ_m remains stationary as the electric and magnetic fields increase, we find that the smooth central manifold M_0 of the infinite-dimensional dynamical system of the boundary-value problem of the theory of the EHD effect can be constructed on the basis of the $(\theta_1^{(1)}, \psi_1^{(1)}, v_{z1}^{(1)}, \theta_0^{(1)})$—eigenvectors of the linear differential operator $\mathscr{L}$, defined by the right-hand side of the first four equations of the system (2) and (3) with $k=1$. The dimension of M_0 does not exceed four, which corresponds to the cases $\mu_F^{(1)} = \mu_1^{(1)} = 0$, $\mu_{2,3}^{(1)} = \pm i\omega$ and $\mu_F^{(1)} = \mu_j^{(1)} = 0, j = 1, 2, 3$. It is obvious that near the origin of the coordinates the manifold M_0 is homeomorphic to Euclidean space.[4] If, as mentioned above, the equation of the Fréedericksz effect for $\theta_0^{(1)}$ can be studied independently of the other modes $\theta_1^{(1)}, \psi_1^{(1)}, \vartheta_{z1}^{(1)}$, then M_0 decomposes into a direct sum of a one-dimensional space M_1, associated with the equation of the Fréerdericksz effect, and the central manifold M_2 of the rest of the dynamical system, and the simultaneous presence of the roots $\mu_F^{(1)}$ and $\mu_j^{(1)}$ on the Im μ axis is not critical for the dynamics of the system. The dimension of M_2 in this case does not exceed three.

BIFURCATIONS OF HIGHER CODIMENSIONS

A dynamical system of nongeneral position is characterized by the fact that some number c of bifurcation conditions (equations which the parameters satisfy) and some conditions of nondegeneracy (inequalities) are satisfied. The bifurcation conditions separate in the parameter space a manifold of codimension $= c$, and the nondegeneracy conditions separate regions on this manifold. The bifurcation diagram at all points of this region is said to be a point of bifurcation of codimension c.

We now consider the degenerate states of the dynamical system for the EHD effect in a uniaxial nematic. We shall find the behavior of the roots $\mu_j^{(1)}$ of the characteristic polynomial from Eq. (8) with the help of the Routh–Hurwitz criterion. When the essential parameters of the problem reach their bifurcation values, the roots $\mu_F^{(1)}$ and $\mu_j^{(1)}, j=1, 2, 3$, can arrive on the imaginary axis from the left half of the μ plane as follows:

codimension = 1

1. Orientational Fréedericksz effect

$$\mu_F^{(1)} = 0, \quad \mathrm{Re}\,\mu_j^{(1)} < 0, \quad j = 1, 2, 3,$$
$$E^2 = \varkappa^{-1}\Gamma_2^{(k)}, \quad D_0^{(1)} < 0, \quad D_1^{(1)} < 0,$$
$$D_2^{(1)} < 0, \quad D_0^{(1)} D_3^{(1)} < D_1^{(1)} D_2^{(1)}. \tag{16}$$

2. Stationary EHD effect

$$\mu_1^{(1)} = 0, \quad \mu_F^{(1)} < 0, \quad \mathrm{Re}\,\mu_{2,3}^{(1)} < 0,$$
$$D_0^{(1)} = 0, \quad D_1^{(1)} < 0, \quad D_2^{(1)} < 0, \quad E^2 < \varkappa^{-1}\Gamma_2^{(1)}. \tag{17}$$

3. Oscillatory EHD effect

$$\mu_{1,2}^{(1)} = \pm i(D_1^{(1)}/D_3^{(1)})^{1/2}, \quad \mu_F^{(1)} < 0, \quad \mu_3^{(1)} < 0,$$
$$D_0^{(1)} D_3^{(1)} = D_1^{(1)} D_2^{(1)}, \quad D_0^{(1)} < 0,$$
$$D_1^{(1)} < 0, \quad D_2^{(1)} < 0, \quad E^2 < \varkappa^{-1}\Gamma_2^{(1)}. \tag{18}$$

codimension = 2

4. Orientational-stationary EHD effect

$$\mu_F^{(1)} = \mu_1^{(1)} = 0, \quad \mathrm{Re}\,\mu_{2,3}^{(1)} < 0,$$
$$E^2 = \varkappa^{-1}\Gamma_2^{(1)}, \quad D_0^{(1)} = 0, \quad D_1^{(1)} < 0, \quad D_2^{(1)} < 0. \tag{19}$$

