Relativistic theory of linearly polarized electromagnetic wave scattering by a nonmagnetized electron beam

Yu. V. Bobylev, M. V. Kuzelev, and V. A. Panin

Moscow State University
(Submitted 21 December 1992)
Zh. Eksp. Teor. Fiz. **104**, 2339–2365 (July 1993)

We consider the linear and nonlinear theories of scattering of linearly polarized electromagnetic waves by an electron beam in the absence of an external magnetic field. We show that the scattering of linearly polarized waves by a beam has a number of fundamental peculiarities absent from the case of circularly polarized waves. We use analytic methods to calculate the amplitudes of the scattering waves and the characteristic scattering-process times. The scattering processes are classified.

1. INTRODUCTION

Induced scattering of electromagnetic waves by relativistic beams of free electrons are widely used at present in free-electron lasers of various types.[1,2] The numerous papers on scattering theory notwithstanding,[3,8] the processes of interaction between linearly polarized waves and a nonmagnetized beam have been far from thoroughly investigated and have a number of fundamental peculiarities due to relativistic as well as nonrelativistic effects. In particular, interaction between linearly polarized waves and a nonrelativistic nonmagnetized beam induces processes that are already of higher order in the parameter $v_\perp/c$ than in the usual scattering theory[9] ($v_\perp$ is the electron velocity transverse to the beam propagation direction). We must therefore consider briefly the nonrelativistic theory and analyze the mechanism of high-order processes.

2. NONRELATIVISTIC NONLINEAR THEORY

It is known that in scattering processes the resonant force exerted on an electron by an electromagnetic field is proportional to the product of the incident- and scattered-wave amplitudes (the combined force). The product of the wave amplitudes can be due here to various causes. Thus, for straight beams in an infinitely strong longitudinal field the force acting on the electron is proportional to the longitudinal electric field

$$F\sim E_z=E_1\exp(i\varphi_1)+E_2\exp(i\varphi_2)$$

and the product E_1E_2 of the amplitudes of the interacting waves is obtained by expanding the phases φ_1 and φ_2 in terms of the fast oscillations of the electron coordinates, followed by averaging[3,5,7] under the assumption that the electron is fast in fields E_1 and E_2 and slow in the combination-wave field E_1E_2.

The situation is different in transversely oscillating beams.[10] Thus, for scattering by a nonmagnetized electron beam of purely transverse electromagnetic waves we have

$$F\sim\frac{\partial}{\partial z}(A_1+A_2)^2,$$

where A_1 and A_2 are the vector potentials of the incident and scattered waves. If the electromagnetic field is circularly polarized the procedure of squaring the sum A_1+A_2 directly without some averaging generates a slow force component ($\sim\partial A_1A_2/\partial z$), since A_1^2 and A_2^2 are both constant for waves with circularly polarized waves.

The situation for linearly polarized waves is more complicated. Rapidly oscillating terms appear on top of the slow $\sim A_1A_2$ ones. However, the described procedure of expanding the phases $\varphi_{1,2}$ followed by averaging, the slow terms also contribute to the slow motion. The mathematics of scattering by a beam of linearly polarized waves includes therefore elements of the theory of scattering of both circularly polarized waves and of the theory of scattering of quasitransverse waves by magnetized beams. The theory of scattering of linearly polarized waves requires therefore a consistent allowance for various types of nonlinearity.

In the most general formulation of the temporal evolution of stimulated scattering processes we use for linearly polarized transverse electromagnetic waves a vector potential in the form

$$\mathbf{A}_1=\{A_{x1},0,0\},\quad \mathbf{A}_2=\{A_{x2},0,0\},\tag{1}$$

where

$$A_{xj}=\tfrac{1}{2}[A_j(t)\exp(-i\omega_jt+ik_jz)+\text{c.c.}],\quad j=1,2\tag{2}$$

and take the initial system of equations describing the dynamics of an electromagnetic field in an electron beam to be

$$\begin{gathered}\left(\frac{\partial^2}{\partial z^2}-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)(A_{x1}+A_{x2})=-\frac{4\pi}{c}j_x,\\ \frac{dz}{dt}=v_z,\\ \frac{dv_z}{dt}=-\frac{e}{m}\frac{\partial\varphi}{\partial z}+\frac{e}{mc}v_x\frac{\partial}{\partial z}(A_{x1}+A_{x2}),\\ v_x=-\frac{e}{mc}(A_{x1}+A_{x2}).\end{gathered}\tag{3}$$

Here z is the coordinate of the electron in the direction of the beam motion, v_z and v_x the longitudinal and transverse velocities, and φ the scalar potential of the field produced upon longitudinal modulation of the beam in density (potential of the charge-density wave).

 1063-7761/93/070042-13$10.00

Following a known scattering-theory approach, we regard as fast the beam-electron motion in fields A_{x1} and A_{x2} and as slow the motion in the fields of the combination wave

$$A_1A_2^* \exp(-i\omega_0 t+ik_0 z)$$

and of the charge density

$$\varphi=\tfrac{1}{2}[\varphi_0(t)\exp(-i\omega_0 t+ik_0 z)+\text{c.c.}]. \tag{4}$$

Here $\omega_0=\omega_1-\omega_2$ and $k_0=k_1-k_2$.

Substituting expressions (2) and (4) in the system (3), we obtain for the longitudinal velocity the expression

$$\begin{aligned}\frac{dv_z}{dt}=&-\frac{i}{2}\frac{e}{m}k_0(\varphi_0\exp(-i\omega_0 t+ik_0 z)-\text{c.c.})\\&-\frac{i}{4}\left(\frac{e}{mc}\right)^2 k_0(A_1A_2^*\exp(-i\omega_0 t+ik_0 z)-\text{c.c.})\\&-\frac{i}{4}\left(\frac{e}{mc}\right)^2(k_1A_1^2\exp(-2i\omega_1 t+2ik_1 z)\\&+k_2A_2^2\exp(-2i\omega_2 t+2ik_2 z)+k_+A_1A_2\\&\times\exp(-i\omega_+ t+ik_+ z)-\text{c.c.}),\end{aligned} \tag{5}$$

where $\omega_+=\omega_1+\omega_2$, $k_+=k_1+k_2$. It can be seen from Eq. (5) that the first and second terms contribute to the slow motion, the second being the result of squaring the sum $A_{x1}+A_{x2}$. The last term of (5) oscillates rapidly. Applying to this term the corresponding phase-expansion and averaging procedure[3,5,7] we can obtain the additional contribution to the slow motion.

To this end we represent the coordinate z and the longitudinal electron velocity v_z in the form

$$z=ut+z'+\tilde{z},\quad v_z=v'+\tilde{v}. \tag{6}$$

Here $\tilde{z}$ and $\tilde{v}$ are the rapid oscillations of the coordinate and the velocity, while v' and z' are the slow ones (u is the velocity of the unperturbed beam). The subsequent averaging procedure, and hence also the representation (6), is meaningful if the following inequality is satisfied:

$$\max[t_0^{-2},\omega_b^2]\ll(\omega_{+,1,2}-k_{+,1,2}u)^2,\quad k_0^2u^2, \tag{7}$$

where t_0 is the characteristic variation time of the amplitudes of the interacting waves, and ω_b is the Langmuir frequency of the beam electrons. Moreover, for the averaging procedure to be correct it is necessary to satisfy the easily realizable assumption that the amplitudes of the fast oscillations of the electrons are small

$$|k_{+,1,2}\tilde{z}|\ll 1. \tag{8}$$

It is just the condition (8) which allows us to write the expression for the fast electron-coordinate oscillations in the form

$$\begin{aligned}\tilde{z}=&\frac{i}{16}\left(\frac{e}{mc}\right)^2\Omega_0^{-2}(k_1A_1^2\exp(-2i\Phi_1)+k_2A_2^2\\&\times\exp(-2i\Phi_2)+k_+A_1A_2\exp(-i\Phi_+)-\text{c.c.}),\end{aligned} \tag{9}$$

where

$$\begin{aligned}&\Phi_1=(\omega_1-k_1u)t-k_1z',\quad \Phi_2=(\omega_2-k_2u)t-k_2z',\\&\Phi_+=(\omega_+-k_+u)t-k_+z',\\&\Omega_0=\omega_1-k_1u\approx\omega_2-k_2u=\tfrac{1}{2}(\omega_+-k_+u).\end{aligned} \tag{10}$$

As a result, the slow electron motions are described by the equations

$$\begin{aligned}\frac{dz'}{dt}=&v'-u,\\\frac{dv'}{dt}=&-\frac{i}{2}\frac{e}{m}k_0(\varphi_0\exp(-i\tilde{D}t+ik_0z')-\text{c.c.})\\&-\frac{i}{4}\left(\frac{e}{mc}\right)^2k_0(A_1A_2^*\exp(-i\tilde{D}t+ik_0z')-\text{c.c.})\\&+\frac{1}{4}\left(\frac{e}{mc}\right)^2\langle\tilde{z}(2k_1^2A_1^2\exp(-2i\Phi_1)\\&+2k_2^2A_2^2\exp(-2i\Phi_2)+k_+^2A_1A_2\exp(-i\Phi_+)\\&-\text{c.c.})\rangle,\end{aligned} \tag{11}$$

in which the angle brackets denote averaging over the fast oscillations, and $\tilde{D}=\omega_0-k_0$ is the detuning. In the one-particle-scattering regime we have $\tilde{D}=0$, and in the collective-scattering regime, when for example resonance obtains with the beam slow charge-density wave, we have $-\tilde{D}=-\omega_b$.

