# A classical representation for the one-dimensional Schrödinger equation

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A new representation for the one-dimensional Schrödinger equation is derived employing the Abel transformation for the square of the wave function. This representation is naturally called classical since the probability density of a classical state with energy $\varepsilon$ in potential $V(x)$ serves as the kernel of the respective integral operator. In meaning the representation is most closely associated with the Wigner distribution method, but differs from the latter in formal structure.

The relationship between classical mechanics and quantum mechanics is one of the central problems of physics. Actually, we always connect the understanding of quantum mechanical results with the possibility of interpreting them in classical terms, with the interpretation based on the WKB approximation. This approximation, however, does not make it possible to completely reduce the problem to the classical one. For instance, the behavior of a particle near a turning point is quantum mechanical in principle, and a complete solution in this approximation is obtained by matching solutions in the regions of quantum and classical motion. The impossibility of a purely classical approximate solution is usually related to the existence of the tunneling effect, which in its very essence is incompatible with the classical approach. For instance, if we consider the problem of a particle scattered by a parabolic barrier, we find that, on the one hand, there is the effect of subbarrier transition (tunneling) of the particle and, on the other, the time-dependent Green's function, which carries the full information about the behavior of the quantum system, can be exactly expressed in terms of the action along classically allowed trajectories. Thus, all the features of the behavior of the particle, including the tunneling effect, can be reduced to analyzing classical trajectories. This analysis shows that with a quantum state with a negative energy (below the top of the barrier) there is associated an ensemble of classical trajectories with any energy values and that the tunnel effect is related to the presence in this ensemble of classical states with positive energies. Here the contribution of these states is exponentially small in comparison to the Planck constant $\hbar$, which reflects the fact that the barrier penetrability is exponentially small. This suggests an approach that is an alternative to the WKB approximation and uses classical trajectories with any energy values. A variant of such an approach is considered below. In its meaning the approach most closely resembles the Wigner distribution method,[1] but differs from the latter in the formal structure.

From general considerations it is clear that to reconcile quantum and classical descriptions more completely we must start by seeking the relation between probability densities. A linear equation for the quantum mechanical probability density $\rho_n(x)=\Psi_n^2(x)$, which we will need in our future discussions, can be obtained from the Schrödinger equation

$$\left\{-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}+V(x)\right\}\psi_n(x)=E_n\psi_n(x) \tag{1}$$

in the following manner. Substituting $\psi_n(x)$ in the form

$$\psi_n(x)=\rho_n^{1/2}(x)$$

into Eq. (1) leads to the following nonlinear equation for the probability density

$$-\frac{\hbar^2}{4m}\left\{\rho_n^{-1/2}\frac{d^2\rho_n}{dx^2}-\frac{1}{2}\rho_n^{-3/2}\left(\frac{d\rho_n}{dx}\right)^2\right\}+V(x)\rho_n^{1/2}=E_n\rho_n^{1/2}.$$

To reduce this equation to the linear form, we must multiply both sides by $\rho_n^{3/2}(x)$ and differentiate the product with respect to $x$. The result is the desired equation

$$-\frac{\hbar^2}{4m}\frac{d^3\rho_n}{dx^3}+2[V(x)-E_n]\frac{d\rho_n}{dx}+\frac{dV(x)}{dx}\rho_n(x)=0, \tag{2}$$

which is a third-order equation and has three linearly independent equations. As shown in Ref. 2, two of these are related to the solutions of the Schrödinger equation and the third to the solution of the Milne equation having the meaning of the quantum wavelength. The orthogonality condition for bound states is obtained from Eq. (2) in the standard way and has the form

$$\int_{-\infty}^{\infty}dx\rho_k(x)\frac{d}{dx}(\rho_n(x))=0. \tag{3}$$

This condition was obtained for the first time by Jost and Kohn,[3] who constructed a family of potentials all having the same spectrum.

