Parametric resonances in a gas of three-level atoms in an external field

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(Submitted 12 November 1992)
Zh. Eksp. Teor. Fiz. **103**, 1548–1570 (May 1993)

A gas of three-level atoms is placed in a weakly modulated bichromatic field quasoresonant to one of the transitions and is probed by a probing wave on an adjacent transition. A fairly weak additional component of the external field is shown to lead, under a threshold condition imposed on the intensity of the external field, to parametric resonances for certain groups of atoms. The absorption coefficient of the additional wave acquires, beside the constant component, oscillating components that have a frequency D for one type of parametric resonances and frequencies D and $2D$ for the other, where D is the frequency detuning of the external-field components. The absorption-coefficient component irregularities associated with the parametric resonances have effective widths of the order of the atomic level widths and a contrast of the order of unity. The paper also discusses possible applications of these phenomena in sub-Doppler spectroscopy and for developing methods of frequency stabilization and the transfer of frequency values within a certain range.

1. INTRODUCTION

A gas of three-level atoms placed in an external field and probed by a probing wave constitutes one of the basics systems in laser spectroscopy.[1,2] Such systems are widely used in sub-Doppler spectroscopy and to develop methods of frequency stabilization and frequency transfer within a certain range. What is important here is that the choice of the external-field parameters is entirely in the hands of the researcher. By selecting the parameters by one method or another we can change the absorption spectrum of the probing wave. Also, the external-field parameters, such as the amplitudes of individual harmonics and the frequency detuning between the harmonics, can be changed within a broad range, by several orders of magnitude. This leads to a broad spectrum of physically realizable asymptotic situations. The requirements of the practical applications mentioned above pose the problem of finding and analyzing situations in which the absorption spectrum of the probing wave acquires sharp peaks with effective widths smaller than the Doppler linewidth. In this paper we consider from this viewpoint parametric resonances obtained by a suitable choice of external-field parameters.

The onset of parametric resonances in problems related to the dynamics of a two- or three-level atom placed in an external quasiresonant polychromatic field is quite natural. Indeed, let us consider an atom in a monochromatic field (the primary wave) that is in quasiresonance with one of the transitions. It is well known[1,2] that in such a system the characteristics of the atom undergo Rabi oscillations. Now suppose that an additional monochromatic wave that is also in quasiresonance with one of the transitions acts on the atom. Then we can select the parameters of this wave in such a way that even with a fairly small amplitude the wave, landing in resonance with the Rabi oscillations, considerably changes the behavior of the atom, namely, parametric resonance sets in. If a gas of atoms is irradiated, there will be a group of atoms which meet the conditions for parametric resonance. The large number of initial parameters and the possibility of realizing various three-level-atom solutions lead to a variety of emerging parametric resonances. In this paper we discuss how such parametric resonances affect the absorption spectrum of the probing wave. We will see, among other things, that the following phenomenon is important for applications. Suppose that the Rabi parameter of the additional wave is of the same order of magnitude as the widths of the atomic levels, and that the Rabi parameter of the primary wave is much larger than these level widths, that is, the primary wave is strong. The frequency detuning of the components of the external field is assumed of the order of the Rabi parameter of the primary wave. Then the additional structures that emerge appear in the absorption spectrum of the probing wave in the event of a parametric resonance and have a contrast of the order of unity possess effective widths of the same order as the atomic levels.

Most often (in our case, too) the situation studied pertains to an external field that is in quasiresonance with one of the atomic transitions (Fig. 1), and an approximation linear in the probing wave (known as probing on an adjacent transition) is studied. Here the analysis of the dynamics of a three-level atom is broken down into two stages (details given below). The first stage studies the behavior of a two-level atom in an external field. On the basis of the results of this study one can determine the absorption spectrum of the probing wave. The dynamics of a two-level atom in an external quasiresonant bichromatic or polychromatic field is being intensively studied both theoretically and experimentally (see, e.g., Refs. 3–16). Some results of such studies and of those concerning a three-level atom can be found in Refs. 1 and 2. The possible consequences of the presence of paramagnetic resonances in such systems have also been studied. For instance, Thomann[3,6] linked the behavior of an atom in a strongly

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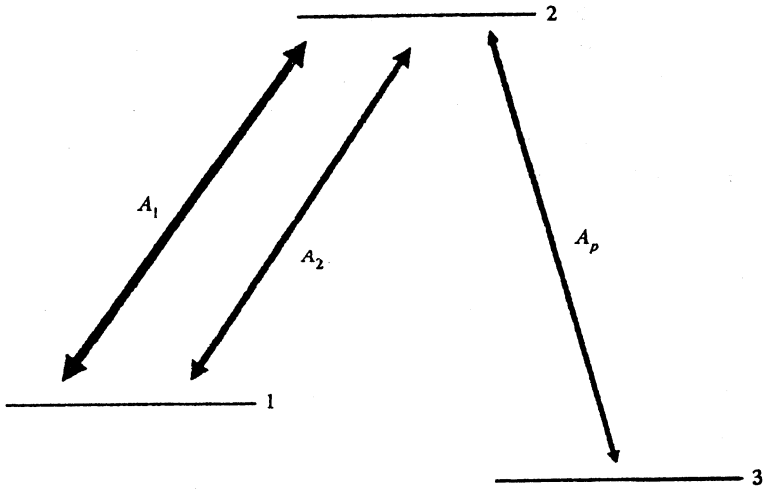

FIG. 1. Level diagram.

modulated bichromatic field with the presence of parametric resonances. Toptygina and Fradkin[11] used the example of a two-level atom in a polyharmonic field to discuss also parametric resonances, which were interpreted, apparently, as phenomena based on quasilevel crossing in the physical system. In this paper we consider a weakly modulated external field and, following Refs. 17 and 18, interpret parametric resonances as resonances that emerge in the system of differential equations describing the dynamics of an atom as a result of a weak periodic perturbation. Well-known methods allow this situation to be studied fairly completely in analytical terms. Such parametric resonances were studied in Refs. 12 and 15. For one thing, in Ref. 12 the dynamics of a two-level atom in a weakly modulated biharmonic field was discussed without allowing for relaxation, and it was found that a parametric resonance splits the Rabi spectrum. The results of Ref. 12, where the problem was fairly simple and, hence, the results could easily be interpreted, show that in the event of a parametric resonance in the sense of Ref. 17 there occurs a phenomenon that can be interpreted as quasilevel crossing in the physical system (more precisely, the levels asymptotically move closer). In Ref. 15 this problem was discussed in connection with a gas of two-level atoms with allowance for relaxation. The paper described additional structures in the absorption spectrum of the probing wave (corresponding to the splitting of the Rabi spectrum) that appeared in the event of a parametric resonance. Our present analysis of the stage referring to a two-level atom is similar to the one done in Ref. 15 with one important exception: we assume that the constants of the longitudinal relaxation of levels are different, while in Ref. 15 the constants were assumed equal. The reason is that in applications involving three-level atoms the most interesting situation is that in which the common level is much wider than the other two levels.

2. STATEMENT OF THE PROBLEM

The simplest case leading to parametric resonances emerges when the external field, which is in quasiresonance with one of the transitions, has two components, primary and additional, with frequencies Ω_1 and Ω_2, amplitudes A_1 and A_2, and wave numbers k_1 and k_2, respectively. The subscript "p" designates the similar parameters of the probing wave. We assume that the primary and additional waves propagate in the same direction. Here we consider the Λ configuration of levels, but the results can easily be generalized to the case of a V configuration or a cascaded scheme. We assume that the thermal velocity u_T of the atoms, the frequency detuning $D=\Omega_2-\Omega_1$, and the effective relaxation constant γ_0 of an atom satisfy the relation $|D|u_T/c \ll \gamma_0$, where c is the velocity of light. Then with an accuracy sufficient for our purpose we can assume that $k_1=k_2=k$ (Ref. 16). Let us write the system of Bloch equations describing the dynamics of the density matrix $\rho(v,z)$ in our situation (v is the velocity of an atom, and the one-dimensional coordinate z parametrizes the spatial position of an atom):

$$\frac{d}{dt}\rho_{11}=-\gamma_1\rho_{11}-2i\{A_1\cos(\Omega_1 t-kz)+A_2\cos(\Omega_2 t-kz)\}(\rho_{21}-\rho_{12})+\lambda_1,$$

