That inexhaustible Robinson–Trautman...

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It is shown that a class of exact solutions of the Einstein equations found by Robinson and Trautman reduces to the generalized Liouville equation on the Lobachevsky plane, when the corresponding Weyl tensor is Petrov type III. Particular solutions of this equation have a moving logarithmic branch point, i.e., the Kovalevskaya–Painlevé integrability test does not hold. The existence of solutions with complex topology is demonstrated. The solutions can be nonsingular only on one copy of the Lobachevsky plane, which does not cover the whole solution of the generalized Liouiville equation.

The gravitational field equations of Einstein posses remarkable properties of internal symmetry, which, in particular, manifest themselves in two-dimensional reductions of these equations (the GRT solutions admitting the two-parameter Abelian group of motions on V_2). The GRT system of equations, viewed as a distributed Hamiltonian system with constraints, is in a certain sense integrable.[1,2] One of the empirical "tests" of integrability of distributed systems is the property that the one-dimensional reductions of these systems have no moving singularities upon continuation of the independent variable in these reductions from the real axis to the complex plane. Evidently the one-dimensional reductions of certain classical integrable systems posses this property. Kovalevskaya demonstrated a brilliant example of integration based precisely on this property, of the equations of a heavy top. Painleve and Gambier have given an exhaustive classification of ordinary differential equations of second order of the type $w''=f(w,w',z)$ (where f is a function rational in w and w' and analytic in z), which have no moving branch point. However at the present time a direct proof of the validity of the Kovalevskaya–Painlevé test for integrable physical systems does not exist. This increases the interest in finding moving logarithmic branch points in one-dimensional reductions of Einstein equations with a Weyl tensor of Petrov type III, which are particular cases in the class of Robinson–Trautman solutions.

The general solution of Einstein's equations in vacuum, for spaces admitting congruences of null geodesics that are shear-free and twist-free, is given by the Robinson–Trautman metric[3,4]

$$ds^2=2dudr+2Hdu^2-2r^2\mathscr{P}^{-2}d\zeta d\zeta^*, \qquad (1)$$

where the function $\mathscr{P}$ is independent of r.

For this solution, using the standard notation of Newman–Penrose[5] in optical frames

$$l^i(1,0,0,0),\ n^i(-H,1,0,0),\ m^i(0,0,P/r,0)$$

$$x^1=r,\quad x^2=u,\quad x^3=\zeta,\quad x^4=\zeta^*,$$

it is easy to count up the nonvanishing spin coefficients

$$\rho=-1/r,\quad \mu=-H/r-(\ln P)_{,u},\quad \nu=-PH_{,\zeta*}/r,$$

$$\alpha=-P_{,\zeta*}/2r,\quad \beta=-\alpha^*,\quad \gamma=H_{,r}/2.$$

In particular, the spin coefficients k and σ vanish, and $\rho=\rho^*$, which testifies to the geodesic and shear-free nature of the normal of the congruence of the coordinate lines r. For the nonzero tetrad components of the Ricci tensor we have the expressions

$$2r^2\Phi_{11}=P^2(\ln P)_{,\zeta\zeta*}-\tilde{H}+r^2H_{,rr}/2,$$

$$2r\Phi_{12}=P\tilde{H}_{,r\zeta},\quad \tilde{H}\equiv H+r(\ln P)_{,u},$$

$$r^2\Phi_{22}=P^2H_{,\zeta\zeta*}+r\tilde{H}_{,u}+(\ln P)_{,u}(r\tilde{H}_{,r}-2\tilde{H}r).$$

From the equations in the vacuum $R_{ik}=0$ readily follow these consequences: the coefficient H in the metric (1) is given in terms of P:

$$H=P^2(\ln P)_{,\zeta\zeta*}-r(\ln P)_{,u}-m(u)/r, \qquad (2)$$

while the function P satisfies the equation

$$P^2(P^2(\ln P)_{,\zeta\zeta*})_{,\zeta\zeta*}+3m(\ln P)_{,u}-m_{,u}=0. \qquad (3)$$

