The conductivity of two-dimensional electrons in a strong magnetic field

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A system of noninteracting two-dimensional electrons in a strong magnetic field and a smooth random potential is considered. The structure of delocalized electronic states concentrated near equipotential lines is examined. Low temperature estimates of the diffusion coefficient and ohmic conductivity of electrons are obtained and the activated conductivity at fractional filling factors is qualitatively discussed.

The conductivity of electrons in strong magnetic fields has been the subject of many theoretical and experimental studies. Particular interest has been shown in the two-dimensional case in which the quantum Hall effect is exhibited.[1] Previous ohmic conductivity calculations either used the self-consistent Born approximation[2] or were based on the renormalization group approach[3] and involved a qualitative analysis of the integral lines of the RG equations. Application of numerical methods should also be mentioned in this connection.

It is of interest to derive a simple physical picture for ohmic conductivity in a strong magnetic field, which is based on considering smooth random potentials in the framework of the so-called percolation model.[4] This approach allows a qualitative explanation of the integral quantum Hall effect and demonstrates its relation to the localization problem.

The density of states in a smooth random potential is determined by the statistical distribution of potential values at a given point. States far above and far below the mean potential value appear localized in the purely classical picture of an electron drifting along an equipotential line. Only states near the mean potential value (the percolation threshold) are delocalized and contribute to the ohmic and the Hall conductivities. A pure classical analysis of the localization length increase may be found in Ref. 5. Reference 6 estimates the mean value of the quasiclassical decay coefficient for an electronic wave function with energy close to the percolation threshold. Reference 7 gives a quasiclassical treatment of dissipative transport but is restricted to the pure drift approximation. In the present paper, in contrast, the structure of delocalized one-electron states is discussed for the case of a nonzero magnetic length—a situation when hopping between various components of a equipotential line is important.

1. EQUIPOTENTIAL LINE RELIEF NEAR THE PERCOLATION LEVEL

If a smooth static potential varies slowly on a scale of the magnetic length, it is readily shown that the wave functions of the first Landau level concentrate near the equipotential lines and are of the form

$$\Psi_E(x,y)=e^{i\varphi(s)}f_0(n),\quad \varphi(s)=\frac{e}{c}\int^s \mathbf{A}_h d\mathbf{s}, \tag{1}$$

where $\mathbf{A}_h$ is the vector potential of the applied magnetic field; $f_0(n)$ is the oscillator function; n is the distance along the normal from the equipotential line $V(x,y)=E$ of the random potential; and s is the length along the equipotential line.

It thus follows that the properties of a one-electron system are entirely determined by the geometry of its equipotential lines. Equation (1) shows that the electron moves freely along the closed components of the equipotential line and that it tunnels between the components if their separation is less than, or of the order of, the magnetic length l_H. We assume that the potential correlation radius satisfies $\lambda \gg l_H$, so that only tunneling in the vicinity of saddle points is possible. More precisely, the presence of a saddle point at $\mathbf{r}_0$, $\nabla V(\mathbf{r}_0)=0$, $V(\mathbf{r}_0)=E$, has the effect that the separation between the components of the equipotential line

$$V(\mathbf{r})=E+\Delta E$$

is reduced to about $\sqrt{|\Delta E|/V''(\mathbf{r}_0)}$. Estimating the second derivative of the potential as

$$V''(\mathbf{r}_0)\sim\frac{\sqrt{\langle V^2\rangle}}{\lambda^2},$$

we see that tunneling between the components of the equipotential line $V(\mathbf{r})=E$ is possible only if there is a saddle point in the energy interval $E\pm\varepsilon$, where

$$\varepsilon\sim\sqrt{\langle V^2\rangle}\left(\frac{l_H}{\lambda}\right)^2.$$

The angular brackets indicate an average over all possible realizations of the random potential.

It is easily established that the contribution to conductivity comes only from states whose energies are close to the percolation threshold V_c:

$$V_c-\varepsilon<E<V_c+\varepsilon.$$

To see this, define the set $\mathscr{A}$ of points obeying

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$$E-\varepsilon<V(\mathbf{r})<E+\varepsilon$$

and consider two closed components of the equipotential line $V(\mathbf{r})=E$. Clearly an electron of energy E can get from one component to another only if both of them belong to the same connected part of $\mathscr{A}$. If a state of energy E is not localized, $\mathscr{A}$ must contain an infinite connected component in it, and hence

$$E-\varepsilon<V_c<E+\varepsilon.$$

Saddle points lying in the energy interval $V_c\pm\varepsilon$ will be called submergible.