5. Orientational-oscillatory EHD effect

$$\mu_F^{(1)} = 0, \quad \mu_{1,2}^{(1)} = \pm i(D_1^{(1)}/D_3^{(1)})^{1/2}, \quad \mu_3^{(1)} < 0,$$
$$E^2 = \varkappa^{-1}\Gamma_2^{(1)}, \quad D_0^{(1)} D_3^{(1)} = D_1^{(1)} D_2^{(1)}, \quad D_0^{(1)} < 0,$$
$$D_1^{(1)} < 0, \quad D_2^{(1)} < 0. \tag{20}$$

6. Stationary-oscillatory EHD effect

$$\mu_{1,2}^{(1)} = \pm i(D_1^{(1)}/D_3^{(1)})^{1/2}, \quad \mu_3^{(1)} < 0, \quad \mu_F^{(1)} < 0,$$
$$D_0^{(1)} = D_2^{(1)} = 0, \quad D_1^{(1)} < 0, \quad E^2 < \varkappa^{-1}\Gamma_2^{(1)}. \tag{21}$$

This type of bifurcation is engendered only by the threshold field E_s and the wave number q_{xs} of the stationary EHD instability. The corresponding characteristics of the oscillatory instability E_0 and q_{x0} are simply absent, since the dispersion relation (14) engendering them becomes an identity in this case.

7. Doubly degenerate stationary EHD effect

$$\mu_{1,2}^{(1)} = 0, \quad \mu_3^{(1)} < 0, \quad \mu_F^{(1)} < 0,$$
$$D_0^{(1)} = D_1^{(1)} = 0, \quad D_2^{(1)} < 0, \quad E^2 < \varkappa^{-1}\Gamma_2^{(1)}. \tag{22}$$

codimension = 3

8. Orientational-stationary-oscillatory EHD effect

$$\mu_F^{(1)} = \mu_1^{(1)} = 0, \quad \mu_{2,3}^{(1)} = \pm i(D_1^{(1)}/D_3^{(1)})^{1/2},$$
$$E^2 = \varkappa^{-1}\Gamma_2^{(1)}, \quad D_0^{(1)} = D_2^{(1)} = 0, \quad D_1^{(1)} < 0. \tag{23}$$

9. Orientational-doubly degenerate stationary EHD effect

$$\mu_F^{(1)} = \mu_{1,2}^{(1)} = 0, \quad \mu_3^{(1)} < 0,$$
$$E^2 = \varkappa^{-1}\Gamma_2^{(1)}, \quad D_0^{(1)} = D_1^{(1)} = 0, \quad D_2^{(1)} < 0. \tag{24}$$

10. Triply degenerate stationary EHD effect

$$\mu_j^{(1)} = 0, \quad j = 1, 2, 3, \quad \mu_F^{(1)} < 0\,,$$
$$D_0^{(1)} = D_1^{(1)} = D_2^{(1)} = 0, \quad E^2 < \varkappa^{-1}\Gamma_2^{(1)}\,. \tag{25}$$

codimension = 4

11. Orientational-triply degenerate stationary EHD effect

$$\mu_F^{(1)} = \mu_j^{(1)} = 0, \quad j = 1, 2, 3\,, \tag{26}$$
$$E^2 = \varkappa^{-1}\Gamma_2^{(1)}, \quad D_0^{(1)} = D_1^{(1)} = D_2^{(1)} = 0\,.$$

The cases (16)–(26) above complete the list of local bifurcations with codimension$\geqslant$2 that can be engendered by the linear part of the four-dimensional dynamical system on M_0. The nonlinear part of the system supplements these cases with nonlocal and local bifurcations by means of the degeneracy of some nonlinear terms,[13,21] which can only increase the codimension of the bifurcation.