$$\frac{d}{dt}\rho_{22}=-\gamma_2\rho_{22}+2i\{A_1\cos(\Omega_1 t-kz)+A_2\cos(\Omega_2 t-kz)\}(\rho_{21}-\rho_{12})-2iA_p\times\cos(\Omega_p t-k_p z)(\rho_{32}-\rho_{23})+\lambda_2,$$

$$\frac{d}{dt}\rho_{33}=-\gamma_3\rho_{33}+2iA_p\cos(\Omega_p t-k_p z)(\rho_{32}-\rho_{23}),$$

$$\frac{d}{dt}\rho_{32}=-(\gamma_{32}+i\omega_{32})\rho_{32}+2iA_p\cos(\Omega_p t-k_p z)(\rho_{33}-\rho_{22})+2i\{A_1\cos(\Omega_1 t-kz)+A_2\cos(\Omega_2 t-kz)\}\rho_{31},$$

$$\frac{d}{dt}\rho_{31}=-(\gamma_{31}+i\omega_{31})\rho_{31}-2iA_p\cos(\Omega_p t-k_p z)\rho_{21}+2i\{A_1\cos(\Omega_1 t-kz)+A_2\cos(\Omega_2 t-kz)\}\rho_{32},$$

$$\frac{d}{dt}\rho_{21}=-(\gamma_{21}+i\omega_{21})\rho_{21}-2iA_p\cos(\Omega_p t-k_p z)\rho_{31}$$

$$+2i\{A_1\cos(\Omega_1 t-kz)+A_2\cos(\Omega_2 t-kz)\}$$
$$\times(\rho_{22}-\rho_{11}),$$
$$\rho_{13}=\overline{\rho_{31}},\quad \rho_{12}=\overline{\rho_{21}},\quad \rho_{23}=\overline{\rho_{32}},\qquad (1)$$

where $d/dt=\partial/\partial t+v\partial/\partial z$, γ_n and γ_{nm} are the longitudinal and transverse relaxation constants, ρ_{nn} and ρ_{nm} the level populations and the respective dipole moments, ω_{nm} the frequencies of transitions between levels, with $n,m=1,2,3$, and $\lambda_{1,2}$ the parameters of pumping to the respective level. Similar systems of equations for an atom in a monochromatic field can be found in Refs. 1 and 2.

Note that there are other possibilities in the level diagrams that can lead to parametric resonances; for instance, when different components of the external field are in quasiresonance with different transitions of the atom. These variants will be examined elsewhere.

In an approximation linear in the probing-wave amplitude the equation for ρ_{33} is decoupled from the other equations, with the remaining system of equations in (1) splitting into two systems, one for ρ_{nm}, with $n,m=1,2$, and the other for the other functions. The first system coincides with the system of equations for a two-level atom in an external field with parameters A_n and Ω_n, with $n=1,2$. By introducing new variables (here we restrict our discussion to an analysis of medium characteristics averaged over the volume),

$$\rho_{21}=R_{21}\exp[-i(\Omega_1 t-kz)],\quad \tau=Dt,\quad R_{12}=\overline{R_{21}},$$

and the rotating-wave approximation we can write the first system as

$$\frac{\partial}{\partial\tau}\mathbf{R}=Q\mathbf{R}+\mathbf{C},\qquad (2)$$

where

$$\mathbf{R}=\begin{pmatrix}\rho_{11}\\ \rho_{22}\\ R_{21}\\ R_{12}\end{pmatrix},\quad C=\begin{pmatrix}\lambda_1\\ \lambda_2\\ 0\\ 0\end{pmatrix},$$

$$Q=\begin{pmatrix}-\Gamma_1 & 0 & -i[a_1+a_2\exp(i\tau)] & i[a_1+a_2\exp(-i\tau)]\\ 0 & -\Gamma_2 & i[a_1+a_2\exp(i\tau)] & -i[a_1+a_2\exp(-i\tau)]\\ -i[a_1+a_2\exp(-i\tau)] & i[a_1+a_2\exp(-i\tau)] & -(\Gamma_{21}+i\sigma_{21}) & 0\\ i[a_1+a_2\exp(i\tau)] & -i[a_1+a_2\exp(i\tau)] & 0 & -(\Gamma_{21}-i\sigma_{21})\end{pmatrix},$$

$$\Gamma_m=\frac{\gamma_m}{D},\quad a_m=\frac{A_m}{D},\quad (m=1,2),$$

$$\Gamma_{21}=\frac{\gamma_{21}}{D},\quad \sigma_{21}=\frac{\omega_{21}-\Omega_1+kv}{D}.$$

The definition of a_1 adopted here differs from the one adopted in Ref. 15 by a factor of $\sqrt{2}$. The second system of equations with the same assumptions has the form

$$\frac{\partial}{\partial\tau}\mathbf{W}=T\mathbf{W}+X\mathbf{R},\qquad (3)$$

where

$$W=\begin{pmatrix}R_{32}\\ R_{31}\end{pmatrix},$$

$$T=\begin{pmatrix}-(\Gamma_{32}+i\sigma_{32}) & i[a_1+a_2\exp(i\tau)]\\ i[a_1+a_2\exp(i\tau)] & -(\Gamma_{31}+i\sigma_{31})\end{pmatrix},$$

$$X=ia_p\begin{pmatrix}0 & -1 & 0 & 0\\ 0 & 0 & -1 & 0\end{pmatrix},$$

$$R_{32}=\rho_{32}\exp[i(\Omega_p t-k_p z)],$$

$$R_{31}=\rho_{32}\exp\{i[(\Omega_p-\Omega)t-(k_p-k)z]\},$$

$$\Gamma_{nm}=\frac{\gamma_{nm}}{D},\quad a_p=\frac{A_p}{D},$$

$$\sigma_{32}=\frac{\omega_{32}-\Omega_p+vk_p}{D},\quad \sigma_{31}=\frac{\omega_{31}-\Omega_p+\Omega_1+(k_p-k)v}{D}.$$

Here it is assumed that the optical frequencies ω_{31}, ω_{21}, and ω_{32}, which exceed all other frequency parameters of the problem, are related by the formula $\omega_{31}+\omega_{21}=\omega_{32}$, from which it follows that $\sigma_{31}+\sigma_{21}=\sigma_{32}$. In what follows we will also need the quantities

$$\Delta_{21}=\frac{\omega_{21}-\Omega_1}{D},\quad \varkappa=\frac{k}{D},\quad \varkappa_p=\frac{k_p}{D}.$$

The quantity measured in the experiment is the probing-wave absorption coefficient

$$\chi(\Omega_p)=-C_0\int\frac{dv}{A_p}\exp\left(-\frac{v^2}{u_T^2}\right)\mathrm{Im}\,R_{32}(v),\qquad (4)$$

whose study is our main goal. The factor C_0 is independent of the external-field parameters. From the above reasoning it follows that the evolution of the function ρ_{33} in the given approximation has no effect on the final result. We assume that the parameters of the problem obey the following asymptotic constraints:

$$\Gamma_1=\varepsilon,\quad \Gamma_2=\varepsilon g,\quad \Gamma_{nm}=\varepsilon g_{nm},\quad a_2=\varepsilon a,$$

$$0<\varepsilon\ll 1,\quad g,g_{nm},a,a_1,\lambda_{1,2},\sigma_{nm}=O(1).\qquad (5)$$

The final result (the relative values of the absorption coefficient) does not depend on the choice of the order of magnitude of the parameters $\lambda_{1,2}$.

Remark 1. Actually, the constraints (5) mean that the frequency difference D between the primary and additional waves and the Rabi parameter A_1, being of the same order of magnitude, considerably exceed the atom relaxation constants and the Rabi parameter of the additional wave. The choice of parameters D and A_1 is in the hands of the experimenter, and such a requirement can easily be realized experimentally.[15] Here one must bear in mind that, as practice shows, asymptotic calculations lead to a correct qualitative result up to values of the small parameter ε equal to 0.3, and for $\varepsilon \leqslant 0.1$ asymptotic relations even lead to a satisfactory quantitative description.

Remark 2. As noted earlier, from the practical viewpoint the most interesting situations occur when the values of Γ_2 [and, respectively, of Γ_{2m}, with $m=1,3$, $\Gamma_{nm} \geqslant \frac{1}{2}(\Gamma_n+\Gamma_m)$] are considerably larger than those of Γ_1 and Γ_3. Here the discussion is restricted to the case where all the relaxation constants are asymptotically of the same order of magnitude; within this assumption they may differ considerably in magnitude.

Remark 3. This paper considers the case where the primary field contains only one harmonic. A similar technique makes it possible to investigate the case where the primary field is polyharmonic in the sense of Refs. 11 and 14.