It is currently believed that percolation exponents are universal and do not depend on the particular problem being studied (a site or bond problem, a specific lattice type, or the continuous limit). Assuming that the probability of a given potential value is symmetric,

$$\mathscr{P}(\{V\})=\mathscr{P}(\{-V\}),$$

and that the correlation radius λ is finite, the continuous problem of determining the statistics of regions $V<E$ is made discrete by introducing a lattice with a constant $a\ll\lambda$ and discarding the sites with $V>E$ (see Ref. 8). In this case the percolation level corresponds to the zero of energy, and from this point on we set $V_c=0$. According to percolation theory,[8] for each value of E there exists a so-called critical cluster such that the number of larger clusters is exponentially small. The size L_c of the critical cluster can only be estimated approximately and is proportional to a power of the energy as measured from the percolation threshold, i.e., $L_c\sim E^{\nu}$. Here ν is the percolation theory exponent having the numerical value of $\nu\approx-1.33$. Critical clusters are dense in the sense that their number per unit area is

$$n_c\sim 1/L_c^2.$$

Let us take a closer look at the behavior of equipotential lines with energies $|E|<\varepsilon$ near the percolation threshold. Consider first the case $E=\varepsilon>0$. The set of points with potential values $V(\mathbf{r})<E$ (let us paint them white) contains finite-size connected regions and an infinite connected region, this latter enabling "white percolation" through the system. Making E smaller diminishes both the finite regions and the infinite one, whereas the perimeters of the finite regions $V(\mathbf{r})>E$ (we paint these black) will become increasingly larger. At $E=0$, the finite black regions start to coalesce into an infinite black-percolation region, and at $E<0$ the infinite white cluster breaks up so that only finite white clusters will remain. (Note that the original finite white clusters have been continuously shrinking with decreasing E.) On the other hand, noting that the random potential distribution is sign symmetric, we see that the critical white clusters with $E=-\varepsilon$ must be the same size as the original critical black clusters with $E=\varepsilon$.

It thus follows that at $E=-\varepsilon$ critical white regions may only result from breaks in the original infinite white region. This means that the boundaries of the white critical regions with $E=-\varepsilon$ pass within the original white infinite cluster. The symmetric picture which results is shown in Fig. 1, where the dashed lines $E=\varepsilon$ and the solid lines $E=-\varepsilon$ show the boundaries of large clusters of critical size L_c. Between these lines there are saddle points ensuring that the finite regions (of either color) close on one another. Needless to say, the infinite region between the solid and dashed lines also contains shorter equipotential lines, down to about the random-potential correlation length; it is mainly the dips between these lines which form the submergible saddle points.

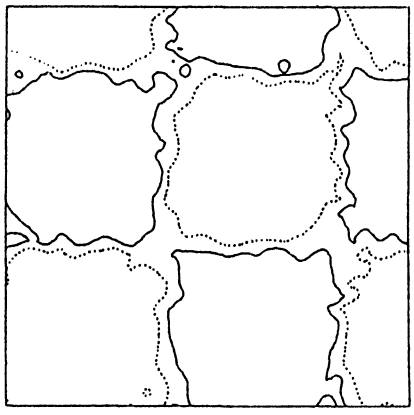

FIG. 1. Pattern of critical clusters. Solid and dashed lines show equipotential lines $V(\mathbf{r})=\pm\sqrt{\langle V^2\rangle}(l_H/\lambda)^2$ enclosing critical clusters.

We can apply relevant percolation theory results to estimate the width of the submergible region between the critical white and black clusters. If the conductivity of the white clusters with $V(\mathbf{r})<E$, $E>0$, is a constant and the black clusters are insulators, it has been shown numerically that the bulk conductivity has a power-law E dependence with exponent $\tau\approx 1.15$. This means that most of the infinite cluster is ineffective from the point of view of conductivity (i.e., is occupied by dead ends, see Ref. 8). The effective width of the conducting region can be estimated by invoking the concept of a current-carrying wire mesh of dimension L_c with wire thickness $h\sim E^{\tau+\nu}\ll L_c$. The quantity h must be of the same order of magnitude as the thickness of the submergible neck connecting critical-size regions of the same color. A schematic illustration of a such a "contact" is given in Fig. 2.