As mentioned in the preceding section, in the case when in the dynamical system (2) and (3) the Fréedericksz effect equation is separated into an equation that is independent of the other modes $u_1^{(1)}$, the dimension of the central manifold M_2 of the remaining dynamical system does not exceed three and the number of different types of bifurcations with codimension $\geqslant$2 decreases to three:

codimension = 2

1. Stationary-oscillatory EHD effect
2. Doubly degenerate stationary EHD effect

codimension = 3

3. Triply degenerate stationary EHD effect

As before, the three types of codimension-1 bifurcations enumerated in Eqs. (16)–(18) remain.

As shown in Refs. 6 and 16, the development of the EHD effect in p- and h-NLCs exhibits characteristics which are peculiar to each type of crystal. For example, h-NLCs are convenient for observing EHD oscillations and the low threshold E_S of the stationary EHD effect makes p-NLCs convenient for observing Williams domains. For this reason, it makes sense to find separately the necessary conditions for the existence of bifurcations (19)–(26) at the threshold of the EHD effect in planar and homeotropic uniaxial nematics with the help of Eq. (9), using the expressions derived in Ref. 16 for the threshold fields E_F, E_S, and E_0. We introduce the following notation to be employed below:

$$B = \sum_{i=1} B_i = \alpha_1 + 2(\alpha_3 + \alpha_4 + \alpha_5), \quad l = \frac{L}{L_0}, \quad m = \frac{H}{H_F},$$
$$H_F = q_z\left(\frac{K}{\chi_a}\right)^{1/2}, \quad \nu = \frac{\rho K}{|\alpha_2| B_2} \sim 10^{-4},$$
$$L_0 = \frac{1}{2}\left(\pi\frac{\varepsilon_\perp}{\sigma_\perp}\frac{K}{|\alpha_2|}\right)^{1/2} \sim 10\ \mu\text{m},$$
$$A_p = \frac{\sigma_\parallel}{\sigma_\perp} + \frac{B_1 + B_4}{B_2} - 2\frac{\alpha_3}{B_2},$$
$$A_h = \zeta\frac{\sigma_\perp}{\sigma_\parallel} + \frac{B_1 + B_4}{B_2} - 2\frac{\alpha_3}{B_2}, \tag{27}$$
$$\zeta = 1 + \frac{\alpha_2}{B_2}, \quad U = \frac{\varepsilon_a\sigma}{\varepsilon\sigma_a}\nu^{-1}\frac{B_3}{\alpha_3}, \quad V = \frac{\varepsilon_a\sigma}{\varepsilon\sigma_a}\nu^{-1}\frac{\zeta}{1-\zeta};$$

L_0 and ν are, respectively, the characteristic thickness of the NLC slab and the smallness parameter which were introduced in Ref. 6, and H_F is the Fréedericksz threshold in a magnetic field.

PLANAR NEMATIC

Orientational-stationary EHD effect

$$\frac{\sigma_a\varepsilon}{\sigma\varepsilon_a} = 1 + \frac{B}{\alpha_2}\left[1 - \frac{\sigma_\perp + \sigma_\parallel}{\sigma_\perp}\left(1 + \frac{K_3}{K_1}\frac{1}{m^2\cos 2\lambda + 1}\right)\right]. \tag{19p}$$

Orientational-oscillatory EHD effect

$$\nu(m^2\cos 2\lambda + 1)^2 + [2 + (\nu + \frac{\alpha_3}{\alpha_2}U^{-1})l^2](m^2\cos 2\lambda + 1)$$
$$+ l^2 - (A_p + \frac{K_3}{K_1}l^{-2})U = 0\,. \tag{20p}$$

Stationary-oscillatory EHD effect

$$\frac{\sigma_a\varepsilon}{\sigma\varepsilon_a} = 1 + \frac{B}{\alpha_2} + 2\nu\frac{K_3/K_1 + 1 + m^2\cos 2\lambda}{1 + \alpha_2/B}. \tag{21p}$$