From now on for the sake of simplicity we consider the problem of a particle in a symmetric potential $V(x)=V(-x)$, where $V(0)=0$ and $V(x)$ increases monotonically to infinity on the semiaxis $x>0$.

In accordance with the above reasoning, we must represent the quantum state $|n\rangle$ as an ensemble of classical states in the potential $V(x)$, states distributed in the energy $\varepsilon$ with a certain probability density $f_n(\varepsilon)$. The probability of discovering a classical particle with energy $\varepsilon$ in an in-

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terval $dx$ is proportional to the time that the particle spends in this interval. Hence, the probability density in $x$ normalized by the condition

$$\int_{x_1}^{x_2} dx W(\varepsilon, x) = 1$$

is

$$W(\varepsilon, x) = \frac{2}{T(\varepsilon)} \frac{dt}{dx} = \frac{2}{T(\varepsilon)} \left[ \frac{2}{m} (\varepsilon - V(x)) \right]^{-1/2}, \quad (4)$$

where

$$T(\varepsilon) = 2 \int_{x_1}^{x_2} dx \left[ \frac{2}{m} (\varepsilon - V(x)) \right]^{-1/2} \quad (5)$$

is the oscillation period, and $x_1$ and $x_2$ the turning points. The probability density at point $x$ for the entire ensemble is obtained by averaging $W(\varepsilon, x)$ over the distribution $f_n(\varepsilon)$ in the classically allowed energy region $\varepsilon \geqslant V(x)$:

$$\rho_n(x) = \int_{V(x)}^{\infty} d\varepsilon W(\varepsilon, x) f_n(\varepsilon) = \int_{V(x)}^{\infty} d\varepsilon \frac{\phi_n(\varepsilon)}{(\varepsilon - V(x))^{1/2}}, \quad (6)$$

where

$$\phi_n(\varepsilon) = (2m)^{1/2} \frac{f_n(\varepsilon)}{T(\varepsilon)}. \quad (7)$$

The function $\phi_n(\varepsilon)$ has the meaning of the mathematical expectation of the frequency $\omega(\varepsilon) = 2\pi/T(\varepsilon)$ and is actually the renormalized energy distribution. We will use it instead of $f_n(\varepsilon)$ when convenient.

A remarkable property of the representation (6) is its reciprocity, owing to which for each solution of Eq. (2) one can build a classical ensemble with the same probability density. This becomes especially evident if instead of $x$ we take the potential $V$ as the independent variable, which corresponds in a one-to-one manner to $x$ on the semiaxis $x > 0$. Such a change of variable is standard in classical mechanics[4] and in the given case reduces the problem of finding $\phi_n(\varepsilon)$ from the integral equation (6) to the well-known Abel problem,[5] whence

$$\phi_n(\varepsilon) = -\frac{1}{\pi} \frac{d}{d\varepsilon} \left( \int_{\varepsilon}^{\infty} dV \frac{\rho_n(x(V))}{(V-\varepsilon)^{1/2}} \right), \quad (8a)$$

where $x(V)$ is the inverse of function $V(x)$. Integrating by parts and differentiating with respect to $\varepsilon$ transforms this expression into

$$\phi_n(\varepsilon) = -\frac{1}{\pi} \int_{\varepsilon}^{\infty} dV \frac{1}{(V-\varepsilon)^{1/2}} \frac{d\rho_n}{dV}. \quad (8b)$$

The Abel transformations given by Eqs. (6) and (8) are exact and reversible and fix a certain new representation in quantum mechanics, which is naturally called classical since the probability density (4) of a classical state with energy $\varepsilon$ in potential $V(x)$ is the kernel of the corresponding integral operator. However, in the new representation a literal interpretation of a quantum state as an ensemble of classical trajectories is impossible since the quantity $f_n(\varepsilon)$ calculated by Eq. (8) can assume negative values. From the formal viewpoint the Abel transformation plays the same role and possesses the same properties as the Fourier transformation in the momentum representation. The basic difference is that it is used for the probability density rather than for the amplitude.