Similar conclusions hold for the perimeters of the critical clusters on the length scale $\ll L_c$. Figure 3a shows a typical portion of such a perimeter as well as the variation of the potential along a normal to the perimeter. Any value of V will be assumed an odd number of times and the drift velocities of the electrons will alternate in conformity with the sign of $\partial V/\partial n$. To elucidate the qualitative features of the electronic wave functions, in the next section we discuss the motion of an electron in a model channel in which each $V=\varepsilon$ level is encountered no more than three times, as shown in Fig. 3b.

Our results rest on the assumption that a moving electron does never go far away from the perimeter of a critical

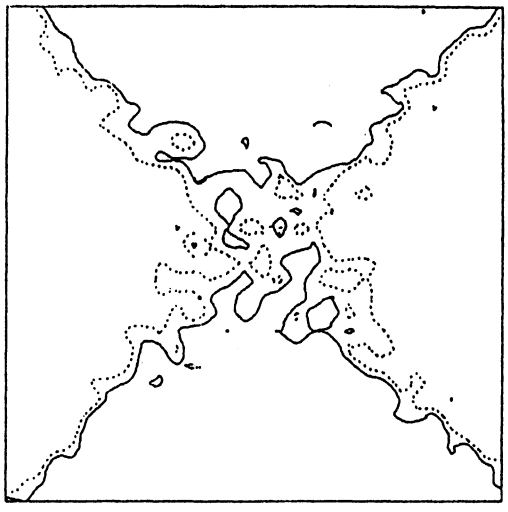

FIG. 2. Typical geometry of a contact between critical clusters.

cluster. To prove this hypothesis, strictly requires a more detailed study of the equipotential line map of the system. For us here, it will suffice to refer to the special case of a long chain of low-lying islands with submergible saddles between them. In this case one-dimensional localization makes the wave function decay exponentially along the chain and so prevents the electron from going away.

2. ELECTRONIC WAVE FUNCTION IN A NARROW CHANNEL

While at a regular point of an equipotential line the wave function is given by Eq. (1), near a submergible saddle the lines are strongly curved, the electron is scattered, and so a transition from one closed line to another may occur. In the vicinity of a saddle point the potential can be written accurate to second order as

$$V(x,y)=V_0+\frac{\alpha x^2}{2}-\frac{\beta y^2}{2}, \quad \alpha,\beta>0. \tag{2}$$

Recently,[9] scattering has been analyzed in such a potential in a strong magnetic field. The simplest way to solve this problem is to use the zero-Landau-level projection, in which case the kinetic energy of the electron becomes a constant $\hbar\omega_c/2$, ω_c being the cyclotron frequency. The problem reduces to the study of the asymptotic behavior of parabolic cylinder functions with a single scattering parameter

$$\tilde{\varepsilon}=2\frac{2(E-V_0)+(\alpha-\beta)l_H^2}{\sqrt{\alpha\beta}l_H^2}. \tag{3}$$

Here we will only reproduce the final expression for the transfer matrix. Suppose to the left of a saddle point the asymptotic behavior is

$$\psi\to Af(1)+Bf(2),$$

and to the right

$$\psi\to Af'(3)+B'f(4),$$

where $f(i)$ is the quasiclassic asymptotic form (1) along the ray i and the rays are numbered as follows:

(1) $y=-x$, $x<0$, (2) $y=x$, $x<0$,

(3) $y=x$, $x>0$, (4) $y=-x$, $x>0$.