Doubly degenerate stationary EHD effect

$$\frac{\sigma_a\varepsilon}{\sigma\varepsilon_a} = 1 + \frac{l^2(1 + B/\alpha_2)}{l^2(1 + \alpha_2/B) + K_3/K_1 + 1 + m^2\cos 2\lambda}. \tag{22p}$$

Orientational-stationary-oscillatory EHD effect

$$\frac{\sigma_a\varepsilon}{\sigma\varepsilon_a} = 1 + 2\nu\frac{K_3}{K_1}\left(1 + \frac{\alpha_2}{B}\right)^{-1} - \frac{B}{\alpha_2}\frac{\sigma_\parallel}{\sigma_\perp},$$
$$1 + m^2\cos 2\lambda = -\frac{1}{2\nu}\frac{\sigma_\parallel + \sigma_\perp}{\sigma_\perp}\left(1 + \frac{B}{\alpha_2}\right). \tag{23p}$$

Orientational-doubly degenerate stationary EHD effect

$$\frac{\sigma_a\varepsilon}{\sigma\varepsilon_a} = 1 - \frac{\sigma_\parallel}{\sigma_\perp}\frac{l^2(1 + B/\alpha_2)}{l^2(1 + \alpha_2/B) + K_3/K_1},$$
$$1 + m^2\cos 2\lambda = -\frac{\sigma_\parallel + \sigma_\perp}{\sigma_\parallel}\left[\frac{K_3}{K_1} + l^2\left(1 + \frac{\alpha_2}{B}\right)\right]. \tag{24p}$$

Triply degenerate stationary EHD effect

$$\frac{\sigma_a\varepsilon}{\sigma\varepsilon_a} = 1 - 2\nu l^2,$$
$$1 + m^2\cos 2\lambda = -\frac{K_3}{K_1} - \left(1 + \frac{B}{\alpha_2}\right)\left(\frac{1}{2\nu} + \frac{\alpha_2}{B}l^2\right). \tag{25p}$$

Orientational-triply degenerate stationary EHD effect

$$\frac{\sigma_a \varepsilon}{\sigma \varepsilon_a} = 1 - \frac{\sigma_\parallel}{\sigma_\perp} \frac{B}{\alpha_2} + 2\nu \frac{K_3}{K_1}\left(1 + \frac{\alpha_2}{B}\right)^{-1},$$

$$1 + m^2 \cos 2\lambda = -\frac{1}{2\nu} \frac{\sigma_\parallel + \sigma_\perp}{\sigma_\perp}\left(1 + \frac{B}{\alpha_2}\right), \tag{26p}$$

$$l^2 = \frac{1}{2\nu} \frac{\sigma_\parallel}{\sigma_\perp} \frac{B}{\alpha_2} - \frac{K_3}{K_1}\left(1 + \frac{\alpha_2}{B}\right)^{-1}.$$

The physical conditions of the problem ($l^2 > 0$) impose stringent conditions on the Leslie coefficients of viscosity, following from Eq. (26p)

$$1 > \frac{B}{|\alpha_2|} > 1 - 2\nu \frac{\sigma_\perp}{\sigma_\parallel} \frac{K_3}{K_1}; \tag{28}$$

assuming that for the known nematics $a_2 < 0$. We do not know of any NLCs whose parameters satisfy the condition (28). For all NLCs described in the literature $B > |a_2|$, i.e., local codimension-4 bifurcations of the type (26p) cannot be observed. On the other hand, the five independent Leslie coefficients of viscosity a_i appearing in Eqs. (27) and (28) can, in principle, ensure that the conditions (28) are satisfied.

In the parametric space ($\sigma_a \varepsilon / \sigma \varepsilon_a, m^2, l^2$) the relations (19$p$)–(22$p$) define a surface (codimension = 2), the relations (23p)–(25p) define a line (codimension = 3), and the relation (26p) defines a point (codimension = 4). We note that in order for the conditions (19)–(26) to be satisfied it is essential to include in the model of the EHD effect the dielectric anisotropy ε_a and the magnetic field **H**, i.e., these parameters, together with the magnitude E of the electric field and the reduced thickness l of the liquid crystal slab, are essential parameters. For a dielectrically isotropic NLC, in the absence of a magnetic field, as follows from the relations (19p)–(26p), all bifurcations with codim$\geqslant$2 are forbidden for all known uniaxial nematics.

