Anomalous electromagnetic waves in a crystal near the zeros of the dielectric constant

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The properties of electromagnetic waves in a crystal in a frequency range where the principal values of the dielectric-constant tensor have opposite signs or vanish are studied. It is shown that in this range there can exist electromagnetic waves, strongly scattered in the medium, with abnormally large wave vectors and with a nonrelativistic phase velocity. The range of existence of such waves in uniaxial and biaxial crystals is found. The waves are generated by the Cherenkov radiation of a nonrelativistic charge.

1. INTRODUCTION

In anisotropic media without spatial dispersion the principal values $\varepsilon_x(\omega)$, $\varepsilon_y(\omega)$, and $\varepsilon_z(\omega)$ of the dielectric-constant tensor are frequency-dependent and vanish, generally speaking, at different frequencies Ω_x, Ω_y and Ω_z. For frequencies intermediate between these values, e.g., $\Omega_x<\omega<\Omega_y$, the values of ε_x and ε_y may have different signs. This leads to peculiar features of the plane waves of such frequency, in particular, to abnormally large values of the wave vector at optical frequencies. Plane waves of frequency ω coinciding with Ω_x, Ω_y or Ω_z, i.e., reducing to zero one of the principal values of the dielectric constant, also have some characteristic features. In particular, the longitudinal waves can propagate only strictly along the relevant axis. Thus, the properties of crystal plane waves in the frequency range mentioned above are of great interest.

2. DIELECTRIC CONSTANT OF A TRANSPARENT CRYSTAL

Consider a crystal in which the imaginary part of the dielectric constant vanishes in a wide range of frequencies between the values ω_1 and $\omega_2 \gg \omega_1$. We align the coordinate axes with the principal axes of the crystal's dielectric-constant tensor ε_{is}. Then, for example, the ith component $\mathbf{D}_i$ of the electric induction depends only on the ith component $\mathbf{E}_i$ of the electric field and, according to the causality principle, only on the values of $\mathbf{E}_i$ at preceding times. Then, similar to the reasoning for an isotropic medium,[1] we can find the Kramers–Kronig dispersion relations for the principal values $\varepsilon_x=\varepsilon'_x+i\varepsilon''_x$, $\varepsilon_y=\varepsilon'_y+i\varepsilon''_y$, and $\varepsilon_z=\varepsilon'_z+i\varepsilon''_z$ of the dielectric-constant tensor,

$$\varepsilon'_i(\omega)-1=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\varepsilon''_i(\omega)}{\omega'-\omega}d\omega' \quad (i=x,y,z), \tag{1}$$

$$\varepsilon''_i(\omega)=-\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\varepsilon'_i(\omega)-1}{\omega'-\omega}d\omega'. \tag{2}$$

For transparent media and ω frequencies satisfying the condition $\omega_2\gg\omega\gg\omega_1$, similar to the case of isotropic media, we can find from (1) and (2)

$$\varepsilon_i(\omega)=\varepsilon_i^0[1-(\Omega_i/\omega)^2], \tag{3}$$

where

$$\varepsilon_i^0=1+\frac{2}{\pi}\int_{\omega_2}^{\infty}\varepsilon''_i(\omega')\frac{d\omega'}{\omega'},$$

$$\varepsilon_i^0\Omega_i^2=\frac{2}{\pi}\int_0^{\omega_1}\omega' d\omega' \varepsilon''_i(\omega'). \tag{4}$$

If for each principal value ε_i we introduce its own oscillator strength

$$f_i(\omega)=\frac{m}{2\pi^2e^2}\omega\varepsilon''(\omega), \tag{5}$$

then

$$\varepsilon_i^0=1+\frac{4\pi e^2}{m}\int_{\omega_2}^{\infty}f_i(\omega')\frac{d\omega'}{\omega'^2};$$

$$\varepsilon_i^0\Omega_i^2=\frac{4\pi e^2}{m}\int_0^{\omega_1}f_i(\omega')d\omega'. \tag{6}$$

It follows from (3) that each principal value $\varepsilon_i(\omega)$ vanishes at $\omega=\Omega_i$ if the transparency region is wide enough, i.e., $\omega_1\ll\omega\ll\omega_2$. Thus, $\varepsilon_x(\omega)$, $\varepsilon_y(\omega)$ and $\varepsilon_z(\omega)$ vanish, generally speaking, at different frequencies.