The coefficients are related by the transfer matrix R, which is defined by

$$\begin{pmatrix}A\\B\end{pmatrix}=\hat{R}\begin{pmatrix}A'\\B'\end{pmatrix}$$

and whose elements depend on $\tilde{\varepsilon}$ alone:

$$\hat{R}=\begin{pmatrix}e^{i\gamma}e^{-\pi\tilde{\varepsilon}/4}(e^{\pi\tilde{\varepsilon}/2}+e^{-\pi\tilde{\varepsilon}/2})^{1/2} & ie^{-\pi\tilde{\varepsilon}/2}\\ -ie^{-\pi\tilde{\varepsilon}/2} & e^{-i\gamma}e^{-\pi\tilde{\varepsilon}/4}(e^{\pi\tilde{\varepsilon}/2}+e^{-\pi\tilde{\varepsilon}/2})^{1/2}\end{pmatrix}, \tag{4}$$

Here $\gamma=-\tilde{\varepsilon}/2\ln 2+\arg\Gamma(1+i\varepsilon/2)$, where $\Gamma(x)$ denotes the Euler gamma function. We assume $\tilde{\varepsilon}>0$, which implies scattering from the left to the right equipotential line. For $\tilde{\varepsilon}<0$ the electron is scattered from the upper to the lower line, and we must replace $i\to -i$ and $\varepsilon\to-\tilde{\varepsilon}$ in the matrix $\hat{R}$.

Qualitatively, the behavior of the wave function near the boundary of a critical cluster may be understood from a simplified model which permits the random potential to coincide with the given level three times at most (Fig. 3b). Since the use of the exact transfer matrix, Eq. (4), would be extremely cumbersome, we restrict ourselves to the weak-scattering case, as is customary in one-dimensional analyses. Thus we consider one-dimensional three-channel motion under the condition that in one of the channels the propagation direction of the electron is opposite to that in the other two in accordance with the sign of $\partial V/\partial n$. Even with this simplification the problem is quite a challenge if

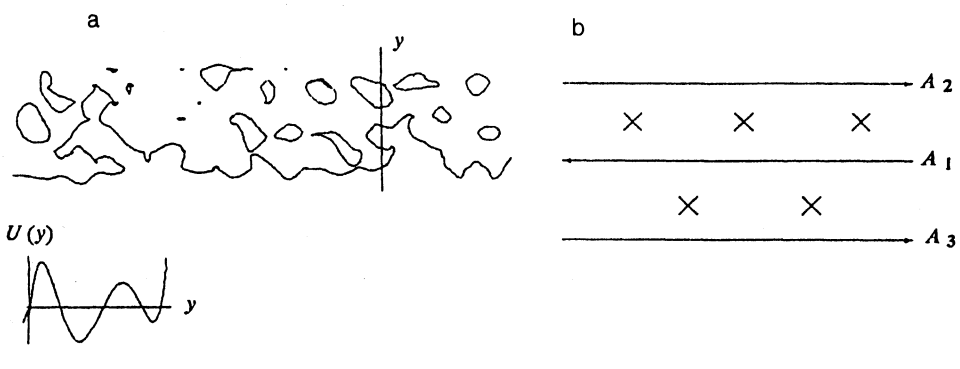

FIG. 3. (a) Perimeter of a critical cluster and potential variation along the perimeter normal; (b) Model for elucidating the behavior of the electronic wave function at the perimeter of a critical cluster. Crosses denote submergible saddle points.

we wish to take into account the real statistical properties of the quantity ε which determines the permeability of the saddles and the discrete nature of their positions. Since we are only interested in qualitative results, we consider instead what may be called the Born case in which the scattering matrix defined at individual points is replaced by a continuously distributed quasiunitary transfer matrix (see, e.g., Ref. 10). The solution to the Schrödinger equation is a three-dimensional vector $\mathbf{A}=(A_1,A_2,A_3)$ whose components characterize the amplitudes of quasiclassical wave functions in each of the three channels considered (i.e., on each of the three equipotentials of Fig. 3b). The Schrödinger equation itself transforms into a system of equations for the vector $\mathbf{A}$,

$$\frac{d\mathbf{A}}{dx}=\hat{U}(x)\mathbf{A}, \tag{5}$$

and the form of the transfer matrix over a small length δx is

$$\hat{T}=1+\hat{U}\delta x, \quad \hat{U}=i\begin{pmatrix}\gamma_0 & -\delta_1 & -\delta_2\\ \delta_1 & \gamma_1 & 0\\ \delta_2 & 0 & \gamma_2\end{pmatrix}, \tag{6}$$