HOMEOTROPIC NEMATIC

Orientational-stationary EHD effect

$$\frac{\sigma_a \varepsilon}{\sigma \varepsilon_a} = -1 + \frac{B_3}{\alpha_2} + \frac{B_3}{\alpha_3} \frac{\alpha_2/\alpha_3 - m^2 \cos 2\lambda}{1 - m^2 \cos 2\lambda}. \tag{19h}$$

Orientational-oscillatory EHD effect

$$\nu(m^2 \cos 2\lambda - 1)^2 + l^2 \nu^{-1}(m^2 \cos 2\lambda - 1)$$

$$- \zeta l^2 - \left(A_h + \frac{K_1}{K_3} l^{-2}\right) = 0. \tag{20h}$$

Stationary-oscillatory EHD effect

$$\frac{\sigma_a \varepsilon}{\sigma \varepsilon_a} = -1 + \frac{B_3}{\alpha_2} + \frac{B_3}{\alpha_3} \frac{\alpha_2/\alpha_3 - m^2 \cos 2\lambda}{(\alpha_2/\alpha_3)(1 + \nu^{-1}) - m^2 \cos 2\lambda + l^2}. \tag{21h}$$

Doubly degenerate stationary EHD effect

$$\frac{\sigma_a \varepsilon}{\sigma \varepsilon_a} = -1 + \frac{B_3}{\alpha_2} + \frac{B_3}{\alpha_3} \frac{\alpha_2/\alpha_3 - m^2 \cos 2\lambda}{\alpha_2/\alpha_3 - m^2 \cos 2\lambda + l^2}.$$

Orientational-stationary-oscillatory EHD effect

$$\frac{\sigma_a \varepsilon}{\sigma \varepsilon_a} = -1 + \frac{B_3}{\alpha_2} + \frac{B_3}{\alpha_3} \frac{\alpha_2/\alpha_3 - m^2 \cos 2\lambda}{1 - m^2 \cos 2\lambda}, \tag{23h}$$

$$l^2 = 1 - \frac{\alpha_2}{\alpha_3}(1 + \nu^{-1}).$$

Orientational-doubly degenerate stationary EHD effect

$$\frac{\sigma_a \varepsilon}{\sigma \varepsilon_a} = -1 + \frac{B_3}{\alpha_2} + \frac{B_3}{\alpha_3} \frac{\alpha_2/\alpha_3 - m^2 \cos 2\lambda}{1 - m^2 \cos 2\lambda}, \tag{24h}$$

$$l^2 = 1 - \frac{\alpha_2}{\alpha_3}.$$

Triply degenerate stationary EHD effect

$$\frac{\sigma_a \varepsilon}{\sigma \varepsilon_a} = -1 + \frac{B_3}{\alpha_2} + \frac{B_3}{\alpha_3} \frac{\alpha_2/\alpha_3 - m^2 \cos 2\lambda}{\alpha_2/\alpha_3 - m^2 \cos 2\lambda + l_2}, \quad \nu^{-1} = 0. \tag{25h}$$

This bifurcation in an h-NLC is also ensured by the very unusual condition

$$\frac{\sigma_a \varepsilon}{\sigma \varepsilon_a} = -1 + \frac{B_3}{\alpha_2} + \frac{B_3}{\alpha_3} \frac{m^2 \cos 2\lambda}{m^2 \cos 2\lambda - l^2}, \quad \frac{\alpha_2}{\alpha_3} = 0.$$

The last condition is excluded in the traditional rod-shaped uniaxial nematic, but it is apparently satisfied in an exotic disk-shaped uniaxial nematic.[22,23]