3. RELATION BETWEEN THE CRYSTAL DIELECTRIC CONSTANT AND THE POLARIZABILITY OF A SINGLE MOLECULE

As is well-known, the polarizability $\alpha_{ik}(\omega)$ of a single molecule is[2]

$$\alpha_{ik}(\omega)=\sum_n\left\{\frac{(1|d_i|n)(n|d_k|1)}{\omega_{\mathrm{nl}}-\omega-i0}+\frac{(1|d_k|n)(n|d_i|1)}{\omega_{\mathrm{nl}}+\omega+i0}\right\}. \tag{7}$$

Consider an axially symmetric molecule whose length is larger than its width. At frequencies for which the quantum energy $\hbar\omega$ is small in comparison with the ionization potential, the inner electrons make a small contribution to α_{ik}, since for them the intermediate states with $|\omega_{\mathrm{nl}}|\sim\omega$ are occupied by other electrons and only the intermediate states with $|\omega_{\mathrm{nl}}|\gg\omega$ are available. Therefore the main con-

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tribution to (7) comes from valence electrons whose wave functions pertain simultaneously to several atoms and are, therefore, stretched along the molecule axis.

For such wave functions we can assume that

$$|(1|z|n)|\gg|(1|x|n)|$$

if the molecule length is large in comparison with its width. If ω is far enough from any of the frequencies $\omega_{\rm nl}$, we can rewrite (7) in the form

$$\alpha_{ik}(\omega)=\alpha(\omega)(\delta_{ik}-e_ie_k)+\beta(\omega)e_ie_k, \tag{8}$$

where we have used the following notation:

$$\alpha(\omega)=e^2\sum_n|(1|x|n)|^2\frac{2\omega_{\rm nl}}{\omega_{\rm nl}^2-\omega^2}, \tag{9}$$

$$\beta(\omega)=e^2\sum_n|(1|z|n)|^2\frac{2\omega_{\rm nl}}{\omega_{\rm nl}^2-\omega^2}. \tag{10}$$

Here e_i is a unit vector in the direction of the molecule axis. Consider now a crystal consisting of axially symmetric molecules whose axes are equally oriented along the vector $\mathbf{e}$, forming a uniaxial crystal with the principal optical axis along $\mathbf{e}$.

In macroscopic electrodynamics the crystal is regarded as a homogeneous anisotropic medium. This means that the spatial distribution of molecules can be considered uniform, i.e., we can neglect the existence of the lattice while considering the spatial distribution of the molecule centers of inertia. The lattice is allowed for by considering the directions of molecule orientation. Therefore we assume below that all the molecules are oriented along $\mathbf{e}$ and are uniformly distributed in space.

The polarization $\mathbf{P}$ of the crystal is proportional to the mean local field $\mathbf{E}^{\rm loc}$ acting upon each molecule, so that the Fourier-transforms of these quantities obey the relation

$$P_i(\mathbf{r},\omega)=n_0\alpha_{is}(\omega)E_s^{\rm loc}(\mathbf{r},\omega), \tag{11}$$

where n_0 is the number of molecules per unit volume. To calculate the mean field, consider first the microscopic field $\mathbf{E}^{\rm mic}$ acting on the molecule at a point $\mathbf{R}_a$. This field is the sum of a primary field $\mathbf{E}^0$ (which has not yet interacted with any molecule) and secondary fields created by dipole moments of all other molecules, $\mathbf{d}(\mathbf{R}_b,t)$. Therefore we can write for the field Fourier-transforms[3] ($\mathbf{R}_{ab}\equiv\mathbf{R}_a-\mathbf{R}_b$):