The general properties of the distribution $f_n(\varepsilon)$ are listed below, and the respective derivation is given in Appendix A. The function $f_n(\varepsilon)$ obeys the normalization condition

$$\int_0^{\infty} d\varepsilon f_n(\varepsilon) = \int_{-\infty}^{\infty} dx \rho_n(x) = 1, \quad (9)$$

which, generally speaking, was obvious from the very beginning from the meaning of the transformations. In view of the nonunitarity of transformations (6) and (8), the orthogonality condition (3) in the classical representation assumes the form

$$\int_0^{\infty} d\varepsilon \gamma_k(\varepsilon) \phi_n(\varepsilon) = 0, \quad (10)$$

where $\gamma_k(\varepsilon)$ is the image of $\phi_k(\varepsilon)$ in the adjoint function space. In addition to the orthonormalization conditions, the condition

$$\int_0^{\infty} d\varepsilon f_n(\varepsilon) \varepsilon = E_n, \quad (11)$$

which is not trivial from a physical standpoint and means that the average energy of the corresponding ensemble of classical states corresponds to the bound-state energy $E_n$, is generally also met.

Now let us derive the Schrödinger equation in the classical representation for the function $\phi_n(\varepsilon)$, which is related to the distribution $f_n(\varepsilon)$ through the simple relation (7). The linear equation (2) for the square of the wave function was introduced for this very reason. Since the equation contains a third derivative and the kernel of transformation (6) has a root singularity at one of the ends of the integration interval, we rewrite (6), using integration by parts, in a form suitable for further differentiation,

$$\rho_n(x) = -\frac{8}{15} \int_{V(x)}^{\infty} d\varepsilon (\varepsilon - V(x))^{5/2} \frac{d^3 \phi_n(\varepsilon)}{d\varepsilon^3}, \quad (12)$$

and substitute it into the first term of Eq. (2) and (6) into the other terms. After multiplying Eq. (2) by $(V(x)-\mu)^{-1/2}$ and integrating with respect to $x$ from $x(\mu)$ to infinity, we arrive at an integral equation for $\phi_n(\varepsilon)$,

$$(\mu - E_n)\phi_n(\mu) = \frac{\hbar^2}{15\pi m} \int_{\mu}^{\infty} d\varepsilon Q(\varepsilon, \mu) \frac{d^3}{d\varepsilon^3} \phi_n(\varepsilon), \quad (13)$$

with the kernel

$$Q(\varepsilon, \mu) = \int_{x_1}^{x_2} dx (V(x) - \mu)^{-1/2} \frac{d^3}{dx^3} (\varepsilon - V(x))^{5/2}, \quad (14)$$

where the limits of integration are defined by the conditions $V(x_1)=\mu$ and $V(x_2)=\varepsilon$. The boundary conditions for bound states in the classical representation consist of the convergence condition for integral (6),

$$\phi_n(\varepsilon) \underset{\varepsilon\to\infty}{\to} 0 \tag{15}$$

and the condition related to the $g$- and $u$-symmetries,

$$\left.\frac{d\rho_n(x)}{dx}\right|_{x=0} = 2\psi_n(0)\left.\frac{d\psi_n(x)}{dx}\right|_{x=0} = 0. \tag{16}$$

Another form of Eq. (13) can be obtained if the first term in Eq. (2) is reduced to a simpler form. To this end we represent (2) in the form

$$\frac{\hbar^2}{4m}\frac{d^3\rho_n}{dVdx^2} = 2(V-E_n)\frac{d\rho_n}{dV}+\rho_n. \tag{17}$$