Expressing next the amplitude φ_0 of the potential of the beam's longitudinal oscillations in terms of the charge-density wave amplitude ρ_1,

$$\begin{aligned}&\varphi_0=\frac{4\pi en_b}{k_0^2}\rho_1\exp(-i\tilde{D}t),\\&\rho_1=\frac{2}{L}\int_0^L\exp(-ik_0z')dz_0,\end{aligned} \tag{12}$$

where $L=2\pi/k_0$, and averaging, we obtain for the equation of motion of the electrons

$$\begin{aligned}\frac{dz'}{dt}=&v'-u,\\\frac{dv'}{dt}=&-\frac{i}{2}\frac{\omega_b^2}{k_0}(\rho_1\exp(ik_0z')-\text{c.c.})\\&-\frac{i}{4}\left(\frac{e}{mc}\right)^2k_0(A_1A_2^*\exp(-i\tilde{D}t+ik_0z')-\text{c.c.})\\&-\frac{i}{64}\left(\frac{e}{mc}\right)^4\frac{k_0}{\Omega_0^2}(2k_1k_2A_1^2A_2^{*2}\exp(-2i\tilde{D}t\\&+2ik_0z')+k_1k_+|A_1|^2A_2^*A_1\exp(-i\tilde{D}t\\&+ik_0z')+k_2k_+A_1|A_2|^2A_2\exp(-i\tilde{D}t+ik_0z')\\&-\text{c.c.}).\end{aligned} \tag{13}$$

Let us dwell in greater detail on the structure of the second equation. Its first term is the force due to modulation of the beam charge, and the second and third terms are the resonant forces from the directions of the incident and scattered waves, respectively. The second term, as already noted, stems from squaring the sum $A_{x1}+A_{x2}$, and the third from averaging over the fast oscillations. Note that in addition to the algebraic and nonlinearities, due to the structure of the Lorentz force, of the second and fourth order in the amplitudes A_1 and A_2, the equation of motion contains also a transcendental nonlinearity $\exp[ik_0z'(t,z_0)]$ that will be shown below to be reducible in a number of cases to an algebraic one.

We shall now describe the dynamics of the electromagnetic field. Following a substitution of the representation (2) in the first equation of the system (3) and elementary averaging, this equation breaks up into two[1)] equations for the amplitudes A_1 and A_2:

$$\frac{2i\omega_j}{c^2}\frac{dA_j}{dt}-\left(k_j^2-\frac{\omega_j^2}{c^2}\right)A_j$$
$$=-\frac{4\pi}{c}\frac{2}{L}\int_0^L j_x(t,z)\exp(i\omega_j t-ik_jz)dz,$$
$$j=1,2. \tag{14}$$

This equation was obtained with account taken of the fact that A_1 and A_2 are slow functions of the time.

Recognizing that for a single-velocity beam the transverse current is defined as

$$j_x(t,z)=en_0\int v_x(t,z_0)\delta[z-z(t,z_0)]dz_0, \tag{15}$$

we rewrite (14) in the form

$$\frac{2i\omega_j}{c^2}\frac{dA_j}{dt}-\left(k_j^2-\frac{\omega_j^2}{c^2}\right)A_j=-\frac{4\pi en_b}{c}\frac{2}{L}\int_0^L v_x(t,z_0)$$
$$\times\exp(i\omega_jt-ik_jz(t,z_0))dz_0, \tag{16}$$

with $v_x(t,z_0)$ given by

$$v_x(t,z_0)=-\frac{e}{2mc}[A_1\exp(-i\omega_1t+ik_1z)+A_2$$
$$\times\exp(-i\omega_2t+ik_2z)+\text{c.c.}]|_{z=z(t,z_0)}. \tag{17}$$

The subsequent derivation of the equations for the amplitudes A_1 and A_2 reduces to substitution of (17) in (16) following by an averaging similar to that described above. Omitting therefore the intermediate steps, we have for the final result

$$\frac{dA_1}{dt}=-i\frac{\omega_b^2}{4\omega_1}\Big[A_2\rho_1\exp(i\tilde{D}t)$$
$$+\frac{1}{16}\left(\frac{e}{mc}\right)^2\Omega_0^{-2}(2k_1k_2A_1^*A_2^2\rho_2\exp(2i\tilde{D}t)$$
$$+2k_1k_+|A_1|^2A_2\rho_1\exp(i\tilde{D}t)$$
$$+k_1k_+A_1^2A_2^*\rho_1^*\exp(-i\tilde{D}t)$$
$$+k_2k_+|A_2|^2A_2\rho_1\exp(i\tilde{D}t)\Big],$$

$$\frac{dA_2}{dt}=-i\frac{\omega_b^2}{4\omega_2}\Big[A_1\rho_1^*\exp(-i\tilde{D}t)$$
$$+\frac{1}{16}\left(\frac{e}{mc}\right)^2\Omega_0^{-2}(2k_1k_2A_1^2A_2^*\rho_2^*\exp(-2i\tilde{D}t)$$
$$+2k_2k_+|A_2|^2A_1\rho_1^*\exp(-i\tilde{D}t)$$
$$+k_2k_+A_2^2A_1^*\rho_1\exp(i\tilde{D}t)$$
$$+k_1k_+|A_1|^2A_1\rho_1^*\exp(-i\tilde{D}t))\Big]. \tag{18}$$

Here

$$\rho_2=\frac{2}{L}\int_0^L\exp(-2ik_0z')dz_0.$$

Equations (13) and (18) constitute the complete system of nonrelativistic equations describing the scattering of linearly polarized waves by a nonmagnetized beam. Discarding from the resultant equations the terms due to averaging over the fast oscillations, we obtain a system describing the usual three-wave interactions.[5,7,8] The processes are then of second order in the parameter $v_\perp/c$. On the whole, however, Eqs. (13) and (18) describe processes of second as well as of at least fourth orders in the parameter $(v_\perp/c)$. Generally speaking, allowance for all the nonlinearities in the averages makes it possible to treat consistently processes of arbitrary accuracy in $v_\perp/c$. In this case, however, the inequality (8) should not be strong.

Further analysis of scattering processes, in both the linear and nonlinear stages, will be based on a more general relativistic system of equations. We shall determine simultaneously the validity regions of Eqs. (13) and (18), and present a general classification of the processes as they relate to the degree of relativism and the density of the beam.