where γ_0, γ_1, γ_2, δ_1, and δ_2 are real functions of the coordinate x and it is assumed that scattering from the second channel into the third is impossible (in the second and third channels the electron propagates in the same direction, opposite to that in the first channel). The wave functions are normalized so that each channel carries a unit current. The phases δ_1 and δ_2 are assumed to be independent, white-noise random quantities having identical distribution:

$$\langle\delta_1(x)\delta_1(x')\rangle=\frac{1}{2l}\delta(x-x').$$

The diagonal elements of the matrix $\hat{U}$ incorporate the systematic change of the phase of the wave function in the absence of scattering. It is assumed that the phase φ of the wave function (1) depends in a prescribed manner on the coordinate and the channel number. It is important that the phase varies rapidly,

$$\gamma_i=k_i(x)+\omega(x),$$

where ω is a white-noise random quantity, and $k_i\sim\nabla\varphi_i$ is large ($k_il\gg1$) in accordance with the considerable change of the phase of the quasiclassical function (1) from saddle to saddle.

As we shall see, the one-dimensional three-channel problem (5) differs significantly from the standard two-channel problem arising for a one-dimensional Schrödinger equation with a random potential.

It is easy to see that the matrix $\hat{T}$ preserves the magnitude of the current in each of the three channels. With the normalization condition adopted the total current is

$$j=-|A_1|^2+|A_2|^2+|A_3|^2.$$

It is natural to introduce the scalar product

$$(\mathbf{AB})=-A_1^*B_1+A_2^*B_2+A_3^*B_3=\mathbf{A}^*\hat{g}\mathbf{B},$$

$$\hat{g}=\begin{pmatrix}-1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}. \tag{7}$$

The conservation of current under the transformation $\mathbf{A}=T\mathbf{A}$ is equivalent to the invariance of the form $(\mathbf{AB})$ under this transformation and may be written as

$$\hat{T}^*\hat{g}\hat{T}=\hat{g}, \quad \hat{g}\hat{T}^*\hat{g}=\hat{T}^{-1},$$

showing that the eigenvalues of the matrices $\hat{T}^*$ and $\hat{T}^{-1}$ are the same. Each matrix has three eigenvalues: λ_1^{-1}, λ_2^{-1}, λ_3^{-1}, and λ_1^*, λ_2^*, λ_3^*, respectively. The case $\lambda_i^{-1}=\lambda_i^*$ is a degenerate version of the more general situation in which $\lambda_1^{-1}=\lambda_2^*$ and $\lambda_3^{-1}=\lambda_3^*$ hold (the choice of numbering is immaterial). Thus we have three eigenvalues whose absolute values are greater than unity, less than unity, and equal to unity, respectively. The above argument uses no other assumptions than the oddness of the number of channels and the conservation of current, so it applies equally well to a transfer matrix for any finite length. The fact that the transfer matrix has an eigenvalue unit distinguishes the problem with an odd number of channels from the standard two-channel model in which all the wave functions turn out to be localized.[10]

In what follows we present arguments demonstrating the absence of localization in the three-channel problem or, more precisely, we show that at any intermediate point of a disordered segment of length L, the average value of the wave function is bounded and L independent. The simplest way to show this is by assuming boundary conditions of the form

$$\mathbf{A}(0)=\begin{pmatrix}t\\ t\\ 1\end{pmatrix}, \quad \mathbf{A}(L)=\begin{pmatrix}u\\ u\\ 1\end{pmatrix}e^{i\theta}, \tag{8}$$

where t and u are arbitrary complex numbers and the real number θ is determined from the condition that Eq. (5) have a solution(s). We believe that our proof is quite general because at large values of L the dependence on the particular form of boundary conditions has to disappear.

Using the definition (7) the boundary conditions (8) may be written as

$$\begin{aligned}&[\xi^+(0)\mathbf{A}(0)]=0, \quad [\xi^-(L)\mathbf{A}(L)]=0,\\ &[\mathbf{A}(0)\mathbf{A}(0)]=1,\end{aligned} \tag{9}$$

where

$$\xi^+(0)=\begin{pmatrix}1\\ 1\\ 0\end{pmatrix}, \quad \xi^-(L)=\begin{pmatrix}1\\ 1\\ 0\end{pmatrix}. \tag{10}$$