In the special case when $m(u)=0$ the calculation of the Newman–Penrose scalars gives

$$\Psi_0=\Psi_1=\Psi_2=0,\quad r^2\Psi_3=-P\tilde{H}_{,\zeta*},$$

$$r^2\Psi_4=(P^2H_{,\zeta*})_{,\zeta*}, \qquad (4)$$

i.e., such solutions have a Weyl tensor of Petrov type III or N. It follows from Eqs. (3) for $m(u)=0$ that the function P satisfies the equation

$$2P^2(\ln P)_{,\zeta\zeta*}=-3(f+f^*), \qquad (3')$$

where $f=f(\zeta,u)$ is an arbitrary function of the complex ζ and real u arguments. Further, it follows from (2), that

$$2H=-3(f+f^*)-2r(\ln P)_{,u}. \qquad (2')$$

In the general case the solution of Eq. (3) can be obtained from the equation

$$2P^2(\ln P)_{,\zeta\zeta*}=-3(\zeta+\zeta^*) \qquad (5)$$

with the help of the obvious replacement

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$$\zeta \to f, \quad P \to P|\partial f/\partial \zeta|.$$

This replacement is not an admissible coordinate transformation for $f_{,u} \neq 0$, since it "spoils" the form of the metric (1) for $f_{,u} \neq 0$. In a particular case the function f in Eq. (3) depends only on u and is real. Then Eq. (3′) is transformed into the Liouville equation, whose general solution has the form

$$P = \frac{(\varphi + \varphi^*)\sqrt{-3f(u)}}{|\partial\varphi/\partial\zeta|}, \quad \varphi = \varphi(\zeta, u). \tag{6}$$

It follows from formula (4) that in that case $\Psi_3 = 0$ and therefore the solution is of Petrov type N, and the function $f(u)$ in the metric (1) can be set equal to a constant in view of the transformation

$$\int \sqrt{f(u)}\,du \to u, \quad r/\sqrt{f(u)} \to r.$$

If the function φ in formula (6) is independent of u then the metric (1) reduces to the metric

$$ds^2 = 2du(du + dr) - r^2 d\varphi d\varphi^*/(\varphi + \varphi^*)^2. \tag{6′}$$

This is a co-moving metric for the two-dimensional cosmological model in the SRT framework (Milne's solution) describing cylindrical homogeneous scatter about some axis. The coordinate r is interpreted as the cosmological time τ. Therefore the solution (6) describes for $\varphi_{,u\zeta} \neq 0$ an outgoing gravitational wave, depending on the retarded time $u = \tau - z$ and the spatial Lagrange variables ζ, and propagating in the direction of the previously mentioned axis z.

The metric of this exact solution is expressed in terms of an arbitrary holomorphic function $\varphi(\zeta, u)$, therefore by studying the function $\varphi(\zeta, u)$ with multi-sheeted Riemann surfaces we can study solutions with a complex topology of two-dimensional space, in particular with the topology of a compact sphere with n handles for algebraic curves. As for as the general case (5), to this time[2] only one exact solution of this equation, found by Robinson, is known

$$P = (\zeta + \zeta^*)^{3/2}. \tag{7}$$

We show here that:

1) Equation (3) can be reduced to the generalized Liouville equation in two-dimensional Lobachevsky space (space of constant negative curvature).

2) One-dimensional reductions of Eq. (3′) in the Lobachevsky plane can be found in semi-geodesic systems of coordinates.

3) The one-dimensional reductions of Eq. (3′) contain, generally speaking, a moving logarithmic singularity, i.e., upon continuation of the canonical parameter in geodesics to the complex plane they contain a moving logarithmic branch point. Therefore these equations do not belong to any of the 50 types discovered by Painlevé and Gambier.[6]

4) For one of the three types of the one-dimensional reductions that were found for Eq. (3′), this equation can be reduced to Emden's equation.

The Robinson solution (4) turns out to be unstable: it coincides with a saddle-type singular point in the phase plane. A subclass of certain asymptotic solutions admits after coordinate transformations an interesting interpretation. The solution of the generalized Liouville equation can be nonsingular on only one copy of the Lobachevsky plane. Thus the solution is not covered by a "single leaf" of Lobachevsky and the two-dimensional cross section of the space-time manifold for fixed u and r can have varied topology (in the compact case—diffeomorphic sphere with n handles).