This form, in which the variables $x$ and $V$ are mixed, will be used to simplify calculations. Integrating Eq. (17) with respect to $V$ and substituting $\rho_n$ in the form (6) into the right-hand side of the result of integration, we get

$$-\frac{\hbar^2}{4m}\frac{d^2\rho_n}{dx^2} = 2\int_{V(x)}^{\infty} d\varepsilon \frac{(E_n-\varepsilon)\phi_n(\varepsilon)}{(\varepsilon-V(x))^{1/2}}. \tag{18}$$

Integrating with respect to $x$ once more, we arrive at the equation

$$\frac{\hbar^2}{4m}\frac{d\rho_n}{dx} = 2\int_{V(x)}^{\infty} d\varepsilon \int_{x}^{x(\varepsilon)} dx' \frac{(E_n-\varepsilon)\phi_n(\varepsilon)}{(\varepsilon-V(x'))^{1/2}},$$

which after multiplication by $(V-\mu)^{-1/2}$ and integration with respect to $x$ from $x(\mu)$ to infinity assumes the form

$$\frac{\hbar^2\pi}{8m}\phi_n(\mu) = \left(\frac{2}{m}\right)^{1/2}\int_{\mu}^{\infty} d\varepsilon\left[\int_{x(\mu)}^{x(\varepsilon)} dx \frac{t(\mu,x)}{(\varepsilon-V(x))^{1/2}}\right] \times(\varepsilon-E_n)\phi_n(\varepsilon), \tag{19}$$

where

$$t(\mu,x) = \left(\frac{m}{2}\right)^{1/2}\int_{x(\mu)}^{x} \frac{dx'}{(V(x')-\mu)^{1/2}}.$$

The quantity $t(\mu,V)$ can be interpreted as the absolute value of the time that it takes a particle to move in the subbarrier region with an energy $\mu<V$ from the turning point $x(\mu)$ to the point $x(V)$.

In accordance with the general reasoning given at the beginning of the paper, we will attempt to write Eq. (19) in terms of the ensemble of classical states. Changing the order of integration and going from $\phi_n(\varepsilon)$ to $f_n(\varepsilon)$, we can write this equation as

$$\frac{\hbar^2}{16}\omega(\mu)f_n(\mu) = \int_{x(\mu)}^{\infty} dx t(\mu,x)(W_n(x)-E_n)\rho_n(x), \tag{20}$$

where $\rho_n(x)$ is the probability density, and

$$W_n(x) = \frac{1}{\rho_n(x)}\int_{V(x)}^{\infty} d\varepsilon \frac{\varepsilon\phi_n(\varepsilon)}{(\varepsilon-V(x))^{1/2}} \tag{21}$$

is the average energy of the classical ensemble at point $x$. The left-hand side of Eq. (20) is proportional to the rate of collisions of particles having an energy of $\mu$ with the wall of the potential well and has the overall appearance of a balance equation in the classical ensemble in terms of "virtual transitions" into states that at the given energy $\mu$ are in the classically forbidden region. Using the definition (21), we can write the initial equation (18) as

$$-\frac{\hbar^2}{8m}\frac{d^2\rho_n}{dx^2}+(W_n(x)-E_n)\rho_n(x)=0.$$

There may be different and simpler forms of the Schrödinger equation in the classical representation that make it possible to give it a clear physical interpretation. The most meaningful approach would seem to be one in which the initial representation [Eqs. (6) and (8)] and the adjoint representation [Eqs. (A1) and (A2)] are both used symmetrically. This idea is suggested by the form of the kernel in Eq. (19), in which the function $t(\mu,x)$ is the "time of motion" of the particle in the subbarrier region, which is a characteristic feature of the adjoint representation, since in this representation all transformations relate to the classically forbidden region (see Appendix A).