Because the scalar product (7) is conserved, the conditions (9) may be transferred to any inner point of the disordered region

$$\begin{aligned}&[\xi^+(x)\mathbf{A}(x)]=0, \quad [\xi^-(x)\mathbf{A}(x)]=0,\\ &[\mathbf{A}(x)\mathbf{A}(x)]=1,\end{aligned} \tag{11}$$

where $\xi^+(x)$ and $\xi^-(x)$ solve (5) with the boundary conditions

$$\xi^+(x=0)=\xi^+(0), \quad \xi^-(x=L)=\xi^-(L).$$

The conditions (11) yield $\mathbf{A}(x)$ to within the phase factor if $\xi^+(x)$ and $\xi^-(x)$ are linearly independent—as indeed they are for the general matrix $\hat{U}(x)$.

Now since

$$[\xi^+(x)\xi^+(x)]=[\xi^-(x)\xi^-(x)]=0,$$

it is natural to set 6

$$\xi^{\pm}(x)=\Lambda^{\pm}\begin{pmatrix}1\\ \xi_2^{\pm}\\ \xi_3^{\pm}\end{pmatrix}, \tag{12}$$

where $\xi_2^{\pm}$ and $\xi_3^{\pm}$ are complex numbers such that

$$|\xi_2^{\pm}|^2+|\xi_3^{\pm}|^2=1.$$

Thus the direction of the vector $\xi^+(\xi^-)$ is determined by a point on a three-dimensional sphere S^3. Using the conditions (11) it is straightforward to obtain an expression for the square of the wave function:

$$|\mathbf{A}(x)|^2=\frac{|\xi_2^+-\xi_2^-|^2+|\xi_3^+-\xi_3^-|^2+|\Delta|^2}{|\xi_2^+-\xi_2^-|^2+|\xi_3^+-\xi_3^-|^2+|\Delta|^2},$$
$$\Delta=\xi_3^+\xi_2^--\xi_3^-\xi_2^+. \tag{13}$$

Now ξ^+ and ξ^- are statistically independent (they are determined by the matrix $\hat{U}(x)$ on nonintersecting line segments), so that the equations for the corresponding functions may be considered separately. Setting

$$\xi_2^+=e^{i\alpha_2}\cos\phi, \quad \xi_3^+=e^{i\alpha_3}\sin\phi,$$

the equation of motion for ξ^+—which, we recall, is exactly Eq. (5)—transforms to the equations

$$\frac{d\alpha_2}{dx}=\tilde{\gamma}_2+\delta_1\cos\alpha_2\frac{1+\cos^2\phi}{\cos\phi}+\delta_2\sin\phi\cos\alpha_3,$$
$$\frac{d\alpha_3}{dx}=\tilde{\gamma}_3+\delta_2\cos\alpha_3\frac{1+\sin^2\phi}{\sin\phi}+\delta_1\cos\phi\cos\alpha_2, \tag{14}$$
$$\frac{d\phi}{dx}=-\delta_1\sin\alpha_2\sin\phi+\delta_2\sin\alpha_3\cos\phi,$$
$$\tilde{\gamma}_2=\gamma_2-\gamma_1, \quad \tilde{\gamma}_3=\gamma_3-\gamma_1.$$

The Langevin equations (14) can be solved by rapid-variable averaging[10] assuming $\tilde{\gamma}_2$, $\tilde{\gamma}_2 \gg \delta_1$, δ_2 and treating the first two equations perturbationally. The distribution over ϕ takes longest to establish. The corresponding Langevin equation is of the form

$$\frac{d\phi}{dx}=-\delta_1\sin\alpha_2^0\sin\phi+\delta_2\sin\alpha_3^0\cos\phi+\frac{1}{2l}(\operatorname{ctg}\phi-\operatorname{tg}\phi), \tag{15}$$

where we have introduced

$$\alpha_i^0=\int^x\tilde{\gamma}_i dx$$

and averaged over the quadratic expressions

$$\langle\delta_1(x)\delta_1(x')\rangle=\frac{1}{2l}\delta(x-x').$$

From Eq. (15) we deduce the Fokker–Planck equation for the probability $P(\phi)$:

$$2l\frac{\partial P}{\partial x}=\frac{\partial}{\partial\phi}\left(-2\operatorname{ctg}2\phi P+\frac{\partial P}{\partial\phi}\right). \tag{16}$$

One readily sees that the stationary distribution

$$P_\infty(\phi,\alpha_2,\alpha_3)=\frac{\sin 2\phi}{4\pi^2}$$

corresponds to a uniform distribution over the surface of a three-dimensional sphere. It can be shown that this distribution is a stable one with exponent $e^{-4x/l}$. Thus the distribution $P(\phi,\alpha_1,\alpha_2;x)$ must rapidly—over a distance of order the mean free path—go over to a uniform distribution over S^3.