1. Following the substitution $P = (\zeta + \zeta^*)^{3/2}\Phi$, Eq. (5) becomes

$$2(\zeta + \zeta^*)^2 \Phi^2 (\ln \Phi)_{,\zeta\zeta^*} = 3(\Phi^2 - 1). \tag{8}$$

This equation can be written in the form

$$2\Delta \ln \Phi = 3(1 - \Phi^{-2}), \tag{9}$$

where Δ is the Laplace operator in the Lobachevsky plane, realized in the upper half-plane $\zeta + \zeta^* > 0$ with the metric

$$ds^2 = d\zeta d\zeta^*/(\zeta + \zeta^*)^2. \tag{10}$$

The lower half-plane $\zeta + \zeta^* < 0$ constitutes with the metric (10) another copy of the Lobachevsky plane. With the help of the obvious substitution $\varphi = -2 \ln \Phi$ Eq. (9) goes over into the generalized Liouville equation in the Lobachevsky plane:

$$\Delta\varphi + 3 \exp \varphi = 3. \tag{11}$$

2. To find the one-dimensional reductions of Eq. (6) [or (II)] we make use of the semi-geodesic orthogonal coordinate systems

$$du^2 + K^2(u,v)dv^2. \tag{12}$$

It follows from the condition of constancy of the negative Gaussian curvature of the surface with internal metric (12) that these metrics can be of one of three types:

$$du^2 + e^{-2u}dv^2 = w^{-2}(dv^2 + dw^2), \quad w = e^u, \quad \zeta = w + iv; \tag{13}$$

$$du^2 + \text{sh}^2\, u dv^2 = \text{sh}^{-2}(w)(dv^2 + dw^2), \quad \text{sh}\, w = 1/\text{sh}\, u, \tag{14}$$

$$du^2 + \text{ch}^2\, u dv^2 = \sin^{-2}(w)(dv^2 + dw^2), \quad \sin w = 1/\text{ch}\, u. \tag{15}$$

The system of coordinates (14) can be interpreted as being polar, where the geodesic lines diverge from a finite point. In the system of coordinates (13) the geodesics diverge from an infinitely distant point.

The system of coordinates (15) is constructed from nonintersecting divergent in both directions geodesic congruences. The family of lines v is constructed orthogonal to the geodesics u. The Laplace operator in the Lobachevsky plane takes the following form in the coordinate systems (13)–(15):

$$\Delta = \partial^2/\partial u^2 + \sigma(u)\partial/\partial u + \mu^{-2}(u)\partial^2/\partial v^2, \tag{16}$$

where $\sigma(u) \equiv d\ln \mu/du$ and for the metrics (13)–(15) the function μ is respectively equal to

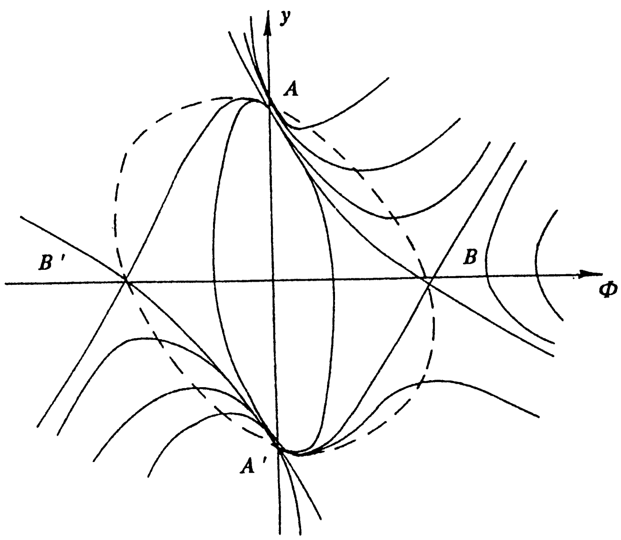

FIG. 1.

