Polarization effects in the reflection of electromagnetic waves from a randomly inhomogeneous plane-layer medium

S. M. Golynskiĭ and V. D. Gusev

Department of Atmospheric Physics, Physics Faculty, M. V. Lomonosov State University, 117465, Moscow

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Plane electromagnetic waves obliquely incident on a randomly inhomogeneous one-dimensional layer are analyzed. A modified traveling-wave method is used to describe the propagation of horizontally and vertically polarized waves in the layer. Statistical characteristics of the transmission and reflection coefficients of the layer are derived in the Markovian diffusion approximation for the case in which the fluctuations in the dielectric constant of the medium are small. These statistical characteristics contain comprehensive information on the polarization of the transmitted and reflected radiation. It is shown in particular that under certain conditions the radiation reflected from a randomly inhomogeneous layer is completely polarized.

INTRODUCTION

The propagation of waves in a randomly varying one-dimensional medium has been the subject of many papers, among which we will cite only the two[1,2] of greatest importance to the problem of the present paper. The great preponderance of this research has been limited to the scalar problem. At the same time, we know[3,4] that when electromagnetic waves are incident obliquely on an inhomogeneous medium there are differences in the reflection of waves of different polarizations. It is thus worthwhile to take the vector nature of the electromagnetic radiation into account and to go through the analysis for the case in which an incident wave is reflected from a randomly varying plane-layer medium.

Let us consider a one-dimensional, randomly varying layer which occupies the region $0<z<L$. Under the assumption of a quasisteady state, the dielectric constant of the medium inside the layer is described by $\varepsilon(z)=\varepsilon_0+\alpha\varepsilon_1(z)$, where $\varepsilon_0=\text{const}$, α is a small dimensionless parameter which is a characteristic of the fluctuation depth ($\alpha\sim\sigma_\varepsilon=\sqrt{\langle\varepsilon_1^2\rangle}$), and $\varepsilon_1(z)$ is a random, uniform, and normal process with a zero mean and a correlation function $B(\xi)$:

$$\langle\varepsilon_1(z)\rangle = 0, \quad B(\xi) = \langle\varepsilon_1(z_1)\varepsilon_1(z_2)\rangle = \sigma_\varepsilon^2 K(\xi), \qquad \xi = z_1 - z_2.$$

Here σ_ε^2 is the variance of the fluctuations in the dielectric constant of the medium, and the angle brackets mean an average over the ensemble of realizations. To streamline the calculations we assume $\varepsilon(z)=1$ outside the layer.

We will refer to a plane wave whose electric vector **E** is perpendicular to the plane of incidence, $XZ(E_x=E_z=0, E_y\neq 0)$, as a "horizontally polarized" wave. Correspondingly, a wave whose magnetic field vector **H** is perpendicular to the plane of incidence ($H_x=H_z=0$, $H_y\neq 0$) is "vertically polarized."

2. DESCRIPTION OF THE PROPAGATION OF A HORIZONTALLY POLARIZED WAVE

The field of a horizontally polarized wave, $E_y(x,z,t)$, inside the layer is described by the wave equation

$$\frac{\partial^2 E_y}{\partial x^2} + \frac{\partial^2 E_y}{\partial z^2} - \frac{\varepsilon}{c^2}\frac{\partial^2 E_y}{\partial t^2} = 0, \tag{2.1}$$

where c is the velocity of light in vacuum. We assume that a monochromatic plane wave of unit amplitude $E_{y0}=\exp\{-i[\omega t - kzc_0 - kxs_0]\}$ is incident at an angle θ_0 on the layer along the positive z direction, where ω is the angular frequency of the incident radiation, $k=\omega/c$, $c_0=\cos\theta_0$, $s_0=\sin\theta_0$. As a result, a wave $E_{yT}=T_1\exp\{-i[\omega t - kzc_0 - kxs_0]\}$ arises in the region $z>L$, having been transmitted through the layer, while a wave $E_{yR}=R_1\exp\{-i[\omega t + kzc_0 - kxs_0]\}$, reflected from the layer, appears in the region $z<0$. The unknown amplitudes T_1 and R_1 determine the complex transmission and reflection coefficients of the layer for the horizontally polarized wave.

The solution of Eq. (2.1) inside the layer can be written in traveling-wave form: $E_y=A_1(z)\exp\{-i[\omega t \pm k\Phi_1(z) - kxs_0]\}$ (Refs. 5 and 6; these are passing or progressive waves[7]). The real amplitude A_1 and the eikonal Φ_1 are related by

$$A_1 = \text{const}(\Phi_1')^{-1/2} \tag{2.2}$$

and satisfy the equation

$$A_1'' + k^2[\varepsilon - s_0^2 - (\Phi_1')^2]A_1 = 0, \tag{2.3}$$

where the prime means differentiation with respect to the variable z.