Let us take the harmonic-oscillator problem as an example in which the classical representation is employed. In this case we have

$$V(x)=\tfrac{1}{2}m\omega^2x^2,$$

and, going from $x$ to $V$ in (14), we can calculate the kernel explicitly:

$$Q(\varepsilon,\mu)=\tfrac{15}{4}\pi m\omega^2(\varepsilon-2\mu).$$

Substituting this into (13) yields an equation for the harmonic oscillator in the classical representation,

$$(E_n-\varepsilon)\phi_n(\varepsilon) = -\frac{\hbar^2\omega^2}{4}\left[\varepsilon\frac{d^2\phi_n(\varepsilon)}{d\varepsilon^2}+\frac{d\phi_n(\varepsilon)}{d\varepsilon}\right], \tag{22}$$

which formally coincides with the radial equation for the Sturm basis functions of the hydrogen atom with the orbital quantum number $l=-1/2$. As $\varepsilon\to 0$, one of the two linearly independent solutions of the equation diverges logarithmically. Hence (and this can easily be verified), here the boundary condition (16) transforms into the condition for the regularity of function $\phi_n(\varepsilon)$ at zero. Combining this with (15), we get

$$E_n=\hbar\omega(n+1/2),$$

$$f_n(\varepsilon)=\frac{\pi}{\omega}\left(\frac{2}{m}\right)^{1/2}\phi_n(\varepsilon)=\frac{2}{\hbar\omega}\exp\left(-\frac{2\varepsilon}{\hbar\omega}\right)L_n\left(\frac{4\varepsilon}{\hbar\omega}\right), \tag{23}$$

where $L_n(x)$ is a Laguerre polynomial.

Another simple example is the problem of a particle in the vee-shaped potential $V(x)=F|x|$. In this case the calculation of the kernel (14) is trivial,

$$Q(\varepsilon,\mu)=\tfrac{15}{8}\pi F^2,$$

and Eq. (13) transforms into the Airy equation

$$(E_n-\varepsilon)\phi_n(\varepsilon)=-\frac{\hbar^2F^2}{8m}\frac{d^2\phi_n(\varepsilon)}{d\varepsilon^2}.$$

As a practical application of the suggested approach we point to the interesting possibilities that emerge, for instance, when in the theory of atomic collisions inelastic transitions are calculated along classical trajectories using the Monte Carlo method (the CTMC method; see Refs. 6 and 7). In this method the initial state of the electron is specified as a microcanonical ensemble of classical states with an energy equal to the energy $E_n$ of the respective atomic state, after which the evolution of this ensemble is calculated according to the classical equations of motion. Such a description of the initial state is approximate, like the solution of the dynamical problem, and is based on the WKB method. The main difficulties in this approach appear when one attempts to isolate in the final ensemble, as $t\to\infty$, the contribution of different quantum states, since as a result of collisions the energy of classical trajectories assumes continuous values as a function of the initial phase, values that do not coincide with the atomic spectrum. On the other hand, in the classical approximation, Eqs. (6) and (8) constitute an exact relation between the standard quantum mechanical description of a bound state and the language of ensembles of classical trajectories. This fact can be used to modify the CMTC method in the treatment of the initial and final states. Here the solution of the dynamical problem remains classical, that is, approximate.

By way of illustration we examine inelastic transitions in the problem of a particle in the potential

$$V(x)=\tfrac{1}{2}m\omega^2x^2+\alpha(t)x,$$

where $\alpha(t)$ is the strength of a homogeneous external field that is time-dependent and tends to zero as $|t|\to\infty$. In this approach, the initial state with the quantum number $n$ is assigned an ensemble of trajectories with the energy distribution $f_n(\varepsilon)$ given by (23). In accordance with the CTMC method, the subsequent evolution of the trajectories is described by the classical equation of motion

$$m\frac{d^2x(t)}{dt^2}=-m\omega^2x(t)-\alpha(t).$$

The general solution of this equation with the initial energy $\varepsilon$ can be obtained in this case explicitly and has the form

$$x(t)=\frac{1}{m\omega}\int_{-\infty}^{t}dt'\alpha(t')\sin[\omega(t-t')]+\frac{(2\varepsilon)^{1/2}}{\omega}\sin[\omega(t-\tau)],\qquad(24)$$