In a similar way, it is shown that the direction ξ^- is also distributed uniformly over S^3. At point x far enough from the ends of the interval, the average values of the squared wave function components $|A_1|^2$, $|A_2|^2$, and $|A_3|^2$ are found by averaging expressions of the form (13) over the independent distributions ξ^+ and ξ^-. This yields identical values $\langle|A_i|^2\rangle=1$, thus showing the absence of an exponential growth in the wave function $\mathbf{A}(x)$.

In the conductivity problem we will need the time interval τ required for an electron to pass the disordered line segment of length L. This may be defined, for example, as a current for a normalized wave function,

$$\left\langle\frac{1}{\tau}\right\rangle=\left\langle\frac{j}{\int_0^L|\mathbf{A}|^2dx}\right\rangle. \tag{17}$$

Generally speaking, to make use of Eq. (17) requires first to obtain the distribution of the wave function normalization integral $\int_0^L|\mathbf{A}|^2dx$. Since we cannot prove that this quantity is ergodic, it is not legitimate to interchange the averaging and integration in the denominator in Eq. (17), but if we do interchange them we find approximately that the drift velocity is reduced by a factor of about three.

There is an alternative method to calculate the time it takes an electron to pass the disordered segment. Consider a wave packet made up of states with close energies

$$\mathbf{A}_{\text{in}}=\int f(p)dp, \quad \mathbf{A}_{\text{out}}=\int f(p)\exp[i\varepsilon t+i\theta(\varepsilon,L)]dp, \tag{18}$$

where ε is the energy and θ the phase introduced in (8). Then it takes a time $\tau=d\theta/d\varepsilon$ for a spatially narrow packet to pass the disordered segment. We may obtain dynamic equations for $d\theta/d\varepsilon$ and then construct the corresponding Fokker–Planck equations, but this requires a knowledge of the simultaneous probability of ξ^+, $d\xi^-/d\varepsilon$, and $d\theta/d\varepsilon$ and is therefore extremely cumbersome. The results give further evidence for the absence of localization

and show an insignificant (in order-of-magnitude) renormalization of the propagation time with respect to the one-channel case.

3. DIFFUSION AND CONDUCTIVITY ESTIMATES

We have shown that close to the percolation threshold the electron propagates with a certain finite velocity v and that within the time interval $\tau \approx L_c/v$ it may be scattered to an equipotential of the same energy around another critical cluster. It is important that scattering by small-size clusters causes the smearing of the electronic wave function along the boundary of a critical cluster: the preceding section illustrates this for the three-channel case. We will assume that the smearing is by an amount h on the order of the thickness of the effective conducting "wire." Thus, referring to the mesh introduced in Section 1, at those sites where critical clusters come close to one another, the electronic wave functions have a significant overlap and the probability w for a transition between critical clusters is of order unity.

Neglecting two-dimensional localization effects, the diffusion coefficient is estimated as

$$D \sim \frac{L_c^2}{\tau} w, \tag{19}$$

where the quantity τ has been introduced in the preceding section and denotes the drift time through $L \sim L_c$. This gives $t_D = \tau/w$ for the diffusive transition time. On the other hand, the average level separation in the vicinity of a critical cluster is $\sim \hbar/\tau$. In order for localization effects to be absent, the Thouless criterion requires that the energy uncertainty $\hbar/\tau$ due to the electron diffusing away be larger than the level separation. In the case discussed, the Thouless number (the ratio of these two quantities) is estimated as $\mathcal{T} \sim w$. We will assume that as the electron energy vanishes, $E \to 0$, the Thouless criterion is satisfied at a certain E_c such that $w(E_c) \sim 1/2$. An alternative possibility arises from inelastic processes occurring at finite temperatures. Since the level separation satisfies $\hbar/\tau \to 0$ as $E \to 0$, it will eventually be overlapped by the level broadening due to these processes. In either case, we will observe finite diffusion and finite conductivity over states close to the percolation threshold. Remembering that the ohmic conductivity and diffusion are related by the Einstein relation, we obtain