Introducing the new variable

$$Q_1 = (\Phi_1')^{-1} = \text{const}\, A_1^2, \tag{2.4}$$

we can put the general solution of our original wave equation, (2.1), in the form of a sum of two linearly independent traveling waves.[5,6] These waves are propagating in opposite directions and do not interact with each other:[3]

$$E_y(x,z,t) = [C_1 Q_1^{1/2}\exp\{ik\Phi_1\} + C_2 Q_1^{1/2}\exp\{-ik\Phi_1\}]\exp\{-i[\omega t - kxs_0]\}, \tag{2.5}$$

where C_1 and C_2 are constants of integration. In this problem, these constants are determined by the conditions that the field E_y and its derivative E_y' are continuous at the boundaries of the layer,

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$$z = 0: \quad 1 + R_1 = Q_{10}^{1/2}(C_1 + C_2),$$

$$ikc_0(1 - R_1) = \frac{1}{2} Q_{10}^{-1/2} Q'_{10}(C_1 + C_2) + ikQ_{10}^{-1/2}(C_1 - C_2);$$

$$z = L: \quad T_1 \exp\{ikc_0 L\} = Q_{1L}^{1/2}[C_1 \exp\{ik\Phi_{1L}\} + C_2 \exp\{-ik\Phi_{1L}\}],$$

$$ikc_0 T_1 \exp\{ikc_0 L\} = \frac{1}{2} Q_{1L}^{-1/2} Q'_{1L}[C_1 \exp\{ik\Phi_{1L}\} + C_2 \exp\{-ik\Phi_{1L}\}] + ikQ_{1L}^{-1/2}[C_1 \exp\{ik\Phi_{1L}\} - C_2 \exp\{-ik\Phi_{1L}\}]. \quad (2.6)$$

Here and below, the subscripts 0 and L specify the values of the corresponding functions at the lower boundary $z = 0$ and the upper boundary $z = L$ of the layer.

Inside the layer, the field of the horizontally polarized wave, (2.5), is thus determined by the one independent function $Q_1(z)$ in (2.4). For this function, we find a nonlinear second-order differential equation after substituting (2.2) into (2.3):

$$2Q_1 Q_1'' - (Q_1')^2 + 4k^2(\varepsilon - s_0^2)Q_1^2 - 4k^2 = 0. \quad (2.7)$$

In general, making an unambiguous choice of a particular solution of (2.7) to describe the wave field (2.5) would require supplementing this equation with two additional conditions (initial conditions, boundary conditions, or conditions of any other sort which follow from physical considerations regarding the particular problem involved). Under the assumption that the fluctuations of the dielectric constant of the medium are small ($\alpha \ll 1$), we would like to use initial conditions at the lower boundary of the layer for the corresponding unperturbed problem in order to solve Eq. (2.7). In the limiting case of wave propagation in a homogeneous medium ($\alpha = 0, \varepsilon = \varepsilon_0$), the unknown function $Q_1(z)$ is determined unambiguously; specifically, $Q_1 = (\varepsilon_0 - s_0^2)^{-1/2} = \text{const}$. Accordingly, an analysis of the general solutions of (2.7) in the case $\varepsilon(z) = \varepsilon_0$ reveals that the choice of initial conditions is unique:

$$Q_{10} = (\varepsilon_0 - s_0^2)^{-1/2}, \quad Q'_{10} = 0. \quad (2.8)$$

We write a formal solution of (2.7) in the form

$$Q_1 = Q_{10}[u_1 + (u_1^2 - 1)^{1/2}\cos\psi_1],$$
$$Q_1' = -2\tilde{k}Q_{10}(u_1^2 - 1)^{1/2}\sin\psi_1, \quad (2.9)$$

where $\tilde{k} = k\tilde{\varepsilon}^{1/2}$, $\tilde{\varepsilon} = \varepsilon_0 - s_0^2$, and the two new functions $u_1(z)$ and $\psi_1(z) = 2\tilde{k}z + \varphi_1(z)$ are described by a known system of first-order differential equations which has been constructed previously by other approaches:[1,2]

$$u_1' = \alpha\frac{\tilde{k}\varepsilon_1}{\tilde{\varepsilon}}(u_1^2 - 1)^{1/2}\sin\psi_1, \quad u_1(z = 0) = 1,$$

$$\psi_1' = 2\tilde{k} + \alpha\frac{\tilde{k}\varepsilon_1}{\tilde{\varepsilon}}[1 + u_1(u_1^2 - 1)^{-1/2}\cos\psi_1], \quad (2.10)$$