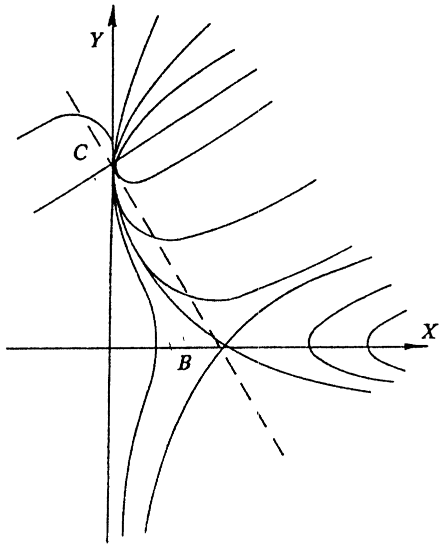

FIG. 2.

-1, cth u, th u. (17)

3. The ordinary differential equations for a function Φ that depends only on the variable u have the form:

$$2\Phi(\ddot{\Phi}+\sigma(u)\dot{\Phi})-2\dot{\Phi}^2=3(\Phi^2-1),\quad \dot{\Phi}\equiv d\Phi/du. \tag{18}$$

Let $u=u_0$ be an arbitrary point where $\sigma(u)$ is analytic [i.e. $u_0\neq0$ in the case that $\sigma(u)=\text{cth}\,u$] and admits the expansion $\sigma=\sigma_0+\sigma_1(u-u_0)+\sigma_2(u-u_0)^2+\ldots$. Then the solution of Eq. (18) with the additional condition $\Phi(u_0)=0$ is not analytic:

$$\Phi=\sqrt{3/2}(u-u_0-\sigma_0(u-u_0)^2/2+A(u-u_0)^3+(3/2-\sigma_0^2-\sigma_1)(u-u_0)^3\ln|u-u_0|/3+\ldots). \tag{19}$$

(Here u_0 and A are arbitrary constants).

The existence of a moving logarithmic branch point for Eqs. (18) means that they do not belong to any of the 50 canonical types of Painlevé and Gambier[6] of second order equations of the form

$$w''=f(w,w',z)$$

with the function f analytic (locally) in z and rational in w and w', which have no moving branch points. Evidently this circumstance provides indirect evidence that Eq. (9) is not integrable, since all known one-dimensional reductions of integrable systems have no moving branch points (Ref. 7).[1)]

In the case $\sigma(u)=-1$ Eq. (18) reduces to the Emden equation that arises in the theory of equilibrium of polytropic gas spheres.[9] In that case we obtain from Eq. (18) for $\dot{\Phi}=y=y(\Phi)$

$$\Phi y dy/d\Phi=3/2(\Phi^2-1)+y^2+y\Phi. \tag{20}$$

The qualitative picture of the integral curves for this equation is shown in Fig. 1.

The point A is a node, near which the integral curves have the asymptotes (19) [near $\sigma(u)=-1$]. The point B is a saddle, and the solution $\Phi=1$ corresponds to Robinson's solution. Thus Robinson's solution is unstable and an arbitrary small deviation from it leads to a solution given by one of the integral curves in Fig. 1, which passes through the neighborhood of the point B.

To complete the picture we consider the behavior of the integral curves for $\Phi\to\infty$. For this purpose we make the substitution $2y=-Y\Phi$ and denote $X=1/\Phi^2$. We then obtain from Eq. (16)

$$XYdY/dX=3X+Y-3. \tag{21}$$

The qualitative picture of the integral curves for this equation is shown in Fig. 2.

The integral curves have near the point C, i.e. as $u\to-\infty$, the following asymptote

$$\Phi\approx\alpha e^{-3u/2}(1+\beta e^u+\beta^2e^{2u}+(\beta^3/6-1/4\alpha^2)e^{3u}+\ldots). \tag{22}$$

For $u\to+\infty$ it follows from the asymptotes of the curves that have the Y axis as the asymptote when $Y\to-\infty$ that:

$$\Phi\approx\alpha\exp(\beta\exp u-3u/2). \tag{23}$$

The integral curves satisfying $\Phi_0\equiv\min\Phi>1$ do not pass through the node A. (see Fig. 1). Such curves have the asymptotes (22) for $u\to-\infty$ and (23) for $u\to+\infty$. If, in addition $\Phi_0\gg1$, then formula (23) gives an approximate solution for all u, and $\min\Phi=\Phi_0\approx\alpha(2\beta e/3)^{3/2}$, which is reached for $u_0=\ln 3/2\beta$.