3. RELATIVISTIC THEORY: DERIVATION OF NONLINEAR EQUATIONS

Allowance for relativistic effects leaves the first two equations of the initial system (3) unchanged, whereas the last two take the form

$$\frac{dv_z}{dt}=-\frac{e}{m\gamma_t\gamma_\parallel^2}\frac{\partial\varphi}{\partial z}-\frac{1}{2}\left(\frac{e}{mc}\right)^2\frac{1}{\gamma_t^2}\left(\frac{\partial}{\partial z}+\frac{v_z}{c^2}\frac{\partial}{\partial t}\right)$$
$$\times(A_{x1}+A_{x2})^2,$$
$$v_x=-\frac{e}{mc\gamma_t}(A_{x1}+A_{x2}), \tag{19}$$

where $\gamma_\parallel=(1-v_z^2/c^2)^{-1/2}$, while

$$\gamma_t=(1-v_z^2/c^2-v_x^2/c^2)^{-1/2} \tag{20}$$

is the total relativistic factor of the electron. Since γ_t depends on v_x and hence also on the amplitudes A_{x1} and A_{x2}, expansion in powers of A_{x1} and A_{x2} is unavoidable not only in the equation for fast oscillations, but also in the equations describing slow motion and containing γ_t. This approach, used in classical scattering theory, cannot be regarded here as satisfactory. To get around the above difficulty, we regard γ_t as an independent variable satisfying the equation

$$\frac{d\gamma_t}{dt}=-\frac{e}{m}\frac{v_z}{c^2}\frac{\partial\varphi}{\partial z}+\frac{1}{2\gamma_t c^2}\frac{\partial}{\partial t}(A_{x1}+A_{x2})^2. \tag{21}$$

Let us dwell, as above, in greater detail on the transformation of the equations of motion. We substitute expressions (2) and (4) in the equations for v_z and γ_t and represent, by analogy with (6), γ_t in the form

$$\gamma_t=\gamma+\tilde{\gamma}, \tag{22}$$

where $\tilde{\gamma}$ and γ denote fast and slow oscillations, respectively. Assuming, as before, inequalities (7) and (8) to be satisfied, we write for the slow electron motion the equations

$$\frac{dz'}{dt}=v-u,$$

$$\begin{aligned}\frac{dv'}{dt}=&-\frac{i}{2}\frac{ek_0}{m}\left(1-\frac{v_z^2}{c^2}\right)^{3/2}\left(\varphi_0\exp\left(-i\tilde{D}t+ik_0z'\right)-\text{c.c.}\right)\\&-\frac{i}{4}\left(\frac{e}{mc}\right)\frac{1}{\gamma^2}\left(k_0-\frac{v}{c^2}\omega_0\right)\left(A_1A_2^*\exp\left(-i\tilde{D}t\right.\right.\\&\left.\left.+ik_0z'\right)-\text{c.c.}\right)+\frac{1}{4}\left(\frac{e}{mc}\right)^2\frac{1}{\gamma^2}\left\langle\tilde{z}\left[2k_1\left(k_1\right.\right.\right.\\&\left.-\frac{v\omega_1}{c^2}\right)A_1^2\exp\left(-2i\Phi_1\right)+2k_2\left(k_2-\frac{v\omega_2}{c^2}\right)A_2^2\\&\times\exp\left(-2i\Phi_2\right)+k_+\left(k_+-\frac{v\omega_+}{c^2}\right)A_1A_2\\&\left.\left.\times\exp\left(-i\Phi_+\right)+\text{c.c.}\right]\right\rangle+\frac{i}{4}\left(\frac{e}{mc}\right)^2\frac{2}{\gamma^3}\\&\times\left\langle\tilde{\gamma}\left[\left(k_1-\frac{v\omega_1}{c^2}\right)A_1^2\exp\left(-2i\Phi_1\right)\right.\right.\\&+\left(k_2-\frac{v\omega_2}{c^2}\right)A_2^2\\&\times\exp\left(-2i\Phi_2\right)+\left(k_+-\frac{v\omega_+}{c^2}\right)A_1A_2\\&\left.\left.\times\exp\left(-i\Phi_+\right)-\text{c.c.}\right]\right\rangle\\&+\frac{i}{4}\left(\frac{e}{mc}\right)^2\frac{1}{c^2\gamma^2}\left\langle\tilde{v}\left[\omega_1A_1^2\exp\left(-2i\Phi_1\right)\right.\right.\\&+\omega_2A_2^2\exp\left(-2i\Phi_2\right)+\omega_+A_1A_2\\&\left.\left.\times\exp\left(-i\Phi_+\right)-\text{c.c.}\right]\right\rangle,\end{aligned} \tag{23}$$

$$\begin{aligned}\frac{d\gamma}{dt}=&-\frac{i}{2}\frac{ek_0}{m}\frac{v}{c^2}(\varphi_0\exp(-i\tilde{D}t+ik_0z')-\text{c.c.})\\&-\frac{i}{4}\left(\frac{e}{mc}\right)^2\frac{\omega_0}{\gamma c^2}(A_1A_2^*\exp(-i\tilde{D}t+ik_0z')\\&-\text{c.c.})+\frac{1}{4}\left(\frac{e}{mc}\right)^2\frac{1}{\gamma c^2}\langle\tilde{z}[2\omega_1k_1A_1^2\exp(-2i\Phi_1)\\&+2\omega_2k_2A_2^2\exp(-2i\Phi_2)+\omega_+k_+A_1A_2\\&\times\exp(-i\Phi_+)+\text{c.c.}]\rangle+\frac{i}{4}\left(\frac{e}{mc}\right)^2\\&\times\frac{1}{c^2\gamma^2}\langle\tilde{\gamma}[\omega_1A_1^2\exp-2i\Phi_1)\\&+\omega_2A_2^2\exp(-2i\Phi_2)+\omega_+A_1A_2\exp(-i\Phi_+)\\&-\text{c.c.}]\rangle,\end{aligned}$$

where the oscillations of the coordinate $\tilde{z}$, the velocity $\tilde{v}$, and the relativistic factor $\tilde{\gamma}$ are given by the relations

$$\begin{aligned}\tilde{z}=&\frac{i}{16}\left(\frac{e}{mc}\right)^2\frac{1}{\gamma^2\Omega_0}\left[\left(k_1-\frac{v\omega_1}{c^2}\right)A_1^2\exp(-2i\Phi_1)+\left(k_2\right.\right.\\&\left.-\frac{v\omega_2}{c^2}\right)A_2^2\exp(-2i\Phi_2)+\left(k_+-\frac{v\omega_+}{c^2}\right)A_1A_2\\&\left.\times\exp(-i\Phi_+)-\text{c.c.}\right],\\\tilde{v}=&\frac{1}{8}\left(\frac{e}{mc}\right)^2\frac{1}{\gamma^2\Omega_0}\left[\left(k_1-\frac{v\omega_1}{c^2}\right)A_1^2\exp(-2i\Phi_1)+\left(k_2\right.\right.\\&\left.-\frac{v\omega_2}{c^2}\right)A_2^2\exp-2i\Phi_2+\left(k_+-\frac{v\omega_+}{c^2}\right)A_1A_2\\&\left.\times\exp(-i\Phi_+)+\text{c.c.}\right],\\\tilde{\gamma}=&\frac{1}{8}\left(\frac{e}{mc}\right)^2\frac{1}{c^2\gamma\Omega_0}[\omega_1A_1^2\exp(-2i\Phi_1)+\omega_2A_2^2\\&\times\exp(-2i\Phi_+)+\omega_+A_1A_2\exp(-i\Phi_+)+\text{c.c.}].\end{aligned} \tag{24}$$

We write next for the longitudinal electron velocity

$$v=v'+u,\quad v'\ll u \tag{25}$$

and, after substituting (24) in (23) and actually averaging, we write the electron-beam equations of motion:

$$\frac{dz'}{dt}=v',$$

$$\begin{aligned}\frac{dv'}{dt}=&-\frac{i}{2}\frac{\omega_b^2}{k_0\gamma_0^3}\left(1-2\gamma_0^2\frac{uv'}{c^2}\right)^{3/2}(\rho_1\exp(ik_0z')-\text{c.c.})\\&-\frac{i}{4}\left(\frac{e}{mc}\right)^2\frac{k_0}{\gamma_0^2\gamma^2}\left(1-\gamma_0^2\frac{uv'}{c^2}\right)(A_1A_2^*\exp(-i\tilde{D}t\\&+ik_0z')-\text{c.c.})-\frac{i}{64}\left(\frac{e}{mc}\right)^4\frac{k_0}{\gamma^4\Omega_0^2\gamma_0^2}\\&\times\left\{2\left[\varkappa_{12}^2-\gamma_0^2\left(k_1k_2-2\frac{\omega_1\omega_2}{c^2}\right.\right.\right.\end{aligned}$$

$$-\frac{\omega_1 k_2^2-\omega_2 k_1^2}{k_0 u}\Big)\frac{uv'}{c^2}\Big]A_1^2A_2^*\exp(-2i\tilde{D}t$$

$$+2ik_0z')+\Big[\varkappa_{1+}^2-\gamma_0^2\Big(k_1k_+-2\frac{\omega_1\omega_+}{c^2}$$

$$-\frac{k_+^2\omega_1-2k_1^2\omega_+}{k_0u}\Big)\frac{uv'}{c^2}\Big]A_1\Big|A_1\Big|^2A_2^*\exp(-i\tilde{D}t$$

$$+ik_0z')+\Big[\varkappa_{2+}^2-\gamma_0^2\Big(k_2k_+-2\frac{\omega_2\omega_+}{c^2}$$