Remark 4. Here we assume that D is positive. The formulas with $D<0$ can easily be derived from our results.

Thus, we can write Eqs. (2) and (3) in the form

$$\frac{\partial}{\partial \tau}\mathbf{R}=[Q_0+\varepsilon Q_1]\mathbf{R}+\mathbf{C}, \tag{6}$$

$$\frac{\partial}{\partial \tau}\mathbf{W}=[T_0+\varepsilon T_1]\mathbf{W}+X\mathbf{R}. \tag{7}$$

Here the elements of matrices Q_0, Q_1, T_0, and T_1 are quantities of the order $O(1)$:

$$Q_0=\begin{vmatrix} 0 & 0 & -ia_1 & ia_1 \\ 0 & 0 & ia_1 & -ia_1 \\ -ia_1 & ia_1 & -i\sigma_{21} & 0 \\ ia_1 & -ia_1 & 0 & i\sigma_{21} \end{vmatrix},$$

$$T_0=\begin{pmatrix} -i\sigma_{32} & ia_1 \\ ia_1 & -i\sigma_{31} \end{pmatrix},$$

$$Q_1=\begin{vmatrix} -1 & 0 & -ia\exp(i\tau) & ia\exp(-i\tau) \\ 0 & -g & ia\exp(i\tau) & -ia\exp(-i\tau) \\ -ia\exp(-i\tau) & ia\exp(-i\tau) & -g_{21} & 0 \\ ia\exp(i\tau) & -ia\exp(i\tau) & 0 & -g_{21} \end{vmatrix},$$

$$T_1=\begin{pmatrix} -g_{32} & ia\exp(-i\tau) \\ ia\exp(i\tau) & -g_{31} \end{pmatrix}.$$

The system of equations (6) and (7) has an "up-triangle" form. First we must solve (6), and then (7). The system has periodic coefficients and the problem has a small parameter ε, that is, we must use an appropriate version of perturbation theory and build the asymptotic expansion of solutions to the system (and of the physical characteristics) in the small parameter ε. The construction and analysis of solutions of systems of type (6) and (7) can be done by methods discussed in Refs. 17 and 18. As Nayfeh's results[17] suggest, for certain parameter ratios, a small perturbation, εQ_1 or εT_1, may change the solution in the leading term, which causes parametric resonances to appear. These may be of different origin.

1. Direct parametric resonances (DPR). These appear when Eq. (6) is being solved if the additional field is in resonance with the system's spectrum: a two-level atom (levels *1* and *2*) plus the primary field. Their appearance is due to the interaction of the external field directly with the appropriate atomic transition.

2. Adjacent parametric resonances (APR). These appear when Eq. (7) is solved, with the additional field in resonance with the induced spectrum in the system of levels *2* and *3*. Ordinary resonances are also possible in (7). These appear when the probing wave is in resonance with the induced spectrum in the system of levels *2* and *3*.

The presence of all these resonances depends on whether certain restrictions imposed on the coefficients of the system of equations (6) and (7) are valid. At fixed values of the parameters of the atom and the external radiation (we call these the external parameters) these coefficients depend on the velocity of the atom. The majority of atoms are "outside" of the parametric resonance, and the condition for parametric resonance is satisfied only for an asymptotically small fraction of the atoms. As follows from Eq. (4), calculating the probing-wave absorption coefficient requires calculating the responses of all the atoms to the external field. If, however, the adopted constraints (5) on the external parameters are valid, the main contribution to the integral in (4) is provided by the atoms with velocities lying within asymptotically small intervals around the values corresponding to the ordinary resonances of (7). These values change with the frequency of the probing

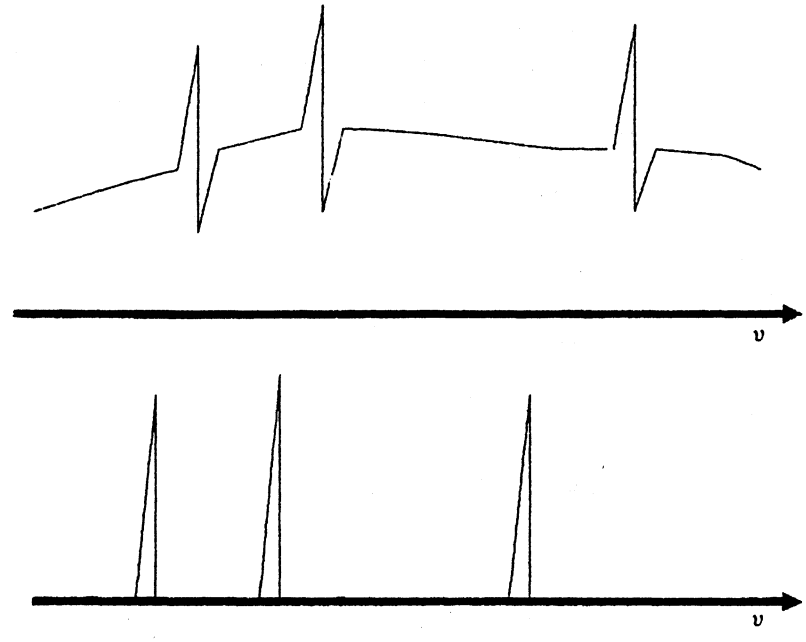

FIG. 2. Singularities of the system consisting of gas atoms and an external field on the velocity scale. The singularities depicted on the upper axis belong to a direct transition and those on the lower axis to an adjacent transition.

wave. By varying the frequency of the probing wave we can, therefore, probe various asymptotically small groups of atoms that respond to the external field differently, including groups of atoms in parametric resonance.

Schematically the situation is depicted in Fig. 2, where on the upper atomic-velocity axis there are seen to be asymptotically small groups of atoms in DPR [this illustrates the behavior of solutions of (6)]. On the lower atomic-velocity axis there are seen to be groups of atoms in ordinary resonance or in APR [this illustrates the behavior of solutions of (7)]. When the external parameters (say, the frequency of the probing wave) vary, these pictures change shape slightly and shift in relation to each other. The probing-wave absorption coefficient experiences sharp changes near the values of the external parameters at which the "singularities" on the upper and lower axes coincide. Multiple coincidences are also possible. Below we show that it is multiple coincidences that are realized.

Thus, there emerges the problem of building solutions to Eqs. (6) and (7) in the presence of parametric resonances. Using the solutions, we can calculate the probing-wave absorption coefficient on the basis of (4). In this paper we discuss the problem of building only the leading term in the solution's asymptotic expansion in the small parameter ε. The plan is as follows. In Sec. 3 we discuss DPR, that is, the features of the solution on the upper velocity axis in Fig. 2. Section 4 is devoted to features characteristic of an adjacent transition, that is, the features on the lower axis. In Sec. 5 we use these results to describe the features of the absorption coefficients related to the passage of the singularities through each other.

In what follows we use the term "neighborhood of a point" for brevity to mean an asymptotically small, neighborhood of a given point, that is, with a width of the order of ε. By the expression "near the point" we understand an asymptotically small neighborhood of the point, and the expression "far from the point" describes a situation when the distance to the given point is a quantity of the order of $O(1)$.

3. DIRECT PARAMETRIC RESONANCES

Let us examine the situation in which parametric resonances appear in system (6). Below we shall see that for all parameter ratios the solution to (6) has the following structure:

$$R(\tau)=\mathbf{B}_1+\exp(i\tau)\mathbf{B}_2+\exp(-i\tau)\mathbf{B}_3+\mathbf{R}_u(\tau), \qquad (8)$$

where the $\mathbf{B}_n$, with $n=1,2,3$, are independent of τ, and by the subscript u from now on we denote terms that become damped with the passage of time. The vectors $\mathbf{B}_2$ and $\mathbf{B}_3$ are nonzero in the vicinity of a DPR and vanish far from such a resonance. Thus, the problem is reduced to determining the vectors $\mathbf{B}_n$.