$$\psi_1(z = 0) = \psi_{10}.$$

With (2.8), the solution of system of equations (2.6) determines expressions for the complex reflection and transmission coefficients of the layer for a horizontally polarized wave:

$$R_1 = -\frac{[1 - c_0Q_{10}][Q'_{1L} - 2ik(1 + c_0Q_{1L})]e^{-ik\Phi_{1L}} + [1 + c_0Q_{10}][Q'_{1L} + 2ik(1 - c_0Q_{1L})]e^{ik\Phi_{1L}}}{[1 + c_0Q_{10}][Q'_{1L} - 2ik(1 + c_0Q_{1L})]e^{-ik\Phi_{1L}} + [1 - c_0Q_{10}][Q'_{1L} + 2ik(1 - c_0Q_{1L})]e^{ik\Phi_{1L}}},$$

$$T_1 = -\frac{8ikc_0Q_{10}^{1/2}Q_{1L}^{1/2}e^{-ikc_0L}}{[1 + c_0Q_{10}][Q'_{1L} - 2ik(1 + c_0Q_{1L})]e^{-ik\Phi_{1L}} + [1 - c_0Q_{10}][Q'_{1L} + 2ik(1 - c_0Q_{1L})]e^{ik\Phi_{1L}}}. \quad (2.11)$$

In the particular case $\varepsilon_0 = 1$, i.e., $\tilde{k} = kc_0$ and $Q_{10} = c_0^{-1}$, these expressions simplify substantially:

$$R_1 = -\frac{Q'_{1L} + 2ik(1 - c_0Q_{1L})}{Q'_{1L} - 2ik(1 + c_0Q_{1L})}e^{2ik\Phi_{1L}},$$

$$T_1 = -\frac{4ikc_0^{1/2}Q_{1L}^{1/2}}{Q'_{1L} - 2ik(1 + c_0Q_{1L})}e^{-ik(c_0L - \Phi_{1L})}. \quad (2.12)$$

For the square moduli of R_1 and T_1, which characterize the reflection and transmission capabilities of the randomly inhomogeneous layer, we find, using (2.9),

$$|R_1|^2 = \frac{(Q'_{1L})^2 + 4k^2(1 - c_0Q_{1L})^2}{(Q'_{1L})^2 + 4k^2(1 + c_0Q_{1L})^2} = \frac{u_1 - 1}{u_1 + 1},$$

$$|T_1|^2 = \frac{16k^2c_0Q_{1L}}{(Q'_{1L})^2 + 4k^2(1 + c_0Q_{1L})^2} = \frac{2}{u_1 + 1}, \quad (2.13)$$

i.e., $|R_1|^2 + |T_1|^2 = I$. This is a natural consequence of the conservation of the energy flux when there is no absorption in the medium.

In the case with $\varepsilon_0 = 1$, which is the simplest from the standpoint of an analytical solution, we can find several statistical characteristics of wave field (2.5), including the reflection and transmission capabilities of the layer, (2.13), simply by examining the distribution of the random function $u_1(z)$ in (2.10).

To conclude this section of the paper we note that the function $m_1(z) = Q_1^{-1}(z)$ serves as an effective refractive index of the medium inside the randomly varying layer.[8] This index uniquely determines the wave field (2.5). The difference between the effective refractive index $m_1(z)$ and the local index $n(z) = \varepsilon^{1/2}(z)$ stems from the retention of the first term in Eq. (2.3) in our approach. That term is ignored in the geometric-optics approximation and various modifications thereof.

3. DESCRIPTION OF THE PROPAGATION OF A VERTICALLY POLARIZED WAVE

For a vertically polarized wave, the symmetry of the problem suggests analyzing the wave equation for the component $H_y\,(x,z,t)$ of the magnetic vector:

$$\frac{\partial^2 H_y}{\partial x^2}+\frac{\partial^2 H_y}{\partial z^2}+\frac{\varepsilon'}{\varepsilon}\frac{\partial H_y}{\partial z}-\frac{\varepsilon}{c^2}\frac{\partial^2 H_y}{\partial t^2}=0. \tag{3.1}$$