Orientational-triply degenerate stationary EHD effect

$$\frac{\sigma_a \varepsilon}{\sigma \varepsilon_a} = -1 + \frac{B_3}{\alpha_2} + \frac{B_3}{\alpha_3} \frac{\alpha_2/\alpha_3 - m^2 \cos 2\lambda}{1 - m^2 \cos 2\lambda}, \tag{26h}$$

$$l^2 = 1 - \frac{\alpha_2}{\alpha_3}, \quad \nu^{-1} = 0.$$

The requirement $l^2 > 0$ excludes from the list (19h)–(26h) of bifurcations at the threshold of the EHD effect in a h-NLC all bifurcations with codimension$\geqslant$3. In the parameter space ($\sigma_a \varepsilon / \sigma \varepsilon_a, m^2, l^2$) the relations (19$h$)–(22$h$) define nonintersecting (in the region $l^2 > 0$) surfaces (codimension = 2). Just as in a p-NLC, in a dielectrically isotropic h-NLC in the absence of a magnetic field all bifurcations (19h)–(26h) with codimension $\geqslant$2 are forbidden for all known nematics with realistic values of the parameters.

The next step, after the degeneracy conditions (16)–(26) of the linear part of the dynamical system have been determined, is to study the restructuring of the phase portrait of the dynamical system near the position of equilibrium. For codimension-2 bifurcations the use of the typical bifurcation diagrams given in Ref. 13 for the types (19) and (22) and in Ref. 24 for the types (20) and (21) greatly simplify the problem. Codimension-3 and -4 bifurcations were discussed in Ref. 25. The results obtained in Ref. 21 for the critical exponents also are relevant here—the maximum ex-

TABLE I.

Equation No.	(16)	(17)	(18)	(19)	(20)	(21)	(22)	(23)	(24)	(25)
η_-	0,5	0,5	0,5	0,75	0,5	0,5	0,75	0,75	0,833	0,833

ponents η_- of hard loss of stability (creation of an unstable limit cycle with radius $r \propto \beta^{\eta_-}$, where β is the supercriticality of the bifurcation parameter, near the position of equilibrium) of the positions of equilibrium considered. Table I gives the exponents η_- for all critical cases with codimension $\leqslant 3$. The numbering of the entries follows the list of bifurcations (16)–(25). Soft loss of stability (creation of a stable limit cycle) from all cases (16)–(25) considered above is possible only for the oscillatory EHD effect with the corresponding exponent $\eta_+ = 0.5$. In all other cases soft loss of stability is impossible, but it happens when the nonlinear part of the dynamical system is degenerate. The critical exponents $\eta_\pm$ were not calculated for codimension-4 bifurcations.

CONCLUSION

An infinite-dimensional dynamical system was constructed on the basis of the linear theory of EHD instability of a uniaxial nematic in a low-frequency electric field and the central manifold M_0 of the system was found. It was shown that the dimension of the smooth manifold M_0 does not exceed four, i.e., a four-dimensional phase space, constructed on the basis of the single-mode approximation, is sufficient for investigating the nonlinear properties of the electrohydrodynamics of a uniaxial NLC near the threshold of the EHD effect. From the physical standpoint, this determines the order in which the spatial modes of the EHD instability in a uniaxial nematic are excited at a transition to turbulence. The parameter space of the problem is four-dimensional. The essential parameters of the problem—the intensities of the electric and magnetic fields, the reduced thickness l of the NLC slab, and the anisotropy parameter $\sigma_a \varepsilon/\sigma\varepsilon_a$—can be chosen as the generators of this space.

Two types of local bifurcations with codimension $\leqslant 4$, which are possible at the threshold of the EHD effect, were described taking into account the Fréedericksz effect and its influence on the EHD effect in an NLC in the region where the threshold fields of these two effects are close to one another. The nonlinear part of the dynamical system expands this list due to nonlocal and local bifurcations via the degeneracy of some nonlinear terms which increase the codimension of the bifurcation. The orientation of a uniaxial nematic at the boundaries of the liquid-crystal slab is important for observing different types of bifurcations at the threshold of the EHD effect. For known NLCs with the planar orientation local bifurcations with codimension $\leqslant 3$ are allowed, while in NLCs with homeotropic orientation local bifurcations with codimension $\leqslant 2$ are allowed. The dielectric anisotropy of NLCs and the presence of a magnetic field are essential for observing high-dimensional bifurcations at the threshold of the EHD effect. The absence of these two factors in the problem reduces significantly the list of degenerate states—only codimension-1 local bifurcations are allowed, irrespective of the type of boundary orientation of the uniaxial nematic. For the types of bifurcations found in this work, the critical exponents $\eta_\pm$ of instability of the equilibrium position accompanying the creation of a limit cycle near this equilibrium position was presented.