To build the solution of system (6) we first solve the homogeneous system of equations, that is, seek the solution to the problem

$$\frac{\partial}{\partial\tau}N(\tau)=[Q_0+\varepsilon Q_1]N(\tau), \quad N(0)=I,$$

where I is the identity matrix here and below. Suppose that matrix U reduces Q_0 to diagonal form:

$$E_0=U^{-1}Q_0U=\mathrm{diag}\{e_1,e_2,e_3,e_4\}, \quad e_1=e_2=0,$$

$$e_3=-e_4=iF, \quad F=[\sigma_{21}^2+4a_1^2]^{1/2},$$

$$U=\begin{vmatrix} (F-\sigma_{21})/2F, & (F+\sigma_{21})/2F, & a_1/F, & a_1/F \\ (F+\sigma_{21})/2F, & (F-\sigma_{21}/2F, & -a_1/F, & -a_1/F \\ a_1/F, & -a_1/F_1, & (\sigma_{21}-F)/(2F), & (\sigma_{21}+F)/(2F) \\ a_1/F, & -a_1/F_1, & (\sigma_{21}+F)/(2F), & (\sigma_{21}-F)/(2F) \end{vmatrix}.$$

Then for $L(\tau)=\exp(-E_0\tau)U^{-1}N(\tau)$ we have the following problem:

$$\frac{\partial}{\partial\tau}L(\tau)=\varepsilon E_1(\tau)L(\tau), \quad L(0)=U,$$

$$E_1(\tau)=\exp(-E_0\tau)U^{-1}Q_1(\tau)U\exp(E_0\tau). \tag{9}$$

Using the multiscale method, we introduce a slow time $\tau_1=\varepsilon\tau$ and seek the solution to (10) in the form

$$L(\tau,\tau_1)=L_0(\tau,\tau_1)+\varepsilon L(\tau,\tau_1)+\dots . \tag{10}$$

Substituting (10) into (9), we arrive at a recurrence system of equations for successive terms in expansion (10). From the first equation it follows that $L_0(\tau,\tau_1)=L_0(\tau_1)$. The following equation for $L_0(\tau_1)$ constitutes the condition for the absence of secular terms in the second equation:

$$\frac{\partial}{\partial\tau_1}L_0=\langle E_1(\tau)\rangle L_0. \tag{11}$$

Here and in what follows $\langle\cdots\rangle$ denotes the averaging procedure, that is, the procedure in which terms oscillating with frequencies of the order of $O(1)$ are discarded. In the case of "parameters in the common position" the averaging of matrix $E_1(\tau)$ is reduced to discarding all oscillating terms. But there may be parameter ratios at which some of the oscillating terms have frequencies that are asymptotically close to zero. The elements of matrix $Q_1(\tau)$ contain exponential functions with exponents 0 and $\pm i\tau$:

$$[E_1]_{nm}(\tau)=[U^{-1}Q_1U]_{nm}\exp[i\tau(e_m-e_n)],$$

$$n,m=1,2,3,4.$$

Thus, parametric resonances can appear in the following situations:

$$F(v)=1+\varepsilon\nu, \quad \nu=O(1),$$

$$F(v)=0.5+\varepsilon\mu, \quad \mu=O(1). \tag{12}$$

From the explicit form of matrix $E_1(\tau)$, which in view of its complexity is not given here, it follows that in the second case there is no resonance.

Remark 5. The absence of this resonance was noted earlier in Refs. 12 and 15. Thus, this fact is not related to the assumption adopted in Ref. 15 that the longitudinal-relaxation constants of a two-level atom coincide.

The presence or absence of atoms satisfying condition (12) is determined by the choice of external parameters, and parametric resonances emerge only if the following threshold condition is met:

$$a_1^2\leqslant\frac{1}{4}. \tag{13}$$

1. Let $1-2a_1=q<0$ and $q=O(1)$. In this case at no atomic velocities is the condition for parametric resonance met. The solution to (6) can be found immediately:

$$\mathbf{B}_1=-(\langle Q(\tau)\rangle)^{-1}\mathbf{C}, \quad \mathbf{B}_2=\mathbf{B}_3=0. \tag{14}$$

2. Let $2a_1=1+\varepsilon\delta$ and $\delta=O(1)$. Then there is only one group of atoms for which condition (13) is met. The velocities of the atoms in this group are parametrized in the following way:

$$v=v^{(r)}\pm\frac{\sqrt{2\varepsilon(\nu-\delta)}}{\varkappa}+O(\varepsilon), \quad v^{(r)}=-\frac{\Delta_{21}}{\varkappa}.$$

Here the parameter ν can take on values not less than δ. The number of such atoms is proportional to $\varepsilon^{1/2}$. Following Ref. 15, we call this situation a type-*I* parametric resonance. Resonances of this type can be interpreted as parametric resonances in the initial stage. Such resonances are shown (see Ref. 15) to lead to singularities in the probing-wave absorption spectrum with widths of the order of $O(\sqrt{D\gamma})$ (in the initial terms). We do not discuss this situation here since we are interested in singularities in the absorption spectrum with widths of the order of $O(\gamma)$.

3. Let $1-2a_1=q>0$ and $q=O(1)$. With this choice of external parameters there are two groups of atoms that meet condition (13). The velocities of these atoms are parametrized as follows:

$$v=v^{(r)}+\varepsilon\alpha\varkappa^{-1}+O(\varepsilon^2), \quad v^{(r)}=\frac{q_1-\Delta_{21}}{\varkappa},$$

$$\alpha=\frac{\nu}{\sqrt{1-4a_1^2}}, \quad q_1=\pm\sqrt{1-4a_1^2}. \tag{15}$$

The different signs of q_1 correspond to different groups of atoms. The number of atoms in each group is proportional to ε. We call this situation a type-II parametric resonance.

Let us describe the solution to problem (9) for those atoms whose velocities obey the DPR condition (12). To this end, following the general procedure described in Ref. 17, we replace $\exp[\pm i(F-1)\tau]$ by $\exp(\pm i\nu\tau_1)$ in the matrix $\langle E_1\rangle$ and substitute in (12):

$$L_0(\tau_1)=Z(\tau_1)M_0(\tau_1),$$

$$Z(\tau_1)=\operatorname{diag}\{1,1,\exp(-i\nu\tau_1),\ \exp(i\nu\tau_1)\}.$$

For the matrix $M_0(\tau_1)$ we obtain a system of equations with constant coefficients:

$$\frac{\partial}{\partial\tau_1}M_0=YM_0,$$

$$Y=\begin{pmatrix}-\theta_1 & \rho & i\xi & -\xi\\ \rho & -\theta_2 & -i\xi & i\xi\\ i\xi & -i\xi & i\nu-\theta_3 & 0\\ -i\xi & i\xi & 0 & -i\nu-\theta_3\end{pmatrix},$$

$$\xi=\frac{a(F+\sigma_{21})}{2F_1},\quad \theta_1=\frac{\sigma_{21}^2+2a_1^2g_{21}}{F_1^2},$$

$$\theta_2=\frac{g\sigma_{21}^2+2a_1^2g_{21}}{F_1^2},\quad \theta_3=\frac{a_1^2+a_1^2g+g_{21}F_1^2}{F^2},$$

$$\rho=\frac{2a_1^2g_{21}}{F_1^2}.$$

Let us introduce the matrix $H=\mathrm{diag}\{0,0,i,-i\}$. Then $\exp(H\tau)=\exp(E_0\tau)Z(\tau_1)$. In these terms we obtain

$$N_0(\tau)=U\exp(H\tau)\exp(\varepsilon Y\tau)U^{-1},$$

which leads us to the following relation:

$$R(\tau)=N_0(\tau)\mathbf{R}(0)+N_0(\tau)\int_0^\tau\frac{\mathbf{C}dx}{N_0(x)}=N_0(\tau)\mathbf{R}(0)$$
$$+U\exp(H\tau)\exp(\varepsilon Y\tau)\int_0^\tau\frac{\mathbf{C}dx}{\exp(\varepsilon Yx)\exp(Hx)U}.$$

We denote the eigenvalues of matrix Y by Φ_n, with $n=1, 2, 3, 4$, where $\Phi_n=O(1)$. In the Appendix we show that $\mathrm{Re}\,\Phi_n<0$. Hence, the term without an integral and the value of the integral at the lower limit can be attributed to $R_u(\tau)$. The steady-state solution $R(\tau)$ is determined by the value of the integral at the upper limit. Moreover, upon integration, only harmonics with asymptotically low frequencies acquire a factor of the order of $O(\varepsilon^{-1})$. The other harmonics can be attributed to the next term in the asymptotic expansion. Thus, we obtain the final form of the leading term in the asymptotic expansion of the solution to (6) for atoms in DPR:

$$R(\tau)=\frac{1}{\varepsilon}U\exp(H\tau)Y^{-1}P_1U^{-1}C,$$

$$P_1=\mathrm{diag}\{1,1,0,0\}. \qquad (16)$$

From this it follows that in the neighborhood of a DPR the solution of (6) has the form of (8).