We again assume that a monochromatic plane wave of unit amplitude, $H_y=\exp\{-i[\omega t-kzc_0-kxs_0]\}$, is incident on the layer at an angle θ_0. The wave transmitted through the layer and the wave reflected by it are characterized by complex transmission and reflection coefficients T_2 and R_2, respectively. In contrast with the case discussed in the preceding section of this paper, the real amplitude $A_2(z)$ and the real eikonal $\Phi_2(z)$ in the traveling-wave solution of (3.1) are related by

$$A_2=\text{const}\,\varepsilon^{1/2}(\Phi_2')^{-1/2}.$$

Introducing the new variable $Q_2=(\Phi_2')^{-1}=\text{const}\,A_2^2$, which satisfies the nonlinear differential equation

$$2Q_2Q_2''-(Q_2')^2+Q_2^2\left[4\tilde{k}^2+2\frac{\varepsilon''}{\varepsilon}-3\left(\frac{\varepsilon'}{\varepsilon}\right)^2\right]-4k^2=0,$$
$$Q_{20}=\tilde{\varepsilon}^{-1/2},\qquad Q_{20}'=0, \tag{3.2}$$

we can write the general solution of wave equations (3.1) as

$$H_y(x,z,t)=[C_3Y^{1/2}\exp\{ik\Phi_2\}+C_4Y^{1/2}\exp\{-ik\Phi_2\}]\exp\{-i[\omega t-kxs_0]\}.$$

Here $Y=\varepsilon Q_2$, and C_3 and C_4 are constants of integration, which are determined by the conditions that H_y and H_y'/ε are continuous at both boundaries of the layer,

$$z=0:\quad 1+R_2=Y_0^{1/2}(C_3+C_4),$$
$$ikc_0(1-R_2)=\frac{1}{2}\varepsilon_0^{-1}Y_0^{-1/2}Y_0'(C_3+C_4)+ikY_0^{-1/2}(C_3-C_4);$$
$$z=L:\quad T_2\exp\{ikc_0L\}=Y_L^{1/2}[C_3\exp\{ik\Phi_{2L}\}+C_4\exp\{-ik\Phi_{2L}\}],$$
$$ikc_0T_2\exp\{ikc_0L\}=\frac{1}{2}\varepsilon_L^{-1}Y_L^{-1/2}Y_L'[C_3\exp\{ik\Phi_{2L}\}+C_4\exp\{-ik\Phi_{2L}\}]+ikY_L^{-1/2}[C_3\exp\{ik\Phi_{2L}\}-C_4\exp\{-ik\Phi_{2L}\}]. \tag{3.3}$$

From the system of equations (3.3) we find the complex reflection and transmission coefficients of the layer for a vertically polarized wave:

$$R_2=-\frac{[Y_0'+2i\Lambda_0(1-c_0Y_0)]\,[Y_L'-2i\Lambda_L(1+c_0Y_L)]\exp(-ik\Phi_{2L})}{[Y_0'+2i\Lambda_0(1+c_0Y_0)]\,[Y_L'-2i\Lambda_L(1+c_0Y_L)]\exp(-ik\Phi_{2L})}$$
$$\frac{[Y_0'-2i\Lambda_0(1+c_0Y_0)]\,[Y_L'+2i\Lambda_L(1-c_0Y_L)]\exp(ik\Phi_{2L})}{[Y_0'-2i\Lambda_0(1-c_0Y_0)]\,[Y_L'+2i\Lambda_L(1-c_0Y_L)]\exp(ik\Phi_{2L})},$$
$$T_2=\frac{8\Lambda_0\Lambda_LY_0^{1/2}Y_L^{1/2}\exp\{-ikc_0L\}}{[Y_0'+2i\Lambda_0(1+c_0Y_0)]\,[Y_L'-2i\Lambda_L(1+c_0Y_L)]\exp(-ik\Phi_{2L})}$$
$$\overline{[Y_0'-2i\Lambda_0(1-c_0Y_0)]\,[Y_L'+2i\Lambda_L(1-c_0Y_L)]\exp(ik\Phi_{2L})}, \tag{3.4}$$

where $\Lambda=k\varepsilon$.

To pursue the analysis of this problem, we need to solve Eq. (3.2). Assuming as before that the fluctuations of the dielectric constant are small ($\alpha\ll 1$), we ignore terms of order α^2 and higher in Eq. (3.2). As a result this equation becomes

$$2Q_2Q_2''-(Q_2')^2+Q_2^2[4\tilde{k}^2+2\varepsilon''/\varepsilon]-4k^2=0 \tag{3.2'}$$

with the same initial conditions.