$$-\frac{2k_2^2\omega_+-k_+\omega_2}{k_0u}\Big)\frac{uv'}{c^2}\Big]A_1\Big|A_2\Big|^2A_2^*\exp(-i\tilde{D}t$$

$$+ik_0z')-\text{c.c.}\Big\}, \tag{26}$$

$$\frac{d\gamma}{d\tau}=-\frac{i}{2}\frac{\omega_b^2u}{k_0c^2}(\rho_1\exp(ik_0z')-\text{c.c.})$$

$$-\frac{i}{4}\left(\frac{e}{mc^2}\right)^2\frac{\omega_0}{\gamma}(A_1A_2^*\exp(-i\tilde{D}t+ik_0z')$$

$$-\text{c.c.})-\frac{i}{64}\left(\frac{e}{mc}\right)^4\frac{k_0u}{\gamma^3c^2\Omega_0^2}\Big[2\Big(\varkappa_{12}^2-\frac{\omega_1\omega_2}{c^2}\frac{v'}{u}\Big)$$

$$\times A_1^2A_2^{*2}\exp(-2i\tilde{D}t+2ik_0z')$$

$$+\Big(\varkappa_{1+}^2-\frac{\omega_1\omega_+}{c^2}\frac{v'}{u}\Big)A_1\Big|A_1\Big|^2A_2^*$$

$$\times\exp(-i\tilde{D}t+ik_0z')+\Big(\varkappa_{2+}^2-\frac{\omega_2\omega_+}{c^2}\frac{v'}{u}\Big)$$

$$\times A_1\Big|A_2\Big|^2A_2^*\exp(-i\tilde{D}t+ik_0z')-\text{c.c.}\Big].$$

Here

$$\gamma_0=(1-(u^2/c^2)^{-1/2},\quad \varkappa_{12}^2=k_1k_2-\omega_1\omega_2/c^2,$$

$$\varkappa_{+1}^2=k_1k_+-\omega_1\omega_+/c^2,\quad \varkappa_{+2}=k_2k_+-\omega_2\omega_+/c^2.$$

Note that by putting formally $c\to\infty$ (but not for the combination e/mc) in Eqs. (23), (24), and (26) we obtain the nonrelativistic equations derived above.

Equations for the amplitudes A_1 and A_2 are obtained by a procedure similar to that in the nonrelativistic theory. The only difference is that the oscillations described by Eqs. (24) have a more complicated structure. We present therefore directly in final form the equations for the amplitudes of the incident and scattered waves

$$\frac{4i\omega_1}{\omega_b^2}\frac{dA_1}{dt}=A_2\hat{\rho}_1\exp(i\tilde{D}t)+\frac{1}{16}\left(\frac{e}{mc}\right)^2\Omega_0^{-2}\frac{2}{L}$$

$$\times\int_0^L\frac{dz_0}{\gamma^3}\Big[2\Big(\varkappa_{12}^2-\frac{k_1\omega_2}{c}\frac{v'}{c}\Big)A_1^*A_2^2$$

$$\times\exp(2i\tilde{D}t$$

$$-2ik_0z')+2\Big(\varkappa_{1+}^2-\frac{k_1\omega_+}{c}\frac{v'}{c}\Big)$$

$$\times\Big|A_1\Big|^2A_2\exp(i\tilde{D}t-ik_0z')+\Big(\varkappa_{1+}^2$$

$$-\frac{k_+\omega_1}{c}\frac{v'}{c}\Big)A_1^2A_2^*$$

$$\times\exp(-i\tilde{D}t+ik_0z')+\Big(\varkappa_{2+}^2$$

$$-\frac{k_+\omega_2}{c}\frac{v'}{c}\Big)\Big|A_2\Big|^2A_2\exp(i\tilde{D}t-ik_0z')\Big], \tag{27}$$

$$\frac{4i\omega_2}{\omega_b^2}\frac{dA_2}{dt}=A_1\hat{\rho}_1^*\exp(-i\tilde{D}t)$$

$$+\frac{1}{16}\left(\frac{e}{mc}\right)^2\Omega_0^{-2}\frac{2}{L}\int_0^L\frac{dz_0}{\gamma^3}\Big[2\Big(\varkappa_{12}^2$$

$$-\frac{k_2\omega_1}{c}\frac{v'}{c}\Big)A_2^*A_1^2\exp(-2i\tilde{D}t+2ik_0z')$$

$$+2\Big(\varkappa_{2+}^2-\frac{k_2\omega_+}{c}\frac{v'}{c}\Big)\Big|A_2\Big|^2A_1\exp(-i\tilde{D}t$$

$$+ik_0z')+\Big(\varkappa_2^2-\frac{k_+\omega_2}{c}\frac{v'}{c}\Big)A_2^2A_1^*\exp(i\tilde{D}t$$

$$-ik_0z')+\Big(\varkappa_{12}^2-\frac{k_+\omega_1}{c}\frac{v'}{c}\Big)\Big|A_1\Big|^2A_1$$

$$\times\exp(-i\tilde{D}t+ik_0z')\Big],$$

where

$$\rho_1=\frac{2}{L}\int_0^L\frac{1}{\gamma}\exp(-ik_0z')dz_0. \tag{28}$$

It is easy to show that Eqs. (26) and (27) lead to flux energy and momentum conservation laws for the electromagnetic waves interacting with the beam

$$\omega_b^2c^2\frac{1}{L}\int_0^L\gamma dz_0+\frac{1}{2}\omega_1^2\left(\frac{e}{mc}\right)^2\Big|A_1\Big|^2+\frac{1}{2}\omega_2^2\left(\frac{e}{mc}\right)^2\Big|A_2\Big|^2$$

$$=\text{const}, \tag{29}$$

$$\frac{\omega_b^2}{m}\frac{1}{L}\int_0^L pdz_0+\frac{1}{2}\omega_1k_1\left(\frac{e}{mc}\right)^2\Big|A_1\Big|^2$$

$$+\frac{1}{2}\omega_2k_2\left(\frac{e}{mc}\right)^2\Big|A_2\Big|^2$$

$$=\text{const},$$

where $p=m\gamma(u+v')$ is the electron momentum.

We introduce next the following dimensionless variables:

$$\tau=\omega_b\gamma_0^{-3/2}t,\quad y=k_0z',\quad \eta=\frac{k_0v'}{\omega_b}\gamma_0^{3/2},\quad \eta_0=\frac{\tilde{D}}{\omega_b}\gamma_0^{3/2},$$

$$
\begin{aligned}
&\varepsilon_{1,2}=\frac{e}{m}k_0\frac{\sqrt{2\omega_{1,2}}}{\omega_b^{3/2}c}\gamma_0^{3/4}A_{1,2}, \quad \nu_0=\frac{1}{2}\frac{\omega_b}{\sqrt{2\omega_1}\sqrt{2\omega_2}}\gamma_0^{3/2},\\
&\hat{\nu}=\frac{1}{16}\frac{\omega_b^3}{\Omega_0^2\gamma_0^{3/2}\sqrt{4\omega_1\omega_2}}\frac{2\varkappa_{12}^2}{k_0^2},\\
&\nu_{1,2}=\frac{1}{16}\frac{\omega_b^3}{\Omega_0^2\gamma_0^{3/2}2\omega_{1,2}}\frac{\varkappa_{1,2}^2}{k_0^2},\\
&\mu_0=2\gamma_0^2\frac{u^2}{c^2}\frac{\omega_b\gamma_0^{-3/2}}{k_0u}, \quad \hat{\mu}=\frac{\omega_1\omega_2}{u^2\varkappa_{12}^2}, \quad \mu_{1,2}=\frac{\omega_{1,2}\omega_+}{u^2\varkappa_{1,2+}^2},\\
&\hat{q}=\frac{k_2^2\omega_1-k_1^2\omega_2}{k_0u\varkappa_{12}^2}, \quad q_1=\frac{k_+^2\omega_1-2k_1^2\omega_+}{k_0u\varkappa_{1+}^2},\\
&q_2=\frac{2k_2^2\omega_+-k_+^2\omega_2}{k_0u\varkappa_{2+}^2},\\
&\beta_1=\frac{k_1u}{\omega_1}, \quad \beta_2=\frac{k_2u}{\omega_2}, \quad \beta_+=\frac{k+u}{\omega_+}
\end{aligned}
\tag{30}
$$