Suppose that the atomic velocity varies near the resonance value $v^{(r)}$ according to (15), so that the parameter ν experiences variations of the order of $O(1)$. Then the parameters of the matrices Q_0, U, and Y vary by asymptotically small quantities (and can be assumed constant) except the two terms proportional to ν on the diagonal of Y. This implies that the elements of vectors B_n, with $n=1, 2, 3$, also vary by quantities of the order of $O(1)$ in the neighborhood of a DPR.

4. ADJACENT PARAMETRIC RESONANCE

Let us examine the behavior of the solutions to (7) as functions of the atomic velocity v, that is, determine the position and type of singularities on the lower velocity axis in Fig. 2. As demonstrated above, irrespective of the value of the atomic velocity, the leading term in the asymptotic expansion in ε of the solution to (6) has the structure of (8). Since (7) is linear, we can analyze the contribution of each term on the right-hand side of (8) separately. For instance, $\mathbf{R}_u(\tau)$ can be shown to lead to the appearance of a damped term in $\mathbf{W}(\tau)$ and has no effect on the steady-state solution to (7). Thus, we need only build the solution of the system

$$\frac{\partial}{\partial\tau}\mathbf{W}=[T_0+\varepsilon T_1]\mathbf{W}+X\mathbf{R}(\tau). \qquad (17)$$

The matrices T_0 and T_1 depend only on the atomic velocity v. When setting up the solutions to (17) we need to know the "special" values of velocities. These, as noted earlier, are related to the following situations.

A. An eigenvector of T_0 may vanish. This results in a resonance with a constant term in the inhomogeneous part of (17).

B. An eigenvalue of T_0 may become equal to $\pm i$. At such values of the atomic velocity there appears a resonance with an oscillating term in the inhomogeneous part of (17).

C. Parametric resonances (APR) may emerge when the contribution of matrix T_1 determines the behavior of the leading term in the asymptotic expansion of the solution to (17).

There are also situations (at certain values of external parameters) when two or three of the above possibilities are realized simultaneously. Below for the sake of brevity we refer to groups of atoms in the neighborhoods of such resonances as atomic groups A, B, and C, respectively.

A remark is in order here. The contribution to the integral in (4) from the C group of atoms is asymptotically small, that is, it is of the next order of smallness in comparison to the contribution of the atomic group A or C, except in cases where group C is asymptotically close to group A or B. The reason is that the presence of an ordinary resonance (under our assumptions concerning the relaxation parameters) changes the asymptotic order of the solution of (7). At the same time the presence of a parametric resonance in this case distorts only the leading term in the asymptotic expansion and not its order. Hence, APRs are of interest only if they are asymptotically close to resonances of the A or B group.

Since it is assumed that $a_1\neq 0$, there are no values of σ_{31} and σ_{32} at which matrix T_0 has a multiple eigenvalue. We can write the condition for an APR. The matrix $T_1(\tau)$ contains harmonics with frequencies 0 and ± 1. Suppose that the matrices U_s and U_s^{-1} reduce T_0 to diagonal form: $U_s^{-1}T_0U_s=\mathrm{diag}\,\{\mu_1,\mu_2\}$. The matrix $U_s^{-1}T_1(\tau)U_s$ contains the same harmonics. A parametric resonance in system (17) emerges when the harmonics of matrix T_1 coincide with the difference of eigenfrequencies of matrix T_0

(see Refs. 17 and 18). Thus, the condition for an APR has the form $\mu_1-\mu_2=\pm i$. Here there are two possible situations in which the presence of an APR leads to a significant variation of the leading term in the asymptotic expansion of the solution to (17).

(a) Coincidence of an APR and an A resonance. This automatically meets the condition for a B resonance:

$$\mu_1=O(\varepsilon),\quad \mu_2=i\eta+O(\varepsilon),\quad \eta=\pm 1. \tag{18}$$

(b) Coincidence of an APR and a B resonance:

$$\mu_1=i\eta+O(\varepsilon),\quad \mu_2=2i\eta+O(\varepsilon),\quad \eta=\pm 1. \tag{19}$$

Here we discuss in greater detail the variant related to condition (18). We start with the relations between the parameters at which the conditions for an APR resonance and an A resonance are exactly the same:

$$\sigma_{31}+\sigma_{32}=\eta,\quad \eta=\pm 1,\quad \sigma_{31}\sigma_{32}=a_1^2. \tag{20}$$

As in the case of a DPR, two different asymptotic situations are possible:

1. $a_1^2=\frac{1}{4}-\varepsilon q$ and $q=O(1)$. Such resonances were called above type-I resonances. The reasons for not examining them are the same.

2. $(\frac{1}{4}-a_1^2)^{1/2}=q_s=O(1)$ and $q_s>0$. Such resonances were called above type-II resonances, and they can be interpreted as resonances in the advanced stage. The reasons for not examining them are the same. Having in mind the goals of our further exposition, we discuss this situation in greater detail here.

From conditions (20) it follows that when the conditions for an APR resonance and an A resonance are exactly the same,

$$\sigma_{32}^{(a)}=\frac{\eta}{2}+h,\quad \sigma_{31}^{(a)}=\frac{\eta}{2}-h,\quad h=\pm q_s.$$

From this we can find the respective values of the atomic velocity and the probing-wave frequency:

$$v^{(r)}=\frac{2hD+\Omega_1-\omega_{21}}{k}$$

$$\Omega_p^{(a)}=\omega_{32}+\frac{(2hD+\Omega_1-\omega_{21})k_p}{k}-(\eta+h)D. \tag{21}$$

Note that the expression for the resonance value of the atomic velocity coincides with (15). The relations in (20) place some restrictions on the parameters. First, σ_{31} and σ_{32} have the same sign. Second, (21) can be realized only if the threshold condition (13) is met. In accordance with different choices of signs of η and h, there can be four different pairs of values of the resonant atomic velocity and the probing-wave frequency corresponding to the situation where the conditions for an APR and an A resonance are exactly the same.

Let us now examine the dynamics of the characteristics of atoms in the neighborhood of the point where the resonances coincide. To this end we build the solution to (17) for the velocities

$$v=v^{(r)}+\frac{\alpha\varepsilon}{\varkappa},\quad \alpha=O(1),$$

and the probing-wave frequencies

$$\Omega_p=\Omega_p^{(a)}+\varepsilon\beta D,\quad \beta=O(1).$$

The respective values of parameters σ_{32} and σ_{31} are

$$\sigma_{32}=\sigma_{32}^{(a)}+\varepsilon\left[\frac{\alpha\varkappa_p}{\varkappa}-\beta\right],$$

$$\sigma_{31}=\sigma_{31}^{(a)}+\varepsilon\left[\frac{\alpha(\varkappa_p-\varkappa)}{\varkappa}-\beta\right].$$

In setting up the solutions to (17) for these values of the parameters we again use the multiscale method, but in a form that differs somewhat from that used above. We regroup the terms in matrices T_0 and T_1 in such a way that the terms proportional to ε are only present in T_1:

$$T=\hat{T}_0+\hat{T}_1,$$

$$\hat{T}_0=i\begin{pmatrix}-\left(\frac{1}{2}\eta+h\right) & a_1\\ a_1 & -\left(\frac{1}{2}\eta-h\right)\end{pmatrix},$$

$$\hat{T}_1=\begin{pmatrix}-\hat{g}_{32} & ia\exp(-i\tau)\\ ia\exp(-i\tau) & -\hat{g}_{31}\end{pmatrix},$$

$$\hat{g}_{32}=g_{32}+i\left[\frac{\alpha\varkappa_p}{\varkappa}-\beta\right],\quad \hat{g}_{31}=g_{32}+i\left[\frac{\alpha(\varkappa_p-\varkappa)}{\varkappa}-\beta\right].$$

The eigenvalues of matrix $\hat{T}_0$ are $\hat{\mu}_1=0$ and $\hat{\mu}_2=-i\eta$. The matrix

$$U_a=\begin{pmatrix}a_1 & -\left(\frac{1}{2}\eta+h\right)\\ \frac{1}{2}\eta+h & a_1\end{pmatrix}$$

and its inverse reduce $\hat{T}_0$ to diagonal form: $U_a^{-1}\hat{T}_0U_a=\mathrm{diag}\{\hat{\mu}_1,\hat{\mu}_2\}\equiv H_a$. We start by solving the homogeneous system corresponding to (17):

$$\frac{\partial}{\partial\tau}N_a=[\hat{T}_0+\varepsilon\hat{T}_1]N_a,\quad N_a(0)=I.$$