This work is the preparatory stage for constructing a nonlinear theory of EHD phenomena in a uniaxial nematic.

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APPENDIX

We present below expressions for the threshold fields $E_s^{(k)}$ and $E_o^{(k)}$ of the corresponding modulated EHD structures for the p and h orientations of a uniaxial nematic liquid crystal. Without loss of generality, we confine our attention to the case of dielectric isotropy.[6,16] For stationary EHD instability

$$E_{sp}^{(k)^2} \simeq \frac{B}{\sigma_a}\frac{\sigma_\parallel + \sigma_\perp}{\sigma_\perp}\frac{1}{\tau_m\tau_o}\left(1+\frac{K_3}{K_1}\right)\left[k^2+\frac{m^2\cos 2\lambda}{1+K_3/K_1}\right], \tag{A1}$$

$$E_{sh}^{(k)^2} \simeq \frac{B_3}{\sigma_a}\left(\frac{\alpha_2}{\alpha_3}\right)^2\frac{1}{\tau_m\tau_o}\left[k^2-\frac{\alpha_3}{\alpha_2}m^2\cos 2\lambda\right], \tag{A2}$$

where the characteristic Maxwell relaxation time τ_m, hydrodynamic relaxation time τ_h, and elasto-orientational relaxation time τ_o of the director $\mathbf{n}$ in the NLC are defined as

$$\tau_m = \frac{\varepsilon}{4\pi\sigma}, \quad \tau_h = \frac{\rho}{B_2 q_z^2}, \quad \tau_o = \frac{\alpha}{K q_z^2}.$$

All other parameters are defined in the text. As follows from Eqs. (A1) and (A2), these expressions can be put into the form Eq. (13).

For the oscillatory type of EHD instability we have

$$E_{op}^{(k)^2} \simeq \frac{B_2}{\sigma_a}\left(\frac{\alpha_2}{\alpha_3}\right)^2\frac{k^2}{\tau_m\tau_h}\left[1+\frac{k^2+m^2\cos 2\lambda}{l^2}\right], \tag{A3}$$

$$E_{oh}^{(k)^2} \simeq \frac{B_2}{\sigma_a}\zeta\frac{k^2}{\tau_m\tau_h}\left[\zeta+\frac{k^2-m^2\cos 2\lambda}{l^2}\right]. \tag{A4}$$

It is easy to show that Eqs. (A3) and (A4) can also be put into the form (15). When the permittivity anisotropy of the NLC is taken into account, the form of the expressions for $E^{(k)}$ in Eqs. (12) and (14) remains the same but the constants C_i are normalized.

[1)]It is interesting that this bifurcation accompanies the codimension-4 bifurcation which has not been noticed in Ref. 12 and corresponds to the doubly degenerate stationary type with additional degeneracy of two quadratic terms, which can be easily verified by comparing the amplitude equation obtained in Ref. 12 to the typical dynamical system for the bifurcation mentioned [see Eqs. (2.16) and (2.25) in Ref. 13].

[2)]Among the twelve numerical parameters describing the development of EHD instability in a uniaxial nematic,[19] some parameters, for example, the anisotropy of the elastic properties of the nematic K_3/K_1, are not important.

[3)]The signature of the metric of the $(n_+ + n_-)$-dimensional surface of this saddle includes n_+ positive and n_- negative signs.

[4)]The component $\theta_1^{(1)}$ of the vector $u_1^{(1)}$ is required to be cyclical $\theta_1^{(1)} + \pi \rightarrow \theta_1^{(1)}$. For this reason, on the whole, M_0 can be homeomorphic to a piece of a cylindrical surface.

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