with $\tau$ the initial phase. The external field vanishes as $t\to\infty$ and the solution (24) describes harmonic oscillations

$$x(t)=\frac{(2\mu)^{1/2}}{\omega}\sin(\omega t-\gamma)$$

with a phase $\gamma$, unimportant to us, and an energy

$$\mu=\varepsilon+\nu+2(\nu\varepsilon)^{1/2}\cos\tau,\qquad(25)$$

with

$$\nu=\frac{1}{2m^2}\left|\int_{-\infty}^{\infty}dt'\alpha(t')\exp(i\omega t')\right|^2.$$

Equation (25) shows that, depending on the value of $\tau$, the energy of the final state assumes continuous values in the interval $\mu_1(\varepsilon)\leqslant\mu\leqslant\mu_2(\varepsilon)$ with the edges

$$\mu_1(\varepsilon)=(\varepsilon^{1/2}-\nu^{1/2})^2,\quad \mu_2(\varepsilon)=(\varepsilon^{1/2}+\nu^{1/2})^2.$$

Since the values of phase $\tau$ in the initial ensemble are equally probable, the energy distribution in the final state at a given $\varepsilon$ has the form

$$r(\mu,\varepsilon)=\frac{1}{\pi}\frac{d\tau}{d\mu}=\frac{\theta(\mu-\mu_1(\varepsilon))\theta(\mu_2(\varepsilon)-\mu)}{\pi[(\mu-\mu_1(\varepsilon))(\mu_2(\varepsilon)-\mu)]^{1/2}}.\qquad(26)$$

This expression, as one can easily check, is symmetric in $\mu$ and $\varepsilon$.

The total energy distribution in the final-state ensemble can be obtained by averaging $r(\mu,\varepsilon)$ over the initial distribution (23). If we allow for the symmetry of $r(\mu,\varepsilon)$, the distribution can be written as

$$F(\mu)=\int_{\varepsilon_1(\mu)}^{\varepsilon_2(\mu)}d\varepsilon r(\mu,\varepsilon)f_n(\varepsilon)=\frac{2}{\pi\hbar\omega}\int_{\varepsilon_1(\mu)}^{\varepsilon_2(\mu)}d\varepsilon\frac{L_n(4\varepsilon/\hbar\omega)\exp(-2\varepsilon/\hbar\omega)}{[(\varepsilon-\varepsilon_1(\mu))(\varepsilon_2(\mu)-\varepsilon)]^{1/2}}.$$

Due to the orthogonality of the eigenfunctions (23), the probability of an inelastic transition to a state with the quantum number $k$ is calculated as the projection of $F(\mu)$ on $f_k(\mu)$:

$$W_{nk}(\nu)=\int_0^\infty d\mu F(\mu)f_k(\mu)=\frac{4}{\pi(\hbar\omega)}\int_0^\infty d\mu\int_{\varepsilon_1(\mu)}^{\varepsilon_2(\mu)}d\varepsilon\times\frac{L_n(4\varepsilon/\hbar\omega)L_k(4\mu/\hbar\omega)\exp(-2(\varepsilon+\mu)/\hbar\omega)}{[2(\mu+\varepsilon)\nu-(\mu-\varepsilon)^2-\nu^2]^{1/2}}.\qquad(27)$$

As shown in Appendix B, this expression reduces to the form

$$W_{nk}(\nu)=L_n^{k-n}\left(\frac{\nu}{\hbar\omega}\right)L_k^{n-k}\left(\frac{\nu}{\hbar\omega}\right)\exp\left(\frac{\nu}{\hbar\omega}\right),\qquad(28)$$

where $L_p^k(z)$ is a generalized Laguerre polynomial. Note that although in the given approach the dynamical problem was solved within the framework of classical mechanics, that is, approximately from the standpoint of quantum mechanics and without allowing for interference effects, (28) coincides with the quantum mechanical expression.[8]