and rewrite Eqs. (26) and (27) in the form

$$
\begin{aligned}
\frac{d\varepsilon_1}{d\tau}=&-i\nu_0\varepsilon_2\hat{\rho}_1\exp(i\eta_0\tau)-i\nu_0\Bigl[\hat{\nu}\Bigl(\hat{\hat{\rho}}_2\\
&-\frac{\mu_0}{2\gamma_0^2}\beta_1\hat{\mu}R_2\Bigr)\varepsilon_1^*\varepsilon_2^2\exp(2i\eta_0\tau)+2\nu_1\Bigl(\hat{\hat{\rho}}_1\\
&-\frac{\mu_0}{2\gamma_0^2}\beta_1\mu_1R_1\Bigr)\Bigl|\varepsilon_1\Bigr|^2\varepsilon_1\exp(i\eta_0\tau)+\nu_1\Bigl(\hat{\hat{\rho}}_1^*\\
&-\frac{\mu_0}{2\gamma_0^2}\beta+\mu_1R_1^*\Bigr)\varepsilon_1^2\varepsilon_2^*\exp(-i\eta_0\tau)+\nu_2\Bigl(\hat{\hat{\rho}}_1\\
&-\frac{\mu_0}{2\gamma_0^2}\beta_+\mu_2R_1\Bigr)\Bigl|\varepsilon_2\Bigr|^2\varepsilon_1\exp(i\eta_0\tau)\Bigr],\\
\frac{d\varepsilon_2}{d\tau}=&-i\nu_0\varepsilon_1\hat{\rho}_1^*\exp(-i\eta_0\tau)-i\nu_0\Bigl[\hat{\nu}\Bigl(\hat{\hat{\rho}}_2^*\\
&-\frac{\mu_0}{2\gamma_0^2}\beta_2\hat{\mu}R_2^*\Bigr)\varepsilon_2^*\varepsilon_1^2\exp(-2i\eta_0\tau)+2\nu_2\Bigl(\hat{\hat{\rho}}_1^*\\
&-\frac{\mu_0}{2\gamma_0^2}\beta_2\mu_2R_1^*\Bigr)\Bigl|\varepsilon_2\Bigr|^2\varepsilon_1\exp(-i\eta_0\tau)+\nu_2\Bigl(\hat{\hat{\rho}}_1\\
&-\frac{\mu_0}{2\gamma_0^2}\beta_+\mu_2R_1\Bigr)\varepsilon_2^2\varepsilon_1^*\exp(i\eta_0\tau)+\nu_1\Bigl(\hat{\hat{\rho}}_1^*\\
&-\frac{\mu_0}{2\gamma_0^2}\beta_+\mu_1R_1^*\Bigr)\Bigl|\varepsilon_1\Bigr|^2\varepsilon_1\exp(-i\eta_0\tau)\Bigr],\\
\frac{d\gamma}{d\tau}=&-\frac{i}{4}\mu_0\gamma_0(\rho_1\exp(iy)-\text{c.c.})\\
&-\frac{i}{4}\mu_0\nu_0\frac{1}{\gamma_0^2\gamma}(\varepsilon_1\varepsilon_2^*\exp(iy-i\eta_0\tau)-\text{c.c.})\\
&-\frac{i}{4}\mu_0\nu_0\frac{1}{\gamma_0^2\gamma^3}\Bigl[\hat{\nu}\Bigl(1-\frac{\mu_0}{2\gamma_0^2}\hat{\mu}\eta\Bigr)\varepsilon_1^2\varepsilon_1^{*2}\exp(2iy\\
&-2i\eta_0\tau)+\nu_1\Bigl(1-\frac{\mu_0}{2\gamma_0^2}\mu_1\eta\Bigr)\varepsilon_1\Bigl|\varepsilon_1\Bigr|^2\varepsilon_2^*\exp(iy\\
&-i\eta_0\tau)+\nu_2\Bigl(1-\frac{\mu_0}{2\gamma_0^2}\mu_2\eta\Bigr)\varepsilon_1\Bigl|\varepsilon_2\Bigr|^2\varepsilon_2^*\exp(iy\\
&-i\eta_0\tau)\Bigr)-\text{c.c.}],
\end{aligned}
\tag{31}
$$

$$
\begin{aligned}
\frac{dy}{d\tau}=&\eta,\\
\frac{d\eta}{d\tau}=&-\frac{i}{2}(1-\mu_0\eta)^{3/2}(\rho_1\exp(iy)-\text{c.c.})\\
&-\frac{i}{2}\nu_0\frac{1}{\gamma_0^2\gamma^2}\Bigl(1-\frac{1}{2}\mu_0\eta\Bigr)(\varepsilon_1\varepsilon_2^*\exp(iy-i\eta_0\tau)\\
&+\text{c.c.})-\frac{i}{2}\nu_0\frac{1}{\gamma_0^2\gamma^4}\Bigl\{\hat{\nu}\Bigl[1-\frac{1}{2}\mu_0\Bigl(1-\frac{\gamma_0^2-1}{\gamma_0^2}\hat{\mu}\\
&-\hat{q}\Bigr)\eta\Bigr]\varepsilon_1^2\varepsilon_2^*\exp(2iy-2i\eta_0\tau)+\nu_1\Bigl[1-\frac{1}{2}\mu_0\Bigl(1\\
&-\frac{\gamma_0^2-1}{\gamma_0^2}\mu_1-q_1\Bigr)\eta\Bigr]\varepsilon_1\Bigl|\varepsilon_1\Bigr|^2\varepsilon_2^*\exp(iy-i\eta_0\tau)\\
&+\nu_2\Bigl[1-\frac{1}{2}\mu_0\Bigl(1-\frac{\gamma_0^2-1}{\gamma_0^2}\mu_2-q_2\Bigr)\eta\Bigr]\varepsilon_1\Bigl|\varepsilon_2\Bigr|^2\varepsilon_2^*\\
&\times\exp(iy-i\eta_0\tau)-\text{c.c.}\Bigr\},\\
\rho_n=&\frac{1}{\pi}\int_0^{2\pi}\exp(-iny)dy_0,\\
\hat{\rho}_n=&\frac{1}{\pi}\int_0^{2\pi}\gamma^{-1}\exp(-iny)dy_0,\\
\hat{\hat{\rho}}_n=&\frac{1}{\pi}\int_0^{2\pi}\gamma^{-3}\exp(-iny)dy_0,\\
R_n=&\frac{1}{\pi}\int_0^{2\pi}\gamma^{-3}\exp(-iny)dy_0, \quad n=1,2.
\end{aligned}
$$

Note that μ_0 is indicative of the degree of relativism of the electron beam, while the product $\nu_0|\varepsilon_{1,2}|$ is inversely proportional to the beam density if the amplitude is fixed. The remaining parameters ($\hat{\mu}_{1,2}$,$\hat{q}_{1,2}$,$\hat{\nu}_{1,2}$,$\beta_{+,1,2}$) are governed to a considerable degree by the dispersion laws of the interacting waves.

Let us consider an actual model: scattering of linearly polarized transverse waves by a nonmagnetized beam of electrons in a decelerating system filled with dielectric. Assuming that the incident and scattered waves obey a linear dispersion law:

$$
\omega_1=k_1v_{\text{ph}}, \quad \omega_2=-k_2v_{\text{ph}}
\tag{32}
$$

and that the approximate resonance condition $\omega_1-\omega_2=(k_1-k_2)u$, is satisfied, we transform the principal parameters in (30) into the following:

$$k_1=\frac{1}{2}\frac{u+v_{\rm ph}}{v_{\rm ph}}k_0,\quad k_2=\frac{1}{2}\frac{u-v_{\rm ph}}{v_{\rm ph}}k_0,\quad k_+=\frac{u}{v_{\rm ph}}k_0,$$

$$\omega_1=\frac{1}{2}(u+v_{\rm ph})k_0,\quad \omega_2=\frac{1}{2}(u-v_{\rm ph})k_0,\quad \omega_+=k_0v_{\rm ph},$$

$$\Omega_0=\frac{1}{2}\frac{v_{\rm ph}^2-u^2}{v_{\rm ph}}k_0,\quad \beta_1=\frac{u}{v_{\rm ph}},\quad \beta_2=-\frac{u}{v_{\rm ph}},\quad \beta_+=\frac{u^2}{v_{\rm ph}^2},$$

$$\varkappa_{12}^2=\frac{1}{4}\frac{u^2-v_{\rm ph}^2}{v_{\rm ph}^2}\left(1-\frac{v_{\rm ph}^2}{c^2}\right)k_0^2,$$

$$\varkappa_{+1}^2=\frac{1}{2}\frac{(u+v_{\rm ph})^4}{v_{\rm ph}^2}\left(1-\frac{v_{\rm ph}}{u}\frac{v_{\rm ph}^2}{c^2}\right)k_0^2, \tag{33}$$