$$E_i^{\rm mic}(\mathbf{R}_a,\omega)=E_i^0(\mathbf{R}_a,\omega)+\frac{1}{2\pi^2}\sum_{R_b}\int d^3lQ_{is}(\mathbf{l},\omega)d_s(\mathbf{R}_b,\omega)\exp(i\mathbf{l}\mathbf{R}_{ab}), \tag{12}$$

where

$$Q_{is}(\mathbf{l},\omega)=(l^2\delta_{is}-l_il_s)/[l^2-(\omega/c)^2]. \tag{13}$$

Equation (12) becomes integral, if we take it into account that

$$d_s(\mathbf{R}_b,\omega)=\alpha_{sk}(\omega)E_k^{\rm mic}(\mathbf{R}_b,\omega). \tag{14}$$

The mean local field is found from the microscopic one by averaging over the coordinates of all molecules except the one considered. As noted above, in crystal optics this averaging can be carried out just as in amorphous media, which gives, instead of (12),

$$E_i^{\rm loc}(\mathbf{R}_a,\omega)=E_i^0(\mathbf{R}_a,\omega)+\frac{1}{2\pi^2}\int d^3lQ_{is}(l,\omega)\alpha_{sk}(\omega)\times\int d^3R_{ab}g(\mathbf{R}_{ab})E_k^{\rm loc}(\mathbf{R}_a-\mathbf{R}_{ab},\omega)\times\exp(i\mathbf{l}\mathbf{R}_{ab}), \tag{15}$$

where $g(\mathbf{R}_{ab})\equiv n_0[1-f(\mathbf{R}_{ab})]$ is the function characterizing the molecule distribution on the average relative to the molecule considered. Since, on the average, the molecules are uniformly distributed, around each of them there is some molecule-free space characterized by the function $f(\mathbf{R}_{ab})$. Multiplying (15) by $n_0\alpha_{ji}(\omega)$ and switching to Fourier-transforms in coordinates, we find

$$P_j(\mathbf{q},\omega)=n_0\alpha_{ji}(\omega)E_i^0(\mathbf{q},\omega)+4\pi n_0\alpha_{ji}(\omega)Q_{is}(\mathbf{q},\omega)P_s(\mathbf{q},\omega)-4\pi n_0\alpha_{ji}(\omega)\int d^3lQ_{is}(\mathbf{l},\omega)f(\mathbf{q}-\mathbf{l})P_s(\mathbf{q},\omega), \tag{16}$$

where

$$f(\mathbf{q}-\mathbf{l})=(2\pi)^{-3}\int d^3R_{ab}f(\mathbf{R}_{ab})\exp[i(\mathbf{q}-\mathbf{l})\mathbf{R}_{ab}]. \tag{17}$$

Equation (16) relates the polarization to the primary field, i.e., the field obeying the Maxwell equations in vacuum with the same current density as in the Maxwell equations in matter for the mean macroscopic field $\mathbf{E}$. This allows us to express the primary field $\mathbf{E}^0$ through $\mathbf{E}$ and $\mathbf{P}$:

$$E_i^0(\mathbf{q},\omega)=E_i(\mathbf{q},\omega)+4\pi P_i(\mathbf{q},\omega)-4\pi Q_{is}(\mathbf{q},\omega)P_s(\mathbf{q},\omega). \tag{18}$$

Eliminating E_i^0 from (16) with the help of (19), we get an equation connecting the mean macroscopic field $\mathbf{E}$ with the polarization:

$$n_0\alpha_{ij}E_j(\mathbf{q},\omega)=P_i(\mathbf{q},\omega)-4\pi n_0\alpha_{ij}P_j(\mathbf{q},\omega)-4\pi n_0\alpha_{ij}\int d^3lf(\mathbf{q}-\mathbf{l})Q_{js}(\mathbf{l},\omega)P_s(\mathbf{q},\omega). \tag{19}$$