Substituting $N_a(\tau)=U_a\exp(H_a\tau)L_a(\tau)$ transforms this system into

$$\frac{\partial}{\partial\tau}L_a(\tau)=\varepsilon P_a(\tau)L_a(\tau),$$

$$P_a=\exp(-H_a\tau)U_a^{-1}\hat{T}_1U_a\exp(H_a\tau).$$

Let us introduce the slow time $\tau_1=\varepsilon\tau$. Substituting into the system

$$\left(\frac{\partial}{\partial\tau}+\varepsilon\frac{\partial}{\partial\tau_1}\right)L_a(\tau,\tau_1)=\varepsilon P_aL_a(\tau,\tau_1)$$

the asymptotic expansion

$$L_a(\tau,\tau_1)=\sum_{k\geqslant 0}\varepsilon^k L_{ak}(\tau,\tau_1),$$

we arrive at a recurrence system of equations for the functions L_{ak}. From the first equation we get $L_{a0}(\tau,\tau_1)=L_{a0}(\tau_1)$. The second equation has the form

$$\frac{\partial}{\partial\tau_1}L_{a0}=P_aL_{a0}-\frac{\partial}{\partial\tau_1}L_{a0}.$$

The requirement that the right-hand side of this equation contains no secular term leads us to the following equation:

$$\frac{\partial}{\partial\tau_1}L_{a0}=Y_aL_{a0},\quad Y_a=\langle P_a\rangle,$$

where $\langle\cdots\rangle$ has the same meaning as before. Without going into the details of simple but tedious computations, we write

$$Y_a=\begin{pmatrix}-\hat g_{32}\left(\frac{1}{2}-\eta h\right)-\hat g_{31}\left(\frac{1}{2}+\eta h\right) & -ia\eta\left(\frac{1}{2}+h\right)\\ -ia\eta\left(\frac{1}{2}+h\right) & -\hat g_{32}\left(\frac{1}{2}+\eta h\right)-\hat g_{31}\left(\frac{1}{2}-\eta h\right)\end{pmatrix}.$$

We denote the eigenvalues of matrix Y_a by $\Phi_n^{(a)}$, with $n=1,2$. Repeating the reasoning given in the Appendix, we can show that $\mathrm{Re}\,\Phi_n^{(a)}<0$, with $n=1,2$. Thus, only the value of the integral at the upper limit of integration participates in forming the solution in the steady-state mode:

$$\mathbf{W}\tau)=U_a\exp(H_a\tau)\exp(\varepsilon Y_a\tau)\times\int^{\tau}dy\exp(-\varepsilon Y_a y)\exp(-H_a y)\,U_a^{-1}X\mathbf{R}(y).\tag{22}$$

All the objects in this final formula for the solution to (17) have been defined above.

Now let us consider the situation with (19). We write the values of the atomic velocity and the probing-wave frequency corresponding to coinciding resonances. According to Viete's theorem applied to the characteristic polynomial of matrix T_0 we have

$$\sigma_{32}+\sigma_{31}=3\eta,\quad \eta=\pm 1,\quad \sigma_{31}\sigma_{32}=2+a_1^2.\tag{23}$$

Combining this with the explicit expressions for σ_{31} and σ_{32}, we get

$$v^{(r)}=\frac{2fD+\Omega_1-\omega_{21}}{\varkappa},$$

$$\Omega_p^{(b)}=\omega_{32}+\frac{(2fD+\Omega_1-\omega_{21})k_p}{k}-D(3\eta+f),$$

$$f=\pm\sqrt{1-4a_1^2}.\tag{24}$$

This situation can be realized only if the threshold condition (13) is met. Note that the expression for $v^{(r)}$ again coincides with (15). The different choice of signs of η and f give four different pairs of values of the probing-wave frequency at which such resonances are observed.

5. BEHAVIOR OF THE PROBING-WAVE ABSORPTION COEFFICIENT

Now we have the tools to study the position of the singularities of the probing-wave absorption coefficient. We are interested in the variations of the absorption coefficient with a contrast of $O(1)$, where the variations are concentrated within intervals of values with widths of the order of $O(\varepsilon)$ for dimensionless parameters, which in the initial notation means intervals with widths of the order of $O(\gamma)$. These phenomena occur because of the above-described passage of the singularities of the solutions to (6) and (7) through each other due to the variations in parameters (the probing-wave frequency). The singularities merge when the respective values of the resonant velocity and probing-wave frequency have the same values.

The starting relation is (4). A brief list of the main aspects pertaining to the behavior of atomic characteristics are listed below in accordance with the results of Secs. 3 and 4.

(I) The order of magnitude of the vector $X\mathbf{B}_1$ forming the constant component of the absorption coefficient does not depend on whether or not an atom is close to a direct resonance. But this factor influences the manner in which the magnitude varies: the variation is slow far from a resonance and rapid near it.

(II) The leading term in the asymptotic expansion of the function $X\mathbf{R}(y,\tau)$ contains no oscillating component (with a unit frequency in dimensionless notation) outside the neighborhood of a DPR. In the neighborhood the amplitude of such a component changes the order of magnitude of the functions and is contained in the leading term.

(III) The main contribution to the constant and oscillating components of the integral in (4) is provided by atoms that are in the neighborhood of A and B resonances.

Let us start with the situation in which the atoms in an A or B resonance are far from a DPR. Then, as we will shortly see, they are also far from an APR. According to (14), the leading term in the asymptotic expansion of the solution $\mathbf{R}(\tau)$ of (6) for such atoms contains only one the time-independent component. Thus, the principal contri-

bution to the integral in (4) is provided by atoms in the neighborhood of an A resonance. Accordingly, the leading term in the asymptotic expansion of the probing-wave absorption coefficient contains only a constant component. Here we arrive at the well known results for the absorption coefficient (see Refs. 1 and 2 and a discussion of this situation in Ref. 15). The coefficient is represented by a smooth function (i.e., without singularities in the sense adopted here) of the parameter Ω_p.

According to what was said earlier, there are two situations in which the absorption spectrum undergoes rapid variations (see Fig. 2).

(i) When the parameters (the probing-wave frequency) are varied, the group of atoms in the neighborhood of a DPR passes through an A resonance. The values of parameters corresponding to the coincidence of resonances are listed below. The velocity values are described by Eq. (15), which corresponds to a DPR. The relation that corresponds to an A resonance and determines the value of Ω_p has the form

$$\sigma_{31}\sigma_{32}=a_1^2.$$

Substituting $\sigma_{31}=\sigma_{32}+\sigma_{21}$ and combining the result with (15), we get

$$\sigma_{32}=\frac{1}{2}(\sigma_{21}\pm 1),$$

which yields

$$\sigma_{32}+\sigma_{31}=\pm 1. \tag{25}$$

This expression for Ω_p coincides with the first formula in (20). At the same time the value of the atomic velocity given by (15) coincides with that given by (21). Thus, if a DPR passes through an A resonance, the condition for an APR is also met.

(ii) The group of atoms in DPR passes through a B resonance. The values of the atom velocity are given by (15). Let us write the equation for Ω_p. The eigenvalue equation for matrix T_0 yields

$$\mu_{1,2}=-\frac{i}{2}(\sigma_{31}+\sigma_{32})\pm\frac{i}{2}.$$

The expression for Ω_p (the condition for a B resonance) can then be written in the form $\mu_{1,2}=i\xi$, with $\xi=\pm 1$. If the choice of the sign of $i/2$ coincides with that of the sign of ξ, we arrive at (25), which means we are in situation (i). This was to be expected, for if the conditions for an APR and an A resonance are met, so is the condition for a B resonance. Selecting different signs, we arrive at the following equation for Ω_p: $\sigma_{31}+\sigma_{32}=3\xi$, which coincides with the first formula in (23). We may conclude, therefore, that the fact that the conditions for a DPR and a B resonance are met implies that this is the condition for an APR.

We can easily show that the probing-wave absorption spectrum has no other singularities than those listed above. Summing up, we arrive at the following conclusions.

There are two distinct phenomena that serve as sources for the singularities observed in the probing-wave absorption spectrum and related to parametric resonances.

1. Simultaneous coincidence of DPR, APR, and A and B resonances. Allowing for certain liberties of speech, we call such situations parametric resonances A (or PRA). Their position on the frequency scale of the probing wave is described by Eqs. (21). Depending on the choice of signs of the parameters η and h, we have four different cases.