$$\sigma_{xx} = \frac{e^2}{\hbar} w,$$

where we have taken into account that the density of states near the percolation threshold is $(1/L_c^2)(\hbar/\tau)$. The conductivity σ_{xx} is nonzero only if $\mathcal{T}$ is sufficiently large that the quantity $w(E)$ is of order unity (in which case the chemical potential μ of the electrons should be close to the percolation threshold).

If the chemical potential is far away from the percolation threshold (the localization region) then activated processes must be considered. Since wave functions at the chemical-potential level have an extremely small overlap with those at the percolation level, it follows that if an electron has occupied a level close to the threshold value, it is very long-lived (matrix elements are small) and so has enough time to be scattered by the random potential. It is these activated electrons which control the conductivity of the material.

To estimate the ohmic conductivity in this case we write the total diffusion current over all the levels of the system:

$$\langle j \rangle = e \sum_n D(E_n) \frac{\partial f(E_n)}{\partial \varepsilon} = e \sum_n D(E_n) \frac{\partial f}{\partial \mu} \frac{\partial \mu}{\partial \varepsilon}, \tag{20}$$

where f is the Boltzmann electron distribution function. Using the Einstein relation this gives

$$\sigma = \frac{e^2}{h} \exp\left(\frac{\mu - V_c}{T}\right) \frac{1}{T} \int w(E) \exp\left(\frac{-E}{T}\right) dE, \tag{21}$$

the integration being limited to a very narrow interval near the percolation threshold in which electron diffusion takes place.

Exactly similar arguments apply to the activated fractional-quantum-Hall conductivity. At filling factors close to a fraction of ν_c an incompressible Laughlin liquid forms and a small number of quasiparticles of charge e^* appear. In the absence of a random potential, quasiparticle states are degenerate because translations fail to commute, and hence the quasiparticle velocity is zero. A random potential lifts this degeneracy and thereby enables the quasiparticles to drift. The theory of the "boundary" states of such quasiparticles in a uniform field has been developed intensely in recent years.[11] Recently, activated conductivity with fractional filling factors has also been discussed.[12] The following qualitative description immediately follows from the existence of a quasiparticle gas. We note that in the system of quasiparticles there also exists a percolation level close to which the Thouless number reaches a certain critical value, thus enabling diffusion to occur. Because of the dilute-Boltzmann-gas nature of the quasiparticles they can be treated very much like electrons and we find

$$\sigma = \frac{e^{*2}}{h} \exp\left(\frac{\mu - V_c}{T}\right) \frac{1}{T} \int w(E) \exp\left(\frac{-E}{T}\right) dE, \tag{22}$$

where μ is the quasiparticle chemical potential and V_c the percolation threshold. Note that the integral in (22) is independent of the quasiparticle charge: while the energy range of the delocalized states is proportional to the quasiparticle charge

$$\int w(E) dE \sim e^* \frac{l_H^2}{\lambda^2}$$

phenomenological arguments suggest that, in full analogy with electrons, the quasiparticle magnetic length is estimated by

$$l_H^2 \sim 1/e^*.$$

We assume that the incompressible Laughlin liquid has no screening effect on the random potential, with the consequence that the correlation length λ is also independent of

ν_c. This agrees with the view[13] that there exists a direct relation between the quasiparticle charge and the preexponential of the activated fractional-quantum-Hall conductivity.

In conclusion, the notion of a narrow interval of localized states near the percolation threshold enables one both to explain the qualitative features and to derive a correct order-of-magnitude estimate of the ohmic conductivity. The physical picture we have developed can provide a basis for a simple numerical simulation model for elucidating the nature of a singularity that occurs when the critical Thouless number is approached. Attempts to derive the localization length increase either from purely classical considerations[5] or from the average decay between critical clusters[6] produce very rough estimates of seemingly no relevance to the true mobility-edge singularity. Finally, our discussion gives a clear illustration of the universal character of activated conductivity at fractional filling factors.

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