The integral over $\mathbf{l}$ in (19) is determined by the function $f(\mathbf{q}-\mathbf{l})$ which is finite only in the region where its argument does not exceed $n_0^{1/3}$ in the order of magnitude. If we consider the field in the optical range, then $\omega/c\ll n_0^{1/3}$ and $|\mathbf{q}|\ll n_0^{1/3}$. This allows to set $f(\mathbf{q}-\mathbf{l})=f(-\mathbf{l})$ and to neglect ω/c in the denominator of (13). Thus, we find

$$I_{js}=\int\frac{d^3lf(\mathbf{q}-\mathbf{l})}{l^2-(\omega/c)^2}(l^2\delta_{js}-l_jl_s)=\int d^3lf(-\mathbf{l})\left(\delta_{js}-\frac{l_jl_s}{e^2}\right)=a\delta_{js}+be_je_s. \tag{20}$$

As mentioned above, the function $f(\mathbf{R}_{ab})$ characterizes the molecule-free volume near the molecule considered. In a uniaxial crystal this volume should be axially symmetric, which is taken into account in (20). If this volume has spherical symmetry, we have $b=0$. Introducing an angle ϑ' between $\mathbf{l}$ and the principal optic axis, we find from (20):

$$2a=\int d^3l f(-\mathbf{l})(1+\cos^2\vartheta');$$
$$2b=\int d^3l f(-\mathbf{l})(1-3\cos^2\vartheta'). \tag{21}$$

Multiplying (19) by e_i and using (8), we can easily find for the induction component along $\mathbf{e}$:

$$\begin{aligned}\mathbf{eD}&=\mathbf{eE}+4\pi\mathbf{eP}\\&=\mathbf{eE}[1+4\pi n_0\beta(a+b)]/[1-4\pi n_0\beta(1-a-b)],\end{aligned} \tag{22}$$

whence

$$\varepsilon_{\|}=\frac{1+4\pi n_0\beta(a+b)}{1-4\pi n_0\beta(1-a-b)}. \tag{23}$$

Denoting the field and polarization components normal to the crystal axis by $\mathbf{E}^{\perp}$ and $\mathbf{P}^{\perp}$ respectively, we can write the result of multiplying (19) by the tensor $(\delta_{ik}-e_ie_k)$ in the form

$$n_0\alpha E_k^{\perp}=P_k^{\perp}\{1-4\pi n_0\alpha(1-a)\}, \tag{24}$$

whence

$$\varepsilon_{\perp}(\omega)=\frac{1+4\pi n_0\alpha a}{1-4\pi\alpha n_0(1-a)}. \tag{25}$$

It follows from (23) and (25) that the frequencies $\Omega_{\perp}$ and $\Omega_{\|}$ which reduce $\varepsilon_{\perp}(\omega)$ and $\varepsilon_{\|}(\omega)$ to zero are given by the equations

$$1+4\pi n_0e^2a\sum_n|(1|x|n)|^2\frac{2\omega_{n1}}{\omega_{n1}^2-\Omega_{\perp}^2}=0,$$
$$1+4\pi n_0e^2(a+b)\sum_n|(1|z|n)|^2\frac{2\omega_{n1}}{\omega_{n1}^2-\Omega_{\|}^2}=0. \tag{26}$$

In the particular case, when $f(\mathbf{R}_{ab})$ is isotropic, we have $b=0$ and $a=\frac{2}{3}$. For a spherically symmetric molecule, $\alpha=\beta$. The frequencies $\Omega_{\perp}$ and $\Omega_{\|}$ coincide only when $\alpha=\beta$ and $b=0$. In a noncubic lattice, even for spherically symmetric molecules, the values $\varepsilon_{\perp}$ and $\varepsilon_{\|}$, and also $\Omega_{\perp}$ and $\Omega_{\|}$ do not, generally speaking, coincide.