Writing a formal solution of (3.2') in the form

$$Q_2=Q_{20}[u_2+(u_2-1)^{1/2}\cos\psi_2],$$
$$Q_2'=-2\tilde{k}Q_{20}(u_2-1)^{1/2}\sin\psi_2,$$

we find a system of first-order equations for the new function $u_2(z)$ and $\psi_2(z)=2\tilde{k}z+\varphi_2(z)$. This system of equations is analogous to (2.10):

$$u_2'=\alpha\left\{\frac{\tilde{k}\varepsilon_1}{\tilde{\varepsilon}}+\frac{\varepsilon_1''}{2\tilde{k}\varepsilon_0}\right\}(u_2^2-1)^{1/2}\sin\psi_2,\qquad u_2(z=0)=1,$$
$$\psi_2'=2\tilde{k}+\alpha\left\{\frac{\tilde{k}\varepsilon_1}{\tilde{\varepsilon}}+\frac{\varepsilon_1''}{2\tilde{k}\varepsilon_0}\right\}[1+u_2(u_2^2-1)^{-1/2}\cos\psi_2],$$
$$\psi_2(z=0)=\psi_{20}. \tag{3.5}$$

For the simplest case, with $\varepsilon_0=1$, and in the approximation that the fluctuations of the dielectric constant are small, the following relations are valid:

$$\varepsilon_L\approx 1,\qquad Y_0\approx Q_{20}=c_0^{-1},\qquad Y_0'\approx 0,$$
$$\Lambda=k,\qquad Y_L\approx Q_{2L},\qquad Y_2'\approx Q_{2L}'.$$

As a result, the expressions (3.4) for the complex reflection and transmission coefficients for a vertically polarized wave simplify substantially, becoming formally the same as the corresponding expressions for a horizontally polarized wave, i.e., (2.12) and (2.13), when we replace Q_1 by Q_2 (and u_1 by u_2). In particular, the reflection and transmission capabilities of the layer are given by identical formulas for the waves of the two polarizations:

$$|R_j|^2 = \frac{u_j - 1}{u_j + 1}, \quad |T_j|^2 = \frac{2}{u_j + 1} \quad (j = 1, 2). \quad (3.6)$$

To analyze the polarization characteristics of the radiation reflected and transmitted by a randomly inhomogeneous layer in this approximation, we can thus use the system of stochastic differential equations in (2.10) and (3.5), which can be written in the unified form

$$\begin{aligned} u_j' &= \alpha\Sigma_j(u_j^2 - 1)^{1/2}\sin\psi_j, \quad u_j(0) = 1, \\ \psi_j' &= 2\tilde{k} + \alpha\Sigma_j[1 + u_j(u_j^2 - 1)^{-1/2}\cos\psi_j], \quad \psi_j(0) = \psi_{j0}, \end{aligned} \quad (3.7)$$

where $\Sigma_1 = \tilde{k}\varepsilon_1/\tilde{\varepsilon}$ and $\Sigma_2 = \Sigma_1 + \varepsilon_1''/2\tilde{k}\varepsilon_0$.

As in the case of a horizontally polarized wave, the effective refractive index of the randomly inhomogeneous medium for the vertically polarized wave is determined by the function $m_2(z) = Q_2^{-1}(z)$.

4. FOKKER–PLANCK EQUATION AND ITS RANGE OF APPLICABILITY

The system of stochastic differential equations in (3.7) can be utilized to go over to the Markovian diffusion approximation, specifically, to derive a Fokker–Planck equation for the one-point probability density $P_2(u_j, \psi_j, z)$, $j = 1,2$.

Skipping over the well-studied case of a δ-correlated random field,[1,8] we take up the more realistic model in which $\varepsilon_1(z)$ is a single-scale random process with a finite correlation radius $r_k = a$, where a is a length scale of the variations of the medium along the z direction.

Introducing the two-dimensional vector $v_j = \{u_j, \psi_j\}$, and separating the deterministic and fluctuation terms on the right side of (3.7),

$$\begin{aligned} F_0 &= \{2\tilde{k}, 0\}, \\ F_j &= \{\Sigma_j(u_j^2 - 1)\sin\psi_j,\ \Sigma_j[1 + u_j(u_j^2 - 1)^{-1/2}\cos\psi_j]\}, \end{aligned} \quad (4.1)$$

we can put (3.7) in the standard form of a stochastic differential equation,

$$v_j' = F_0 + \alpha F_j(v_j, z). \quad (4.2)$$

The Fokker–Planck equation corresponding to (4.2), derived by the Van Kampen method,[9] is (a repeated index ν, $\mu = u_j, \psi_j$ means a summation)

$$\frac{\partial P_2(v_j, z)}{\partial z} = -\frac{\partial}{\partial v_\nu}\{B_\nu(v_j)P_2(v_j, z)\} + \frac{1}{2}\frac{\partial^2}{\partial v_\nu \partial v_\mu}\{B_{\nu\mu}(v_j)P_2(v_j, z)\}, \quad (4.3)$$

where the drift coefficients $B_\nu(v_j)$ and the local dispersions $B_{\nu\mu}(v_j)$ are given by [we are using (4.1)]

$$\begin{aligned} B_\nu(v_j) &= F_{0\nu} + \chi^2 \int_0^\infty \left\langle \frac{\partial F_{j\nu}(v_j, z)}{\partial v_\mu} F_{j\mu}(v_j, z - \xi)\right\rangle d\xi, \\ B_{\nu\mu}(v_j) &= 2\chi^2 \int_0^\infty \langle F_{j\nu}(v_j, z) F_{j\mu}(v_j, z - \xi)\rangle d\xi. \end{aligned} \quad (4.4)$$