2. Simultaneous coincidence of DPR, APR, and B resonances. We call such situations parametric resonances B (or PRB). The respective values of the probing-wave frequency are described by Eqs. (24). Here, too, depending on the choice of signs of the parameters η and f, we have four different cases.

Both types of parametric resonance exist only if the threshold condition (13) is met. The resonance conditions are met for the same groups of atoms whose velocities are in the neighborhood of the value given by (15).

Let us write the final formulas for the probing-wave absorption coefficient for PRA. According to (16) and (22), the steady-state solution of (7) has the form

$$W(\tau)=\varepsilon^{-1}U_a\exp(H_a\tau)\exp(\varepsilon Y_a\tau)\int^{\tau}dy\exp(-\varepsilon Y_a y)$$
$$\times\exp(-H_a y)\,G_a\exp(Hy)\,Y^{-1}P_1U^{-1}\mathbf{C},$$

$$G_a=U_a^{-1}XU.$$

Integration of the harmonics whose frequencies are of the order of $O(\varepsilon)$ yields a factor $O(\varepsilon^{-1})$. Allowing for this, we arrive at the final expression for the leading term in the asymptotic expansion for the given case:

$$\mathbf{W}(\tau)=\frac{1}{\varepsilon^2}U_a\exp(H_a\tau)\,Y_a^{-1}\hat{G}_aY^{-1}P_1U^{-1}\mathbf{C}. \tag{26}$$

Here matrix $\hat{G}_a$ is defined by the relations $[\hat{G}_a]_{11}=G_{11}$, $[\hat{G}_a]_{12}=G_{12}$, $[\hat{G}_a]_{23}=G_{23}(1-\eta)/2$, and $[\hat{G}_a]_{24}=G_{24}(1+\eta)/2$, with all other elements of $\hat{G}_a$ zero. The explicit expressions for Y and Y_a contain parameters α and β, which characterize the exact tuning to a parametric resonance (more precisely, these parameters enter only into the diagonal elements of these matrices). The matrices U_a, $\hat{G}_a$, and U in the leading term of their expansion in ε are independent of α and β and are determined by the values of their parameters in the neighborhood of the resonance point of a PRA. According to our definitions, $R_{23}=P_2\mathbf{W}$, where $P_2=\{1,0\}$ (a row matrix). Substituting (26) into (4) and going from integration over v to integration over α

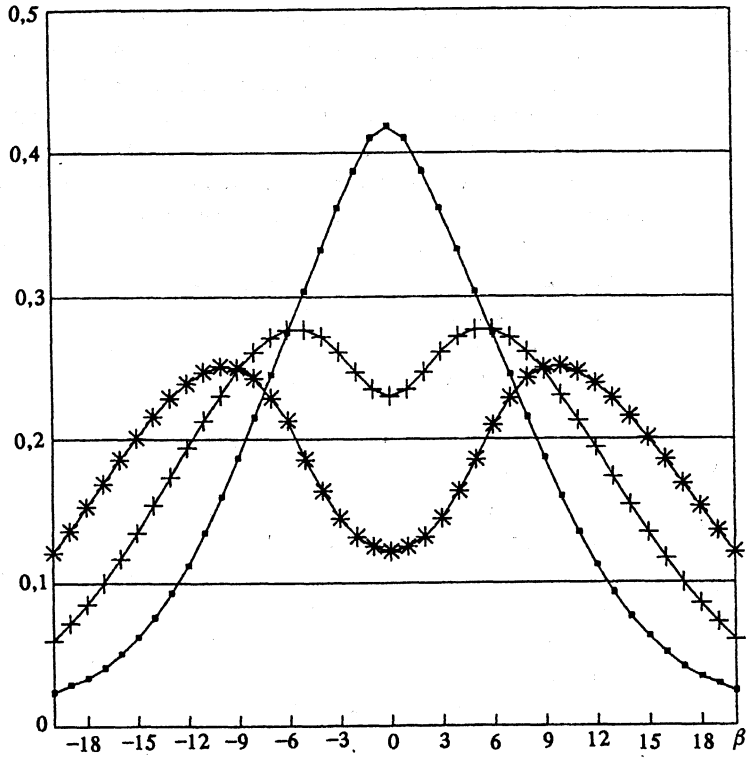

FIG. 3. Amplitudes of the constant component of function $V_a(\beta)$: $g_{32}=2$, $g_{31}=1.5$, $g_{21}=2$, $g=1$, $h=q_s$, $\varkappa=1.2$, $a_1=1/3$, and $\eta=1$. Series 1 corresponds to $a=5$(■), series 2 to $a=7$(+), and series 3 to $a=9$(*).

via (15), we arrive at the final formula for the leading term in the asymptotic expansion of the probing-wave absorption coefficient in the neighborhood of a PRA (recall that $\varepsilon D=\gamma_1$):

$$\chi_a(\beta;\tau)\equiv\chi(\Omega_p)=\chi(\Omega_p^{(a)}+\beta\gamma_1)$$
$$=-\varepsilon^{-1}C_0k^{-1}|q_1|$$
$$\times\exp(-[v^{(r)}/u_T]^2)V_a(\beta),$$

$$V_a(\beta)=\int d\alpha\ \mathrm{Im}\{P_2U_a\exp(H_a\tau)Y_a^{-1}\hat{G}_aY^{-1}P_1U^{-1}\mathbf{C}\}/a_p. \tag{27}$$

Here and in what follows the integral over α is taken from $-\infty$ to ∞ in the principal-value meaning. Note that the factor a_p enters into matrix X and, hence, into matrix $\hat{G}_a$.

Equation (27) implies that at $\beta=O(1)$, that is, in the neighborhood of a PRA and in the steady-state mode the probing-wave absorption coefficient has two components, a time-independent constant component and a component oscillating with frequency D. Equation (27) can be written in explicit form, which is much more complicated. Here we only list the results of computer calculations, via (27), of the peak values of the constant and oscillating components of these functions. The characteristic curve are depicted in Figs. 3 and 4. In calculating the curves it was assumed that elements of all the matrices in (27) except those of Y and Y_a took on values equal to those at the point of paramagnetic resonance (i.e., only the leading terms in the respective asymptotic expansions were considered). Since in a real situation ε assumes specific values, it must be stressed once more that these relations describe well the behavior of the system characteristics up to $|\beta|=O(\varepsilon^{-1})$. Of course, the smaller ε the more precise for each given β these terms are, that is, the smaller the relative contribution of the next terms in the asymptotic expansion.

Let us now discuss in greater detail the behavior of the constant component of the absorption coefficient. The structures associated with the presence of a parametric resonance are symmetric with respect to the point that corresponds exactly to the condition for the parametric resonance (Fig. 3). Their effective halfwidth γ_c is determined by a complex combination of atomic relaxation constants, and to roughly estimate the halfwidth we can use the approximate relation $\gamma_c\sim\gamma(\xi^2+g^2+g_{21}^2+g_{32}^2+g_{31}^2)^{1/2}$. (Here we assume that the width of the Doppler contour is much larger than γ_c.) For fairly low values of ξ these structures are bell-shaped. As ξ increases (and ξ is linearly connected with the amplitude a of the additional wave) up to $(g^2+g_{21}^2)^{1/2}$, there appears a dip on the bell-shaped curve, and with further increase in ξ the contrast and width of the dip increase. This is similar to the behavior of the characteristics of the Bennett hole,[1,2] which appears when a gas of two-level atoms is irradiated by a saturating monochromatic quasiresonance wave. The important difference is that in our problem the relatively stronger wave (the primary wave) creates the conditions necessary for a hole whose characteristics are determined by the relatively weaker wave (the additional wave). From the viewpoint of the results obtained in Ref. 12 this is natural. Indeed, Ref. 12 shows that a weakly modulated bichromatic quasiresonant field splits the Rabi frequency of a two-level atom. For a low-intensity additional wave, when the splitting is fairly weak, it does not manifest itself and we see a bell-shaped curve, which is the result of convolution of func-