4. SLOW EXTRAORDINARY WAVES IN A UNIAXIAL CRYSTAL

Consider extraordinary waves in a uniaxial crystal in the frequency range between $\Omega_{\perp}$ and $\Omega_{\|}$, when $\varepsilon_{\perp}(\omega)$ and $\varepsilon_{\|}(\omega)$ have opposite signs.

The dispersion equation for the extraordinary wave has the form

$$k^2=\left(\frac{\omega}{c}\right)^2\varepsilon_{\perp}\frac{1+\varkappa}{1+\varkappa\cos^2\vartheta},\quad\left(\varkappa\equiv\frac{\varepsilon_{\|}}{\varepsilon_{\perp}}-1\right), \tag{27}$$

where ϑ is the angle between the wave vector $\mathbf{k}$ and the principal optical axis of the crystal. For positive uniaxial crystals ($\varepsilon_{\|}>\varepsilon_{\perp}$, and $\varkappa>0$) an undamped extraordinary wave exists for any ϑ if $\varepsilon_{\perp}>0$, and does not if $\varepsilon_{\perp}<0$. For negative uniaxial crystals ($\varepsilon_{\|}<\varepsilon_{\perp}$, and $\varkappa<0$) the undamped extraordinary wave exists for any ϑ if $\varepsilon_{\perp}>0$ and $|\varkappa|<1$. In the case when $\varkappa=-|\varkappa|<-1$, for $\varepsilon_{\perp}>0$ the extraordinary wave exists in the angle range $\vartheta>\vartheta_0$ and for $\varepsilon_{\perp}<0$ in the range $\vartheta<\vartheta_0$, where ϑ_0 is defined by the relation

$$\cos\vartheta_0=(1/\sqrt{|\varkappa|}). \tag{28}$$

When $\varkappa\sim-1$, the denominator in (27) can be so small that the wave vector $\mathbf{k}$ becomes abnormally large, i.e., $k\gg\omega/c$. The wave phase velocity becomes nonrelativistic:

$$v_{\text{ph}}=(\omega/k)\ll c, \tag{29}$$

so that it is natural to call such waves slow.

For macroscopic electrodynamics to be valid, k must be small in comparison with the inverse lattice constant L^{-1}, so that for slow waves the following inequality should hold:

$$(c/L\omega)^2\gg\varepsilon_{\perp}(1-|\varkappa|)/(1-|\varkappa|\cos^2\vartheta)\gg1. \tag{30}$$

The condition $\varkappa\sim-1$ necessary for the slow wave existence can be satisfied, in particular, in the frequency range where $\varepsilon_{\|}$ is negative and small in absolute value and $\varepsilon_{\perp}>0$. This is satisfied for ω close to $\Omega_{\|}$.

The inequality (30) holds only in a comparatively narrow angle range, $\vartheta\sim\vartheta_0$ [see (28)]. From (30) we can define the range where slow waves exist by the inequality

$$(c/\omega L)^2\gg\sqrt{|\varkappa|-1}/2(\vartheta_0-\vartheta)\gg1. \tag{31}$$

The electric field $\mathbf{E}$ and induction $\mathbf{D}$ in the slow wave, as always in the extraordinary wave, are in the plane defined by the vectors $\mathbf{e}$ and $\mathbf{k}$ (the principal cross-section plane) and the magnetic field $\mathbf{H}$ is normal to this plane.

The short wavelength of slow waves makes them more sensitive to small inhomogeneities and, in particular, to scattering. Since slow waves exist only in a narrow angle range, the result is that a scattered wave leaves this range and becomes an ordinary wave. The slow wave extinction coefficient turns out to be much larger than in the case of ordinary waves, and the attenuation of the slow-wave intensity due to scattering goes much faster then usually. The fact that the extinction coefficient is proportional to the fourth power of the wave vector clearly indicates that scattering can strongly affect slow wave propagation.