For $\sigma(u)=\tanh u$ the approximate solution for the curves satisfying $\min\Phi=\Phi_0\gg1$ has for all u the form

$$\Phi\approx\alpha\,\text{ch}^{3/2}u\exp(\beta\,\text{arctg}\,e^u). \tag{24}$$

The minimum of Φ is reached for $u=u_0$, where u_0 satisfies the equation $3\sinh u_0+\beta=0$.

Solutions similar to the solutions with the global asymptotic behavior (23) and (24) admit of an interesting physical interpretation. Let us consider the approximate solution of Eq. (5) for large P

$$P|\partial f/\partial\zeta| \approx 1 - 3f^* \int \zeta df/2 - 3f \int \zeta^* df^*/2 + \dots, \tag{25}$$

where the arbitrary holomorphic function f does not depend on u. In the metric (1) it is convenient to pass to new independent variables f and f^* in place of ζ and ζ^* and to represent the arbitrary function $\zeta(f)$ [inverse to the function $f(\zeta)$] in the form of a derivative

$$3\zeta(f) = d\varphi/df, \quad \varphi = \varphi(f).$$

Then the metric (1) takes the form

$$ds^2 = 2drdu - (\varphi' + \varphi'^*)du^2 - 2r^2 df df^* (1 + \varphi f^* + \varphi^* f), \quad \varphi' \equiv d\varphi(f)/df. \tag{26}$$

At this point it is appropriate to recall Liouville's theorem according to which every holomorphic function $f'(\zeta)$ bounded in the extended complex plane can only be a constant and therefore the approximate solution (25) is not global in the ζ plane. However, in terms of the coordinates $\zeta(10)$ variation in the complex plane results in two copies of the Lobachevsky plane and to select just one it is necessary to demand that $\zeta + \zeta^*$ be either larger than, or less than, zero. In terms of the coordinates f this leads to the requirement $\varphi' + \varphi'^* > 0$. In this way the metric (26) gives the asymptote of the Robinson–Trautman solution if in the region $-H = \varphi' + \varphi'^* > 0$ the function $\varphi f^* + \varphi^* f$ is sufficiently small. We emphasize that there are no constraints on the coefficient H here.

For example, in the case of the solution (23) we have $2H = -3(\zeta + \zeta^*) = -\varphi' - \varphi'^* = 2\varepsilon \ln|f|$, where $\varepsilon > 0$. From the requirement $\zeta + \zeta^* > 0$ follows that $|f| < 1$. In that region $\varphi f^* + \varphi^* f = 2\varepsilon |f^2| (\ln|f^2| - 1)$ is a small quantity. If one removes from the unit disc $|f| < 1$ a small neighborhood of the origin $|f| < \varepsilon_1$ such that $|\varepsilon \ln \varepsilon_1| \ll 1$, then in the remaining annular region $\varepsilon_1 < f < 1$ the coefficient H is bounded and the metric (26) represents a small perturbation of Minkowski space. Indeed, the unperturbed metric

$$ds^2 = 2drdu - 2r^2 df df^*$$

is transformed by means of

$$rf = x + iy, \quad rf^* = x - iy, \quad r = ct + z,$$

$$u = ct - z - (x^2 + y^2)/(ct + z)$$

into the standard metric of Minkowski space in cartesian coordinates x,y,z,t.

The coefficients of the metric (26) depend on f and f^*, which are respectively equal to the projective variables $(x \pm iy)/(z + ct)$. Here the Poincaré model of the Lobachevsky plane $|f| < 1$ is represented in Minkowski space inside the cone $\sqrt{x^2 + y^2} < |z + ct|$, which moves with the speed of light progressively along the z axis. Therefore the Robinson–Trautman solution in the form (26) for the solution (23) takes place for any given instant of time only inside this cone.

Physical explanation of the accumulated theoretical models of GRT is a necessary condition for their practical utilization.[10]

The authors view this research as one attempt at a study of the topology and physical meaning of one of the most beautiful solutions of Einstein's equations—the exact Robinson–Trautman solution.

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[1)] This property of Eq. (9) distinguishes it radically from the equations obtained from the Einstein equations (as well as from the neutrino electrovacuum equations[8]) with two commuting Killing vectors (see citations in Ref. 8).

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