$$\varkappa_{+2}^2=\frac{1}{2}\frac{(u-v_{\rm ph})^4}{v_{\rm ph}^2}\left(1+\frac{v_{\rm ph}}{u}\frac{v_{\rm ph}^2}{c^2}\right)k_0^2,$$

$$\hat{q}=\frac{1}{1+v_{\rm ph}^2/c^2},\quad q_1=\frac{1-(v_{\rm ph}/u)(1+v_{\rm ph}/u)}{1+v_{\rm ph}v_{\rm ph}^2/uc^2},$$

$$q_2=\frac{1+(v_{\rm ph}/u)(1-v_{\rm ph}/u)}{1+v_{\rm ph}v_{\rm ph}^2/uc^2},$$

$$\hat{\mu}=-\frac{v_{\rm ph}^2/u^2}{1+v_{\rm ph}^2/c^2},\quad \mu_1=\frac{v_{\rm ph}^3/u^3}{1-v_{\rm ph}v_{\rm ph}^2/uc^2},$$

$$\mu_2=-\frac{v_{\rm ph}^3/u^3}{1+v_{\rm ph}v_{\rm ph}^2/uu^2},$$

$$\hat{\nu}=-\frac{1}{8}\left(\frac{\omega_b}{\gamma_0^{1/2}k_0v_{\rm ph}}\right)^3\frac{1+v_{\rm ph}^2/c^2}{(1-u^2/v_{\rm ph}^2)^{3/2}},$$

$$\nu_1=\frac{1}{8}\left(\frac{\omega_b}{\gamma_0^{1/2}k_0v_{\rm ph}}\right)^3\frac{u}{v_{\rm ph}}\frac{1-v_{\rm ph}v_{\rm ph}^2/uc^2}{(1-u^2/v_{\rm ph}^2)^2},$$

$$\nu_2=-\frac{1}{8}\left(\frac{\omega_b}{\gamma_0^{1/2}k_0v_{\rm ph}}\right)^3\frac{u}{v_{\rm ph}}\frac{1+v_{\rm ph}v_{\rm ph}^2/uc^2}{(1-u^2/v_{\rm ph})^2},$$

$$\nu_0=\frac{1}{2}\frac{\omega_b\gamma_0^{3/2}}{(1-u^2/v_{\rm ph}^2)^{1/2}k_0v_{\rm ph}}.$$

Using Eqs. (33), we estimate in Eqs. (31) the terms of type $\hat{\mu}\eta$, $\hat{q}\eta$, and those similar to them. Thus, for example

$$\frac{\mu_0}{2\gamma_0^2}\hat{\mu}\eta=-\frac{v_{\rm ph}^2}{c^2}\frac{1}{1+v_{\rm ph}^2/u^2}\frac{v'}{u}\sim\frac{v'}{u}\ll 1,$$
$$\hat{\hat{\rho}}_2-\frac{\mu_0}{2\gamma_0^2}\beta_1\hat{\mu}R_2\sim\hat{\hat{\rho}}_2. \tag{34}$$

Equations (33) are thus substantially simplified and take the form

$$\frac{d\varepsilon_1}{d\tau}=-i\nu_0\varepsilon_2\hat{\rho}_1\exp(i\eta_0\tau)-i\nu_0(\hat{\nu}\varepsilon_1^*\varepsilon_2^2\hat{\hat{\rho}}_2\exp(2i\eta_0\tau)$$
$$+2\nu_1|\varepsilon_1|^2\varepsilon_2\hat{\hat{\rho}}_1\exp(i\eta_0\tau)+\nu_1\varepsilon_1^2\varepsilon_2^*\hat{\hat{\rho}}_1^*$$
$$\times\exp(-i\eta_0\tau)+\nu_2|\varepsilon_1|^2\varepsilon_2\hat{\hat{\rho}}_1\exp(i\eta_0\tau)),$$

$$\frac{d\varepsilon_2}{d\tau}=-i\nu_0\varepsilon_1\hat{\rho}_1^*\exp(-i\eta_0\tau)-i\nu_0(\hat{\nu}\varepsilon_2^*\varepsilon_1^2\hat{\hat{\rho}}_2^*$$
$$\times\exp(-2i\eta_0\tau)+2\nu_2|\varepsilon_2|^2\varepsilon_1\hat{\hat{\rho}}_1^*$$
$$\times\exp(-i\eta_0\tau)+\nu_2\varepsilon_2^2\varepsilon_1^*\hat{\hat{\rho}}_1\exp(-i\eta_0\tau)),$$

$$\frac{d\gamma}{d\tau}=-\frac{i}{4}\mu_0\gamma_0(\rho_1\exp(iy)-\text{c.c.})-\frac{i}{4}\frac{\mu_0\nu_0}{\gamma_0^2\gamma}(\varepsilon_1\varepsilon_2^*\exp(iy$$
$$-i\eta_0\tau)-\text{c.c.})-\frac{i}{4}\frac{\mu_0\nu_0}{\gamma_0^2\gamma^3}(\hat{\nu}\varepsilon_1^2\varepsilon_2^{*2}\exp(2iy$$
$$-2i\eta_0\tau)+\nu_1\varepsilon_1|\varepsilon_1|^2\varepsilon_2^*\exp(iy-i\eta_0\tau)$$
$$+\nu_2\varepsilon_1|\varepsilon_2|^2\varepsilon_2^*\exp(iy-i\eta_0\tau)-\text{c.c.}),$$

$$\frac{dy}{d\tau}=\eta, \tag{35}$$

$$\frac{d\eta}{d\tau}=-\frac{i}{2}(1-\mu_0\eta)^{3/2}(\rho_1\exp(iy)-\text{c.c.})-\frac{i}{2}\nu_0\frac{1}{\gamma_0^2\gamma}\bigg(1$$
$$-\frac{1}{2}\mu_0\eta\bigg)(\varepsilon_1\varepsilon_2^*\exp(iy-i\eta_0\tau)-\text{c.c.})$$
$$-\frac{i}{2}\frac{\nu_0}{\gamma_0^2\gamma^4}\left(1-\frac{1}{2}\mu_0\eta\right)(\hat{\nu}\varepsilon_1^2\varepsilon_2^{*2}\exp(2iy-2i\eta_0\tau)$$
$$+\nu_1\varepsilon_1|\varepsilon_1|^2\varepsilon_2^*\exp(iy-i\eta_0\tau)$$
$$+\nu_2\varepsilon_1|\varepsilon_2|^2\varepsilon_2^*\exp(iy-i\eta_0\tau)-\text{c.c.}),$$

$$\rho_n=\frac{1}{\pi}\int_0^{2\pi}\exp(-iny)dy_0,$$

$$\hat{\rho}_n=\frac{1}{\pi}\int_0^{2\pi}\gamma^{-1}\exp(-iny)dy_0,$$

$$\hat{\hat{\rho}}_n=\frac{1}{\pi}\int_0^{2\pi}\gamma^{-3}\exp(-iny)dy_0,\quad n=1,2.$$

Equations (35) lead to relations of the Manley–Rowe type:

$$\gamma_0-\frac{1}{2\pi}\int_0^{2\pi}\gamma dy_0=\frac{\mu_0}{8\gamma_0^2}(|\varepsilon_1|^2-|\varepsilon_{10}|^2).$$
$$\gamma_0-\frac{1}{2\pi}\int_0^{2\pi}\gamma dy_0=-\frac{\mu_0}{8\gamma_0^2}(|\varepsilon_2|^2-|\varepsilon_{20}|^2), \tag{36}$$

where $\varepsilon_{10}=\varepsilon_1|_{\tau=0}$, $\varepsilon_{20}=\varepsilon_2|_{\tau=0}$, $\gamma_0=\gamma|_{\tau=0}$; using these relations one can define the effectiveness of the process:

$$k=1-\frac{1}{2\pi}\int_0^{2\pi}\frac{\gamma}{\gamma_0}dy_0=\frac{\mu_0}{8\gamma_0^3}(|\varepsilon_1|^2-|\varepsilon_{10}|^2)$$
$$=-\frac{\mu_0}{8\gamma_0^3(|\varepsilon_2|^2-|\varepsilon_{20}|^2)}. \tag{37}$$