Small phase velocity of slow waves results in that the processes of radiation of such waves differ from those of ordinary waves. For example, if the velocity v of a charge moving uniformly in a crystal is larger than the slow-wave phase velocity v_{ph} but smaller than the phase velocity $c/\sqrt{\varepsilon_{\perp}}$ of an ordinary wave, the Cherenkov radiation of the charge consists of slow waves.

Such a source radiates only slow waves, while in other cases ordinary waves are also emitted.

One has to bear in mind that for the Cherenkov radiation of slow waves to arise, the charge velocity direction with respect to the crystal should be such as to satisfy the inequality (31) for the angle range in which the slow waves exist.

For the Cherenkov radiation to exist, the following equations must hold

$$v_{\mathrm{ph}}=v\cos\alpha,\quad \text{or}\quad \omega=\mathbf{kv}=kv\cos\alpha,$$

where α is the angle between the radiation direction and the particle momentum. Introducing the angles θ and Φ determining the particle velocity direction, we can write

$$\cos\alpha=\cos\vartheta\cos\theta+\sin\vartheta\sin\theta\cos(\varphi-\Phi),$$

(φ is the azimuth of the wave-vector $\mathbf{k}$). Using this relation together with (27), we can write the condition of the Cherenkov-radiation existence in the form

$$\begin{aligned}&\mathrm{tg}^2\vartheta[1-\Lambda\sin^2\theta\cos^2(\varphi-\Phi)]\\&\quad-2\,\mathrm{tg}\,\vartheta\Lambda\sin\theta\cos\theta\cos(\vartheta-\Phi)\\&\quad+(1-|\varkappa|)-\Lambda\cos^2\theta=0.\end{aligned}$$

For a charge moving along the principal axis $\alpha=\vartheta$, $\theta=0$ and

$$\cos^2\vartheta=\frac{c^2}{|\varkappa|c^2-v^2|\varepsilon_{\|}|},$$

this angle is in the range of slow wave existence given by (31).

The emitted energy can be found from an expression derived in Ref. 4 for charge energy losses in an arbitrary crystal. In the case considered the lost energy goes into slow wave formation. Note that using the results of Refs. 4 and 5, one must start from the exact expression for the losses and proceed to the nonrelativistic limit, allowing for smallness of the ratio $\omega/ck\sim v_{\mathrm{ph}}/c\sim v/c$. The expression given in Refs. 4 and 5 for the nonrelativistic case corresponds to $k\sim\omega/c$ and is not applicable to slow waves.

5. SLOW ELECTROMAGNETIC WAVES IN A BIAXIAL CRYSTAL

Slow plane waves with wave vectors satisfying the condition $1/L\gg k\gg\omega/c$ in a biaxial crystal can be easily described by neglecting in the Fresnel equation the terms of order $(\omega/ck)^2$. Then the dispersion equation takes the form

$$k^2=\left(\frac{\omega}{c}\right)^2\times\frac{\varepsilon_x\varepsilon_y+\varepsilon_z(\varepsilon_x\cos^2\varphi+\varepsilon_y\sin^2\varphi)+(\varepsilon_x+\varepsilon_y)\varepsilon_z\,\mathrm{ctg}^2\vartheta}{\varepsilon_x\cos^2\varphi+\varepsilon_y\sin^2\varphi+\varepsilon_z\,\mathrm{ctg}^2\vartheta}. \tag{32}$$

The range where such waves exist is given by the inequality

$$\left(\frac{c}{L\omega}\right)^2\gg\frac{\varepsilon_x\varepsilon_y+\varepsilon_z(\varepsilon_x\cos^2\varphi+\varepsilon_y\sin^2\varphi)+\varepsilon_z(\varepsilon_x+\varepsilon_y)\mathrm{ctg}^2\vartheta}{\varepsilon_x\cos^2\varphi+\varepsilon_y\sin^2\varphi+\varepsilon_z\,\mathrm{ctg}^2\vartheta}\gg1. \tag{33}$$