In conclusion let us examine the possibility of generalizing this approach. Above we discussed a one-dimensional Schrödinger equation with a purely discrete spectrum and a potential monotonic on a semiaxis. These restrictions are not important and were imposed for reasons of simplicity. Strictly speaking, only bound states have physical meaning

in a time-independent problem, and for such states the presence of a continuous spectrum only changes the integration range in the transformation (6), which always lies in the region of finite motion of a classical particle, while the upper limit of integration must coincide with the edge of the continuous spectrum. For instance, for the hydrogen atom the only difference is that the energy values $\varepsilon$ are negative and the upper limit of integration is zero. When a problem is studied in which the behavior of the potential is nonmonotonic, one specifies the regions of monotonicity and carries out all calculations as in the case of the respective inverse problem of classical mechanics. As for multidimensional problems with nonseparable variables, there is a way, at least in principle, to generalize the method by introducing, as the kernel of the integral equation (4), the respective classical probability density. But the transition to such formalism is, apparently, not very constructive because it is impossible to obtain an explicit expression for the classical probability density.

## APPENDIX A

If we substitute the function $T(\varepsilon)$ in the form (5) into (8b) and employ the unit step-function $\theta(\theta(z)=1$ if $z\leqslant 0$ and $\theta(z)=0$ if $z<0)$, we can reduce (8b) to a form with fixed integration limits that is more convenient for calculations $(\rho_n(V')\equiv\rho_n(x(V'))):$

$$f_n(\varepsilon)=-\frac{1}{\pi}\int_{-\infty}^{\infty}dx\,\frac{\theta(\varepsilon-V(x))}{(\varepsilon-V(x))^{1/2}}\times\int_0^{\infty}dV'\,\frac{d\rho_n(V')}{dV'}\,\frac{\theta(V'-\varepsilon)}{(V'-\varepsilon)^{1/2}}.$$

We can now change the order of integration when we calculate the normalization condition

$$\int_0^{\infty}d\varepsilon f_n(\varepsilon)=-\frac{1}{\pi}\int_{-\infty}^{\infty}dx\int_0^{\infty}dV'\,\frac{d\rho_n}{dV'}\int_0^{\infty}d\varepsilon\times\frac{\theta(\varepsilon-V(x))\theta(V'-\varepsilon)}{[(\varepsilon-V(x))(V'-\varepsilon)]^{1/2}},$$

and, bearing in mind that the integral with respect to $\varepsilon$ is exactly $\pi\theta(V'-V(x))$, we arrive at the following normalization condition

$$\int_0^{\infty}d\varepsilon f_n(\varepsilon)=-\int_{-\infty}^{\infty}dx\int_0^{\infty}dV'\,\frac{d\rho_n}{dV'}\,\theta(V'-V(x))=\int_{-\infty}^{\infty}dx\rho_n(x)=1.$$

Similarly, to prove the validity of (11) we write the initial expression in the form

$$\int_0^{\infty}d\varepsilon f_n(\varepsilon)\varepsilon=-\frac{1}{\pi}\int_{-\infty}^{\infty}dx\int_0^{\infty}dV'\,\frac{d\rho_n}{dV'}\times\int_0^{\infty}d\varepsilon\varepsilon\,\frac{\theta(\varepsilon-V(x))\theta(V'-\varepsilon)}{[(\varepsilon-V(x))(V'-\varepsilon)]^{1/2}}.$$

Here the integral with respect to $\varepsilon$ is equal to $\pi(V'+V(x))\theta(V'-V(x))/2$, and the right-hand side is reduced to

$$\int_0^{\infty}d\varepsilon f_n(\varepsilon)\varepsilon=-\frac{1}{2}\int_{-\infty}^{\infty}dx\int_{V(x)}^{\infty}dV'(V(x)+V')\,\frac{d\rho_n(V')}{dV'}=-\frac{1}{2}\int_{-\infty}^{\infty}dx\int_{V(x)}^{\infty}dV'\left[\frac{d((V(x)-V')\rho_n(V'))}{dV'}+2V'\,\frac{d\rho_n(V')}{dV'}+\rho_n(V')\right].$$