Here $\chi = \alpha\tilde{k}a/\tilde{\varepsilon}$ is a new dimensionless parameter. In deriving the Fokker–Planck equation we used

$$\langle F_{j\nu}(v_j, z) F_{j\mu}(v_j, z - \xi)\rangle \approx 0 \quad \text{at} \quad \xi \gtrsim a. \quad (4.5)$$

In general, the coefficients of Fokker–Planck equation are given by quite unwieldy expressions, and it is difficult to solve Eq. (4.3) itself even in the simplest limiting cases. However, the situation simplifies substantially when we note that the variable $\psi_j(z)$ has the structure

$$\psi_j(z) = \psi_{j0} + 2\tilde{k}z + \varphi_j(z)$$

in our case, in which the fluctuations of the dielectric are small ($\alpha \ll 1$).

According to (3.7), the stochastic characteristics of the functions $u_j(z)$ and $\varphi_j(z)$ thus vary slowly over distances on the order of the wavelength $\lambda = 2\pi/\tilde{k}$. On the other hand, there are rapidly oscillating functions on the right side of the system of equations (3.7). To determine the slow variations of the statistical characteristics of the weight field (2.5), in particular, the mean square absolute values of the reflection and transmission coefficients (3.6) of the layer, we can thus average the Fokker–Planck equation (4.3) over the period of the fast oscillations.[1] After this average is taken, the coefficients of the Fokker–Planck equation in (4.4) do not depend on the variable ψ_j. In other words, it becomes possible to integrate the equation over this variable. As a result, we find the well-studied one-dimensional Fokker–Planck equation for the probability density[1,2] $P(u_j, z)$:

$$\begin{aligned} \frac{\partial P(u_j, z)}{\partial z} &= D_j \frac{\partial}{\partial u_j}(u_j^2 - 1)\frac{\partial P(u_j, z)}{\partial u_j}, \\ P(u_j, z = 0) &= \delta(u_j - 1), \quad j = 1, 2, \end{aligned} \quad (4.6)$$

where we have again introduced a variable D_j which plays the role of a diffusion coefficient and which is given by

$$D_j = \frac{1}{2}\int_0^\infty \Omega_j(\xi)\cos 2\tilde{k}\xi d\xi, \quad \Omega_j(\xi) = \langle\Sigma_j(z)\Sigma_j(z - \xi)\rangle. \quad (4.7)$$

In the particular case of a Gaussian correlation function of the random field, $\varepsilon_1(z)$, with $B(\xi) = \sigma_\varepsilon^2 \exp\{-\xi^2/a^2\}$, the diffusion coefficients D_j take the following form, when we incorporate the definition of the functions Σ_j in (3.7):

$$D_1 = \frac{1}{4}\pi^{1/2}\sigma_\varepsilon^2 \frac{\tilde{k}^2 a}{\tilde{\varepsilon}} \exp\{-(\tilde{k}a)^2\} \quad (4.8)$$

for a horizontally polarized wave and

$$D_2 = D_1\left[1 - 2\frac{\tilde{\varepsilon}}{\varepsilon_0}\right]^2 \quad (4.9)$$

for a vertically polarized wave.

For a layer with $\varepsilon_0 = 1$, i.e., for $\tilde{\varepsilon} = 1 - s_0^2 = c_0^2$, $\tilde{k} = kc_0$, we find, in accordance with (4.8) and (4.9),

$$D_1 = \frac{1}{4}\pi^{1/2}\sigma_\varepsilon^2\frac{k^2a}{c_0^2}\exp\{-(kc_0a)^2\}, \qquad D_2 = D_1\cos^2 2\theta_0. \tag{4.10}$$

To evaluate the statistical characteristics of (3.6), we use an integral representation[1] of the solution of the Fokker–Planck equation (4.3):

$$P(u_j, t_j) = \int_0^\infty \mu \,\mathrm{th}(\mu\pi)\exp\left\{-\left(\mu^2 + \frac{1}{4}\right)t_j\right\}\wp_{-1/2+i\mu}(u_j)d\mu. \tag{4.11}$$

Here we have introduced the dimensionless variable

$$t_j = D_j z, \tag{4.12}$$

and $\mathscr{P}_{-1/2+i\mu}(u_j)$ is the Legendre function of the first kind (the cone function).