The second inequality can be satisfied if the denominator is small, i.e., if the principal values ε_x, ε_y, and ε_z of the dielectric constant can be positive as well as negative. This is possible, in particular, at frequencies ω between two zeros of these principal values, e.g., for $\Omega_x<\omega<\Omega_y$ or $\Omega_x<\omega<\Omega_z$. We assume, for definiteness, that $\varepsilon_z>0$, while ε_x and $\varepsilon_y<0$. The denominator in (33) vanishes for the angles $\vartheta_0(\varphi)$ defined by

$$\vartheta_0(\varphi)=\mathrm{arctg}\left\{\frac{1}{\sqrt{|\varepsilon_z|}}\sqrt{(\varepsilon_x-\varepsilon_y)\cos^2\varphi+\varepsilon_y}\right\}. \tag{34}$$

The angles ϑ and φ cannot approach the line $\vartheta_0(\varphi)$ in the $\vartheta\varphi$ plane owing to the left-hand side of inequality (33). On the other hand, they cannot move away too far from it because of right-hand side of the inequality (33).

As is well-known, in a biaxial crystal, given the frequency and the propagation direction, the Fresnel equation has two solutions for the absolute value of the wave vector.

Equation (32) gives only one solution for k^2. This means that out of two waves with the same propagation direction and frequency only one can be turned into a slow wave by adjusting the propagation direction.

The general properties of slow waves in a biaxial crystal is similar to the properties, considered above, of slow extraordinary waves in a uniaxial crystal.

6. PLANE WAVES OF A FREQUENCY THAT REDUCES TO ZERO ONE OF THE PRINCIPAL VALUES OF THE DIELECTRIC CONSTANT

In isotropic media, at a frequency that reduces to zero the dielectric constant, there can be no transverse plane waves, but there exists a longitudinal wave whose wave vector can be found, with allowance for spatial dispersion, from the condition $\varepsilon(\omega,\mathbf{k})=0$.

That one of the principal values of the crystal dielectric constant vanishes does not, generally speaking, disagree with plane wave propagation. To prove this, consider a plane wave of frequency $\omega=\Omega_z$ for which $\varepsilon_z(\Omega_x)=0$. The Maxwell equations yield

$$\begin{aligned}E_x&=\frac{k^2}{k^2-(\omega/c)^2\varepsilon_x}\left(\frac{k_x}{k_z}\right)E_z;\\E_y&=\frac{k^2}{k^2-(\omega/c)^2\varepsilon_y}\left(\frac{k_y}{k_z}\right)E_z.\end{aligned} \tag{35}$$

If the wave propagates along the z axis, then $k_x=k_y=0$, $E_x=E_y=0$, and $\mathbf{H}=0$, so that the wave is longitudinal. Its wave vector $\mathbf{k}$ is parallel to the electric field, and its value $\mathbf{k}$ is determined, with allowance for spatial diapersion, from the equation $\varepsilon_z(\omega,\mathbf{k})=0$. If $\mathbf{k}$ deviates from the z axis, transverse, with respect to **k**, components of the fields **E** and **H** arise. The Fresnel dispersion equation has then the form

$$k^2 = \left(\frac{\omega}{c}\right)^2 \frac{\varepsilon_x \varepsilon_y}{\varepsilon_x \cos^2 \varphi + \varepsilon_y \sin^2 \varphi}. \tag{36}$$

Since, while deriving (36), we have cancelled out $\sin^2 \vartheta$, this expression is inapplicable at $\vartheta = 0$.