The integral of the first term in the brackets is zero and, using the Schrödinger equation in the form (17), we get

$$\int_0^{\infty}d\varepsilon f_n(\varepsilon)\varepsilon=\int_{-\infty}^{\infty}dxE_n\rho_n(x)=E_n.$$

To obtain the orthogonality condition, we must introduce, in view of the nonunitarity of the Abel transformation, a transformation symmetric to this transformation,

$$\rho_k(x)=\int_0^{V(x)}d\varepsilon\,\frac{\gamma_k(\varepsilon)}{(V(x)-\varepsilon)^{1/2}},\tag{A1}$$

$$\gamma_k(\varepsilon)=-\frac{1}{\pi}\frac{d}{d\varepsilon}\left(\int_0^{\varepsilon}dV\,\frac{\rho_k(x(V))}{(\varepsilon-V)^{1/2}}\right),\tag{A2}$$

which specifies the adjoint representation. If we now substitute (6) and (A1) into (3), we get the following chain of transformations:

$$\int_{-\infty}^{\infty}dx\rho_k\,\frac{d\rho_n}{dx}=2\int_0^{\infty}dV\int_0^{V}d\mu\,\frac{\gamma_k(\mu)}{(V-\mu)^{1/2}}\,\frac{d}{dV}\times\left[\int_V^{\infty}d\varepsilon\,\frac{\phi_n(\varepsilon)}{(\varepsilon-V)^{1/2}}\right]=2\pi\int_0^{\infty}d\mu\gamma_k\int_{\mu}^{\infty}d\varepsilon\,\frac{d\phi_n}{d\varepsilon}=-2\pi\int_0^{\infty}d\mu\gamma_k\phi_n,$$

which leads to the orthogonality condition (10).

## APPENDIX B

To prove the validity of Eq. (28) let us consider the generating function of the transition probability,

$$R(x,y,\nu)=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}x^n y^k W_{nk}(\nu).\tag{B1}$$

Substituting $W_{nk}(\nu)$ in the form (27) into (B1) and employing the expression for the generating function of the Laguerre polynomials,[9]

$$(1-x)^{-1}\exp\left(\frac{xq}{x-1}\right)=\sum_{n=0}^{\infty}x^nL_n(q), \tag{B2}$$

we get

$$R(x,y,\nu)=\frac{4}{\pi\hbar\omega(1-x)(1-y)}\int_0^{\infty}d\mu\int_{\varepsilon_1(\mu)}^{\varepsilon_2(\mu)}d\varepsilon \times\frac{\exp\left(\frac{2\varepsilon(1+x)}{\hbar\omega(1-x)}\right)\exp\left(\frac{2\mu(1+y)}{\hbar\omega(1-y)}\right)}{[2(\mu+\varepsilon)\nu-(\mu-\varepsilon)^2-\nu^2]^{1/2}}.$$

This double integral can easily be calculated:

$$R(x,y,\nu)=\frac{\exp\left(\frac{-\nu(1+x)(1+y)}{\hbar\omega(1-xy)}\right)}{(1-xy)}. \tag{B3}$$

On the other hand, if we employ $W_{nk}(\nu)$ in the form (28) in (B1) and combine it with (B2) and the relation[9]

$$\sum_{k=0}^{\infty}y^kL_n^{k-n}(q)L_k^{n-k}(q)=y^n\exp(-yq)L_n\left(\frac{(1+y)^2q}{y}\right),$$

we arrive at the same result (B3), which proves the equivalence of the right-hand sides of Eqs. (27) and (28).


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