In the derivation of the Fokker–Planck equation (4.3) by the Van Kampen method, an expansion was carried out in the small parameter $\chi = a/l$, where $l = (a\tilde{k}/\tilde{\varepsilon})^{-1}$ is the length scale of the variations in the functions $u_j(z)$ and $\varphi_j(z)$. If $\chi \ll 1$ holds, i.e., if

$$\sigma_\varepsilon \tilde{k}a/\tilde{\varepsilon} \ll 1, \tag{4.13}$$

we can partition the layer thickness L into subintervals Δz such that $\Delta z \gg a$ while $\Delta z \ll l$. Consequently, the functions $u_j(z)$ and $\varphi_j(z)$ are essentially constant in a subinterval Δz, while the random process $\varepsilon_1(z)$ has "completely forgotten its past," (4.5). Collecting these inequalities,

$$L \gtrsim l \gg \Delta z \gg a, \tag{4.14}$$

we find the condition for a two-scale problem—the standard condition in the theory of Markovian diffusion processes.[9,10] Inequalities (4.13) and (4.14) thus determine the range of applicability of Fokker–Planck equation (4.3) and of the results which follow from it, (4.6) and (4.11). The reason is that in terms of the coarse spatial scale characterized by Δz, this process can be approximated as a Markovian process.

The probability density $P(u_j, t_j)$ in (4.11) can also be used to study the statistical characteristics of the average effective refractive index of the medium, $m_{\mathrm{eff}\,j}(z)$. After an average is taken over the period of the fast oscillations in the expressions for the functions $Q_j(z)$, e.g., in the first of Eqs. (2.9), this index is given by[8]

$$m_{\mathrm{eff}\,j}(z) = u_j^{-1}(z). \tag{4.15}$$

5. REFLECTIVITY OF THE LAYER FOR HORIZONTALLY AND VERTICALLY POLARIZED WAVES

In our approximation of small fluctuations of the dielectric constant, the functions $u_j(z)$ and $\varphi_j(z)$ described by (3.7) characterize the magnitude and phase of the reflection coefficient of the layer[1] for waves with different types of polarization. Consequently, the Fokker–Planck equation (4.3) can be used, after an average is taken over the period of the fast oscillations, to analyze the statistical characteristics of these quantities; i.e., it contains a complete solution of the problem of the polarization of waves reflected and transmitted by the randomly inhomogeneous layer. The particular features of the behavior of the horizontally and vertically polarized waves stem from the difference between the diffusion coefficients D_1 in (4.8) and D_2 in (4.9). In the limit of normal incidence ($\theta_0 = 0$), these coefficients are naturally the same, since the components of the vectors **E** and **H** of the waves of different polarizations are parallel to the boundaries of the layer and are physically equivalent.

Using the probability density (4.11), and also using Eqs. (3.6) and (4.15), we find general expressions for the reflectivity of this layer,

$$\langle|R_j|^2\rangle = 1 - 4\pi^{-1/2}\exp(-t_j/4)\int_0^\infty x^2\exp(-x^2)[\mathrm{ch}(xt_j^{1/2})]^{-1}dx \tag{5.1}$$

and for the mean value of its effective refractive index,

$$\langle m_{\mathrm{eff}\,j}\rangle = 2\left(\frac{2}{\pi t_j}\right)^{1/2}\exp(-t_j/4)\int_0^\infty \frac{x\exp(-x^2)}{[\mathrm{ch}\,\omega_j]^{1/2}} \times \ln[(\mathrm{ch}\,\omega_j)^{1/2} + (\mathrm{ch}\,\omega_j - 1)^{1/2}]dx, \tag{5.2}$$

where $\omega_j = 2xt_j$ ($j = 1,2$).

Figure 1 shows plots of the functions $\langle|R|^2\rangle$ (Ref. 2; the solid line) and $\langle m_{\mathrm{eff}}\rangle$ (Ref. 8; the dashed line) versus the dimensionless parameter $t = D_0L$ in (4.12) for the case studied previously, in which a wave is incident normally on the layer ($\theta_0 = 0$), with $D_1 = D_2 = D_0$. It follows from an analysis of the expressions for the diffusion coefficients, (4.8)–(4.10), that the functional dependences remain qualitatively the same in the general case in which the waves of the different polarizations are incident obliquely for a fixed initial angle of incidence on the layer, θ_0. For the particular case $\varepsilon_0 = 1$ [see (4.10)], a vertically polarized wave incident at an angle $\theta_0 = \pi/4$ is an exceptional case, in which we have $D_2 = 0$, i.e., $t_2 = 0$. Consequently, we have $\langle|R_2|^2\rangle = 0$, and $\langle m_{\mathrm{eff}\,2}\rangle = 1$ according to (5.1) and (5.2). Note also that the effective average refractive index which we introduced [see (4.15)] gives a qualitatively correct description of the process of inverse scattering in the randomly varying layer.