For $\vartheta \neq 0$ and positive ε_x and ε_y a solution of Eq. (36) exists for any φ. Let now ε_x and ε_y have opposite signs, $\varepsilon_y = -|\varepsilon_y| < 0 < \varepsilon_x$. Then

$$k^2 = \left(\frac{\omega}{c}\right)^2 \frac{\varepsilon_x |\varepsilon_y|}{|\varepsilon_y| - (\varepsilon_x + |\varepsilon_y|)\cos^2 \varphi}$$

and the solution exists only in the angle range

$$\cos^2 \varphi < \frac{|\varepsilon_y|}{|\varepsilon_y| + \varepsilon_x}.$$

In a more narrow range

$$(c/L\omega)^2 \gg [\sqrt{\varepsilon_x |\varepsilon_y|}/2(\varphi - \varphi_0)] \gg 1, \tag{37}$$

where $\varphi_0 = \arccos[|\varepsilon_y|/(\varepsilon_x + \varepsilon_y)]^{1/2}$, the wave becomes slow, i.e., we have $k \gg \omega/c$ and $v_{\text{ph}} = \omega/k \ll c$.

The fact that no transverse waves can propagate along the crystal principal axis for which the principal value of the dielectric constant is zero can result in a curious effect when light of this frequency is reflected by the crystal. If the reflected-wave direction coincides with the direction forbidden for transverse waves, then, naturally, total internal reflection of an incident transverse wave arises. However, even a small deviation of the incident wave ensures the existence of a refracted wave carrying away a part of the incident-wave energy. This means that the reflection coefficient for a plane wave of frequency $\omega = \Omega_z$ (Ω_x or Ω_y) versus the angle of incidence must have a sharp maximum when the angle of incidence corresponds to the direction in which a refracted transverse wave does not exist. For other angles the reflection coefficient varies in an ordinary manner.

7. DISCUSSION

It follows from the above results that, at frequencies close to the zeros of the dielectric-constant principal values, the properties of electromagnetic plane waves in crystals can differ dramatically from the properties of the waves usually considered in crystal optics. For a uniaxial crystal, in particular, the range where such waves exist is determined by the frequencies $\Omega_\perp$ and $\Omega_\parallel$ which decrease to zero the functions $\varepsilon_\perp(\omega)$ and $\varepsilon_\parallel(\omega)$. In order to find $\Omega_\perp$ and $\Omega_\parallel$ from (26) we need information about the substance structure somewhat more detailed than usually, i.e., we need to know the size and shape of the volume surrounding a molecule and containing on the average, no other molecules. However all the information about this volume is given by the parameters a and b in (26). In the absence of additional information about the substance structure, a and b can be regarded as phenomenological parameters on which the dielectric constant depends. Having found the parameters a and b from experiment, we can then use them for prediction of other experimental results pertaining to the same substance.

It follows also that at the frequency ω that reduces to zero one of the principal values of the dielectric constant, e.g., $\varepsilon_z(\omega) = 0$, both longitudinal and transverse waves can exist.

The longitudinal waves can propagate only in one direction, strictly along the z axis. A fixed direction of longitudinal wave propagation in such a crystal creates peculiar conditions for processes connected with longitudinal waves: their intensity strongly depends on the particle velocity orientation.

Other principal values, ε_x and ε_y, can also vanish, but at other frequencies. This means that longitudinal waves frequencies of their own can propagate along each axis, but there are no longitudinal waves in intermediate directions.

The most interesting result above is that when the principal values of the dielectric constant have different signs, we arrive at a conclusion that there must exist slow electromagnetic waves with nonrelativistic phase velocity and short wavelength, sensitive to substance inhomogeneities. In particular, the range where these waves exist lies between the zeros of the uniaxial crystal dielectric constant, $\Omega_\perp$ and $\Omega_\parallel$. Since we have restricted the discussion to the case of real principal values of the dielectric constant, it is necessary also that the imaginary parts of $\varepsilon_\perp$ and $\varepsilon_\parallel$ be small at frequencies between $\Omega_\perp$ and $\Omega_\parallel$.

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