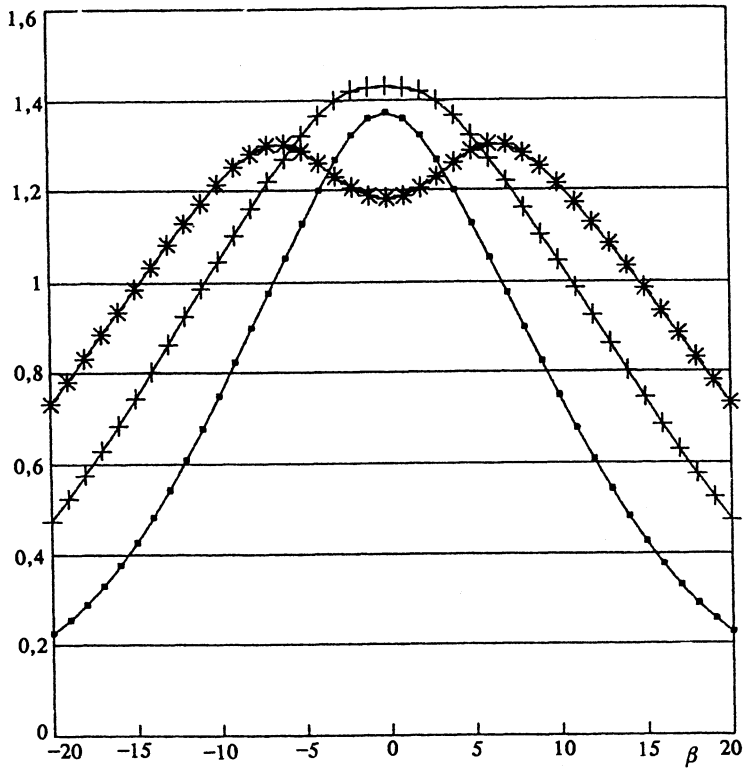

FIG. 4. Amplitudes of the oscillating component of function $V_a(\beta)$: $g_{32}=2$, $g_{31}=1.5$, $g_{21}=2$, $g=1$, $\varkappa=1.2$, $a_1=1/3$, and $\eta=1$. Series 1 corresponds to $a=5$, series 2 to $a=7$, and series 3 to $a=9$.

tions, Lorentzian profiles, with different widths in (27). When the additional-field amplitude is fairly high, the bell-shaped curve acquires a dip corresponding to the splitting of the Rabi spectrum.

Let us write the relations for PRB similar to (27). Let $H_b=\mathrm{diag}\{-i\eta,-2i\eta\}$,

$$U_b=\begin{pmatrix} a & -i\left(\frac{1}{2}\eta-f\right) \\ i\left(\frac{1}{2}\eta-f\right) & a_1 \end{pmatrix}, \quad G_b=U_b^{-1}XU,$$

$$Y_b=\begin{pmatrix} -\left(\frac{1}{2}+\eta f\right)\hat{g}_{32}-\left(\frac{1}{2}-\eta f\right)\hat{g}_{31} & -ia\eta\left(\frac{1}{2}-f\right) \\ -ia\eta\left(\frac{1}{2}-f\right) & -\left(\frac{1}{2}+\eta f\right)\hat{g}_{31}-\left(\frac{1}{2}-\eta f\right)\hat{g}_{32} \end{pmatrix}.$$

The 2×4 matrix $\hat{G}_b$ has only two nonzero elements, $[\hat{G}_b]_{13}=G_{13}(1-\eta)/2$ and $[\hat{G}_b]_{14}=G_{14}(1+\eta)/2$, while $\hat{g}_{32}$ and $\hat{g}_{31}$ have the same meaning as before. Then

$$\chi_b(\beta,\tau)\equiv\chi(\Omega_p^{(b)}+\beta\gamma_1)$$

$$=-\varepsilon^{-1}\exp\left[-\left(\frac{v^{(r)}}{u_T}\right)^2\right]C_0k^{-1}|q_1|V_b(\beta),$$

$$V_b(\beta)=\int\frac{d\alpha}{a_p}\,\mathrm{Im}\{P_2U_b\exp(H_b\tau)Y_b^{-1}\hat{G}_bP_1U^{-1}\mathbf{C}\}. \tag{28}$$

Here the integration variable is introduced, as above, by Eq. (15). Again, in the integrand of (28) only the matrices Y_b and Y (more precisely, their diagonal parts) depend on α and β in the leading term of the asymptotic expansion, while the other objects in the leading term are independent of these parameters. This implies that in the steady-state mode in the neighborhood of a PRB the probing-wave absorption coefficient depends on two oscillating harmonics: one of frequency of 1 and the other of frequency of 2 (D and $2D$ in the initial notation).

5. CONCLUSION

We have discussed the behavior of a gas of three-level atoms that is irradiated by a bichromatic field quasiresonant with one of the transitions, and is probed by a wave on an adjacent transition. We found that under a certain

threshold condition imposed on the intensity of the external field, a fairly weak additional component leads to parametric resonances in certain groups of atoms. These resonances can be of different origin. The probing-wave absorption coefficient in the neighborhood of parametric resonances of the A type (described above) has an oscillating component (with a frequency of D equal to the frequency detuning between the components of the external field) in addition to a constant component. Both components exhibit sharp "singularities" under variations of the probing-wave frequency. The singularities are due to the coincidence of parametric and ordinary resonances of the system of equations describing the dynamics of an atom in the external field. The contrast of the singularities is of the order of $O(1)$ and the width of the order of $O(\gamma)$. The probing-wave absorption coefficient in the neighborhood of parametric resonances of the B type, resonances due to the passage of another ordinary resonance of the system, has two oscillating components, with frequencies D and $2D$, and the behavior of the amplitudes of these components is characterized by similar sharp singularities. Far from parametric resonances the absorption coefficient has no oscillating components.

These phenomena exist if a number of conditions imposed on the parameters of the external field and atoms are met. The most important is the requirement $D \gg \gamma$. These conditions can easily be met in practice (for more details see Ref. 15). It might be interesting to discuss the possible role that the described phenomena play in applications, such as sub-Doppler spectroscopy and the design of optical benchmarks. Recent years have seen an upsurge in the employment of two-photon resonances and resonances on cooled atoms or atomic beams.[2] These methods require complex experimental techniques, and the measured signals have proved to be fairly low. Methods based on the application of parametric resonances have a definite advantage in this respect. But there is a drawback in the use of parametric resonances in comparison to that of two-photon resonances. In the first case all atomic relaxation constants participate in the formation of the effective widths of the singularities in the absorption spectrum, and the larger the constant the greater its role, while for two-photon resonances the width of the singularities is determined by the smallest relaxation constants. This poses the following question: Can an external field (bichromatic or polychromatic) be used to create physical systems with singularities whose widths are determined by the smallest atomic relaxation constants (i.e., ultranarrow resonances)? Next, devices for stabilizing laser frequencies often use the absorption-saturation effect, for instance, in stabilizing a helium-neon laser over an I_2 or CH_4 cell (Ref. 2, Chap. 11). Here the peak used as the reference point has an extremely low contrast, of the order of 0.5% for I_2 and of 1% for CH_4. Hence, methods based on the use of parametric resonances have a certain future.

In this paper we have considered parametric resonances in a gas of atoms. It goes without saying that similar phenomena emerge (and can be treated by a similar asymptotic technique) in situations where no velocity distribution of atoms is present, say, for cooled atoms. In such an experiment, however, in addition to the usual difficulties one encounters a difficulty that lowers the precision of the method compared with the use of a gas of atoms, namely, the low strength of the signal.

The author is grateful to D. A. Firsov and A. S. Kheĭfets for technical support and to M. Z. Smirnov for stimulating discussions.

APPENDIX

Let us show that the eigenvalues of matrix Y satisfy the conditions $\mathrm{Re}\,\Phi_n < 0$. We have $Y = Y_1 + Y_2$, where Y_1 and Y_2 are symmetric:

$$Y_1 = \begin{pmatrix} -\theta_1 & \rho & 0 & 0 \\ \rho & -\theta_2 & 0 & 0 \\ 0 & 0 & -\theta_3 & 0 \\ 0 & 0 & 0 & -\theta_3 \end{pmatrix}.$$

Direct calculations easily show that η_n, the eigenvalues of matrix Y_1, satisfy the conditions $\mathrm{Re}\,\eta_n < 0$. Next, matrix Y_2 differs only in notation from matrix Q_0 and, hence, has eigenvalues on the imaginary matrix. For every four-vector u and an ordinary scalar product, therefore, we have $\mathrm{Re}(\mathbf{u}, Y_1\mathbf{u}) < 0$ and $\mathrm{Re}(\mathbf{u}, Y_2\mathbf{u}) \leqslant 0$. Adding, we obtain $\mathrm{Re}(\mathbf{u}, Y\mathbf{u}) < 0$, from which it immediately follows that $\mathrm{Re}\,\Phi_n < 0$.

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Translated by Eugene Yankovsky