To analyze polarization effects for the incidence of hori-

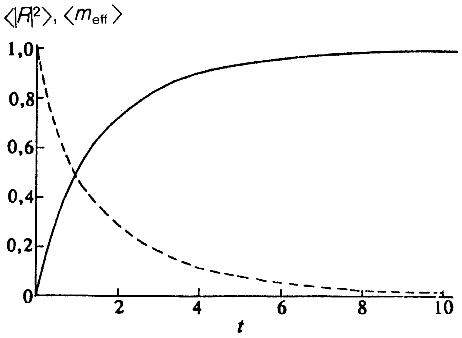

FIG. 1.

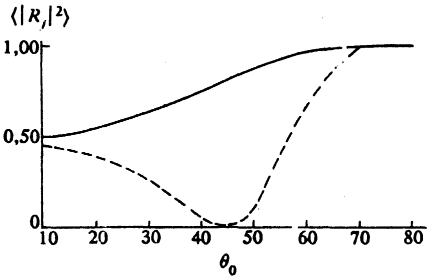

FIG. 2.

zontally and vertically polarized waves on a randomly inhomogeneous layer, we take a closer look at the behavior of the reflection capability of the layer, $\langle |R_j|^2 \rangle$, as a function of the angle of incidence θ_0 as shown in Figs. 2 and 3. The functions corresponding to the incidence of a horizontally polarized wave on the layer are shown by the solid lines, while the dashed lines show the behavior for vertically polarized waves. The curves drawn here correspond to the simple case $\varepsilon_0 = 1$, and the parameters determining the diffusion coefficients in (4.10) have been assigned the values $\sigma_\varepsilon^2 = 10^{-6}$, $ka = 1$, $L/a = 5\cdot 10^6$ in Fig. 2 and $\sigma_\varepsilon^2 = 10^{-6}$, $ka = 5$, $L/a = 4\cdot 10^{13}$ in Fig. 3. It follows from an analysis of these plots that there are certain intervals of the angle of incidence, $\Delta\theta_0$, in which the reflectivities of the layer for the waves with different types of polarization are quite different. These angular intervals depend on the relation among the variances σ_ε^2 of the fluctuations in the dielectric constant of the medium inside the layer, the length scale a of the inhomogeneities of the medium, the layer thickness L, and the wavelength $\lambda = 2\pi/k$ of the incident radiation. In general, these differences lead to degrees of polarization for the transmitted and reflected radiation which are different from that of the incident radiation. In particular, the reflected and transmitted waves turn out to be elliptically polarized because of the difference between the arguments of the transmission and reflection coefficients of the randomly inhomogeneous layer for horizontally polarized waves, (2.11), and vertically polarized waves, (3.4), when a linearly polarized wave is incident on the layer in a direction which is neither normal nor tangent to the plane of incidence.

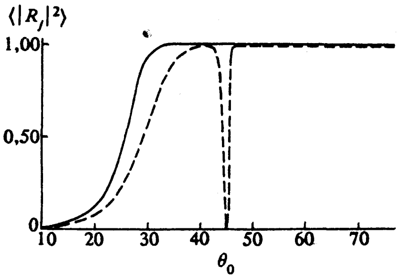

FIG. 3.

Of particular interest under the condition $\varepsilon_0 = 1$ is the case in which an initially plane wave is incident at an angle $\theta_0 = \pi/4$. From (4.10) we have $D_2 = 0$, and the vertically polarized wave passes through the layer without reflection. On the other hand, we have $D_1 \neq 0$, and the reflectivity of the layer for a horizontally polarized wave may be extremely large, up to unity (Fig. 3). Consequently, a "Brewster law" may hold in the reflection from a randomly varying plane-layer medium; i.e., there may exist certain angles of incidence at which the reflected radiation is completely polarized. The absence of reflection for a vertically polarized wave in this case for incidence at an angle $\theta_0 = \pi/4$ is explained by analogy with the classic Brewster law describing the polarization associated with the reflection from a plane interface between two homogeneous and transparent dielectrics. Specifically, the charges which are oscillating in the medium as a result of the electric field of the wave do not radiate along the oscillation direction. Consequently, a randomly varying plane-layer medium can serve as a radiation polarizer under certain conditions.

We note in conclusion that the assumption that the fluctuations of the dielectric constant of the medium inside the layer have a normal distribution has been made for convenience in the intermediate calculations. This is not a necessary condition for the validity of the results of this study.

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