Recall that the terms proportional to the squared amplitudes of the interacting waves in the system (35) are of second order in the parameter $v_\perp$, whereas the terms proportional to $(\varepsilon_1\varepsilon_2)^2$ are respectively of fourth order. We estimate them with the aid of Eqs. (33) in greater detail using the equation for γ as an example. As a result we have

$$\frac{1}{\gamma^2}\hat{\nu}|\varepsilon_1\varepsilon_2^*|\approx\frac{1}{8}\frac{e^2}{m^2c^4}\frac{(c-u)(c+u)}{(v_{ph}-u)(v_{ph}+u)}|A_1A_2^*| \sim\frac{c-u}{v_{ph}-u}\frac{\tilde{v}_\perp^2}{c^2},$$

$$\frac{1}{\gamma^2}\nu_1|\varepsilon_1|^2\approx\frac{1}{8}\frac{e^2}{m^2c^4}\frac{(c-u)(c+u)}{(v_{ph}-u)(v_{ph}+u)}\times\left(u-v_{ph}\frac{v_{ph}^2}{c^2}\right)|A_1|^2 \sim\frac{(c-u)(u-v_{ph}v_{ph}^2/c^2)}{(v_{ph}-u)^2}\frac{v_\perp^2}{c^2}, \tag{38}$$

$$\frac{1}{\gamma^2}\nu_2|\varepsilon_2|^2\approx\frac{1}{8}\frac{e^2}{m^2c^4}\frac{(c-u)(c+u)}{(u-v_{ph})(u+v_{ph})}\times\left(u+v_{ph}\frac{v_{ph}^2}{c^2}\right)|A_2|^2 \sim\frac{c-u}{u-v_{ph}}\frac{\tilde{v}_\perp^2}{c^2},$$

where $\tilde{v}_\perp\approx eA/mc$. It follows hence in the nonrelativistic limit, when $\mu_0\ll1$, $\gamma_0\sim1$, and $u\ll v_{ph}\lesssim c$, all the expressions in (38) are of the same order and are small compared with unity. On the contrary, when $\mu_0\gg1$ and $\gamma_0\gg1$ the situation is determined by the relations between u, v_{ph}, and c. If $v_{ph}=c>u$ (scattering in vacuum), the terms proportional to $\hat{\nu}$, ν_1, and ν_2 in (35) are small and can be disregarded. In the case of the inequality

$$u\approx v_{ph}\lesssim c; \tag{39}$$

however, it follows from (33) that the terms proportional to $\hat{\nu}$, ν_1, and ν_2 can predominate. In other words, processes of higher order than $(v_\perp/c^2)$ will have a substantially larger growth rate.

4. LINEAR THEORY

We assume further that $\omega_1>\omega_2$ and accordingly the wave with amplitude ε_1 is the signal and that with amplitude ε_2 the pump. In this case we have scattering with rise of frequency, with $\eta_0=-1$ in the case of collective scattering (resonance with a slow beam wave).

In the linear approximation, when $|\varepsilon_2|=|\varepsilon_{20}|=\text{const}$, and

$$\varepsilon_1\sim\exp(-i\delta\tau+i\eta_0\tau),\quad \hat{\rho}_1,\hat{\hat{\rho}}_1 \sim\exp(-i\delta\tau),\quad \hat{\hat{\rho}}_2\sim\exp(-2i\delta\tau), \tag{40}$$

where δ is a growth rate normalized to the frequency ω_2, we obtain from (35) the dispersion equation

$$(\delta-\eta_0)(\delta^2-1)=\nu_0'^2(1+\nu_2'^2)[1+\nu_2'^2-\tfrac{1}{2}\mu_0(1+3\nu_2'^2)\delta], \tag{41}$$

where

$$\nu_0'=\nu_0^2\gamma_0^{-5}|\varepsilon_{20}|^2,\quad \nu_2'^2=\nu_2\gamma_0^{-2}|\varepsilon_{20}|^2.$$

We consider first the nonrelativistic limit, when the parameter $\mu_0\ll1$. In the case of high-density beams, when

$$\nu_0'(1+\nu_2'^2)\ll1 \tag{42}$$

there is realized a regime of collective scattering by the beam-density oscillations[11] and the instability growth rate is determined by the expression ($\text{Im}\,\delta\ll\eta_0=-1$)

$$\delta=-1+i\frac{\nu_0'}{\sqrt{2}}\sqrt{(1+\nu_0'^2)\left[1+\nu_2'^2+\frac{1}{2}\mu_0(1+3\nu_2'^2)\right]}. \tag{43}$$

The parameters μ_0 and ν_1' are regarded in (43) as small parameters.

For low-density beams, when $\nu_0'\gg1$, the scattering becomes a single-particle process[11] ($\text{Im}\,\delta\gg\eta_0\sim0$). If the inequality

$$1\ll\nu_0'(1+\nu_2'^2)^{-1/2}\ll\left(\frac{\mu_0}{2}\right)^{-3/2} \tag{44}$$

holds, the usual Thomson scattering of waves is realized, with a growth rate

$$\delta=\frac{-1+i\sqrt{3}}{2}\nu_0'^{2/3}(1+\nu_2'^2)^{2/3}. \tag{45}$$

However, in the case of the inequality

$$\nu_0'(1+3\nu_2'^2)^{3/2}(1+\nu_2'^2)^{-1/2}\gg\left(\frac{\mu_0}{2}\right)^{-3/2} \tag{46}$$

the stimulated scattering is due to energy bunching[8,10] and has a growth rate

$$\delta=i[\tfrac{1}{2}\mu_0(1+3\nu_2'^2)(1+\nu_2'^2)]^{1/2}\nu_0'. \tag{47}$$

The quantity $\nu_2'^2$ in (45) and (47) is a small correction.

Consider now strongly relativistic beams, when $\mu_0\gg1$. For high-density beams, when the inequality

$$\frac{1}{2}\nu_0'\sqrt{(1+\nu_2'^2)(1+3\nu_2'^2)}\ll\mu_0^{-1/2} \tag{48}$$

is satisfied, the collective-scattering regime sets in again, but with another growth rate

$$\delta=-1+i\nu_0'\frac{1}{2}\sqrt{(1+\nu_2'^2)(1+3\nu_2'^2)\mu_0}. \tag{49}$$

If, however, an inequality opposite to (48) obtains, the electromagnetic-wave scattering is due to energy bunching with a growth rate (47).

The equations obtained above for the scattering growth rate make it easy to single out more particular cases, in which the scattering is due to the second order in the parameter $(v_\perp/c)$ $[\nu_2'\ll1]$ or to the fourth $[\nu_2'\gg1]$.

The main results above are listed in Table I.

TABLE I.

Conditions on ν_0' and μ_0	Instability growth rate	Type of process	Stabilization mechanism
$\mu_0 \ll 1,\ \dfrac{\nu_0'}{\sqrt{1+\nu_2'^2}} \ll 1$	$-1+i\dfrac{\nu_0'}{\sqrt{2}}\sqrt{(1+\nu_2'^2)(1+\nu_2'^2+\frac{1}{2}\mu_0(1+3\nu_2'^2))}$	Collective or Raman scattering	Trapping of Langmuir beam wave at $\nu_0' \lesssim 1$; nonlinear frequency shift at $\nu_0' \ll 1$.
$\mu_0 \ll 1,$ $1 \ll \nu_0'(1+\nu_2'^2)^{-1/2} \ll \left(\dfrac{\mu_0}{2}\right)^{-3/2}$	$\frac{1}{2}(-1+i\sqrt{3})\nu_0'^{2/3}(1+\nu_2'^2)^{2/3}$	Single-particle or Thomson scattering	Trapping by combination wave
$\mu_0 \ll 1,$ $\left(\dfrac{\mu_0}{2}\right)^{-3/2} \ll \nu_0'(1+3\nu_2'^2)^{3/2}(1+\nu_2'^2)^{-1/2}$	$i(\frac{1}{2}\mu_0(1+3\nu_2'^2)(1+\nu_2'^2))^{1/2}\nu_0'$	Energy phasing	Beam total momentum modulation
$\mu_0 \gg 1,$ $\frac{1}{2}\nu_0'\sqrt{(1+\nu_2'^2)(1+3\nu_2'^2)} \ll \mu_0^{-1/2}$	$-1+i\dfrac{\nu_0'}{2}\sqrt{(1+\nu_2'^2)(1+3\nu_2'^2)\mu_0}$	Collective or Raman scattering	Trapping of Langmuir beam wave at $\mu_0^{-1/2}\nu_0' \lesssim 1$; nonlinear frequency shift at $\mu_0^{-1/2}\nu_0' \ll 1$.
$\mu_0 \gg 1,$ $\frac{1}{2}\nu_0'\sqrt{(1+\nu_2'^2)(1+3\nu_2'^2)} \gg \mu_0^{-1/2}$	$i(\frac{1}{2}\mu_0(1+3\nu_2'^2)(1+\nu_2'^2))^{1/2}\nu_0'$	Energy phasing	Beam total momentum modulation

5. NONLINEAR STABILIZATION MECHANISM

We consider the mechanisms of nonlinear stabilization of stimulated-scattering processes in accord with the classification given above. For nonrelativistic beams, when the inequality (42) is satisfied is not strong, the scattering is stabilized as a result of capture of electrons by a Langmuir beam wave, breaking of this wave, and turbulization of the beam.[7,10] The beam is in this case fully modulated in density ($|\rho_1|_{\max}\approx 1$) and no analytic solutions can be obtained. If, however, the inequality (42) is satisfied [very dense beams], the stabilization is due to a nonlinear frequency shift.[12] The nonlinear frequency shift in nonrelativistic beams is determined mainly by the deceleration of the electron beam.[7] The beam modulation in density is in this case weak ($|\rho_1|_{\max}\ll 1$) and the deceleration effects are described by nonlinearities of cubic type, so that an analytic solution of the problem can be obtained.

If inequality (44) is valid, single-particle Thomson scattering processes are stabilized, as is well known, by trapping beam electrons in the electromagnetic wave.[3] Under these conditions the beam is fully modulated in density, so that no analytic solutions can be obtained.

If inequality (46) and an inequality inverse to (48) are satisfied, the beam is modulated mainly in energy [or momentum]. This case, least known in the literature,[8,9] is illustrated in Fig. 1, which shows the results of a numerical simulation of the system (35) with $\mu_0=0.8$, $\nu_0' = 1$, ($\hat{\nu}=\nu_1=\nu_2=0$). It can be seen that if the signal-wave amplitude $|\varepsilon_1|$ and the first harmonic of the charge density $|\rho_1|$ grow smoothly enough, the quantity $|\hat{\rho}_1|$ indicative of the beam modulation in energy increases radically. For the values of μ_0 and ν_0' chosen here, the aforementioned inequalities are not yet strong and therefore $|\rho_1|_{\max}\sim 1$. However, with increase of, say, μ_0 the beam modulation in density becomes immaterial ($|\rho_1|_{\max}\ll 1$) and the scattering is stabilized only by total bunching of the beam in energy. We shall show below that in this case analytic solutions are possible.

Finally, if the strong inequality (48) is satisfied and the beam is relativistic, the collective scattering is stabilized as before by a nonlinear frequency shift. Here, however, the principal role is played no longer by the beam deceleration, but by the relativistic dependence of the frequency of its Langmuir oscillations on the amplitude,[8] which is again mathematically described by cubic nonlinearities. Analytic solutions can be obtained in this case, too.

6. EXPANSION IN TERMS OF THE COORDINATES AND MOMENTA FOR THE COLLECTIVE-SCATTERING REGIME

We introduce the electron momentum

$$p=\frac{k_0 u}{\omega_b}\gamma_0^{3/2}\gamma+\eta\gamma, \quad p=\frac{k_0\gamma_0^{3/2}}{m\omega_b}\tilde{p}, \tag{50}$$

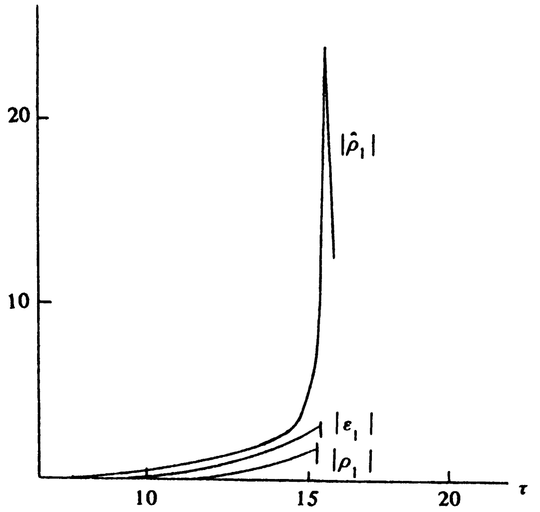

FIG. 1.

where $\tilde{p}$ is the dimensional momentum and rewrite the equations for the coordinate y and for the velocity η from the system (35) in the form

$$\frac{d^2y}{d\tau^2}=\frac{1}{\gamma_0^2}\frac{1}{\gamma}\frac{dp}{d\tau}, \tag{51}$$

$$\frac{dp}{d\tau}=-\frac{i}{2}\gamma_0^3(\rho_1\exp(iy)-\text{c.c.})-\frac{i}{2}\frac{\nu_0}{\gamma}(\varepsilon_1\varepsilon_2^*\exp(iy-i\eta_0\tau)-\text{c.c.})-\frac{i}{2}\frac{\nu_0}{\gamma^3}(\hat{\nu}\varepsilon_1^2\varepsilon_2^{*2}\exp(2iy-2i\eta_0\tau)+\nu_1\varepsilon_1|\varepsilon_1|^2\varepsilon_2^*\exp(iy-i\eta_0\tau)+\nu_2\varepsilon_1|\varepsilon_2|^2\varepsilon_2^*\exp(iy-i\eta_0\tau)-\text{c.c.}).$$

It was taken into account in the equation for the momentum that $\eta\ll 1$. This makes it possible to write the approximate relation

$$p\approx\frac{2\gamma_0^2}{\mu_0}\gamma, \tag{52}$$

which will be used below.

We represent next the coordinate and the momentum (or the relativistic factor) of the electron in the form

$$y=y_0+W(\tau)+\tfrac{1}{2}[a_1(\tau)\exp(iy_0)+a_2(\tau)\times\exp(2iy_0)+\text{c.c.}], \tag{53}$$

$$p=\langle p\rangle(\tau)+\tfrac{1}{2}[b_1(\tau)\exp(iy_0)+b_2(\tau)\exp(2iy_0)+\text{c.c.}].$$

Here W is the average displacement of the electron beam, $\langle p\rangle$ is the average momentum of the electron, while $a_{1,2}$ and $b_{1,2}$ are the averaged coordinate and momentum oscillation amplitudes in the combined-wave field.

Substituting (53) in (35) and (51) and expanding in powers of the wave amplitudes up to cubic linearities inclusive, we obtain the systems of equations

$$\frac{d\varepsilon_1}{d\tau}=-\nu_0\Bigg\{\frac{1}{\gamma_0}\varepsilon_2 f_2\exp(-iW+i\eta_0\tau)+\frac{i}{\gamma_0^3}\Bigg[2\hat{\nu}\varepsilon_1^*\varepsilon_2^2\exp(-2iW+2i\eta_0\tau)\times\left(a_2-\frac{i}{2}a_1^2+\frac{3\mu_0}{4\gamma_0^3}\left(-b_2+\frac{\mu_0}{2\gamma_0^3}b_1^2+ib_1a_1\right)\right)+\nu_1(2|\varepsilon_1|^2\varepsilon_2 f_2\exp(-iW+i\eta_0\tau)-\varepsilon_1^2\varepsilon_2^* f_2^*\exp(iW-i\eta_0\tau)+\nu_2\varepsilon_2|\varepsilon_2|^2 f_2\times\exp(-iW+i\eta_0\tau)\Bigg]\Bigg\},$$

$$\frac{d\varepsilon_2}{d\tau}=\nu_0\Bigg\{\frac{1}{\gamma_0}\varepsilon_1 f_1^*\exp(iW-i\eta_0\tau)+\frac{1}{\gamma_0^3}\Bigg[2\hat{\nu}\varepsilon_2^*\varepsilon_1^2\exp(2iW-2i\eta_0\tau)\left(a_2^*+\frac{i}{2}a_1^2+\frac{3\mu_0}{4\gamma_0^3}\left(-b_2^*+\frac{\mu_0}{2\gamma_0^3}b_1^{*2}-ib_1a_1\right)\right)+\nu_2(2|\varepsilon_2|^2\varepsilon_1 f_2^*\exp(iW-i\eta_0\tau)-\varepsilon_2^2\varepsilon_1^* f_2\exp(-iW+i\eta_0\tau)+\nu_1\varepsilon_1|\varepsilon_1|^2 f_2^*\exp(iW-i\eta_0\tau)\Bigg]\Bigg\},$$

$$\frac{d^2a_1}{d\tau^2}+a_1=\frac{1}{2}|a_1|^2a_1+ia_1^*a_2+\left(-\frac{\mu_0}{2\gamma_0^3}(|\varepsilon_1|^2-|\varepsilon_{10}|^2)+\frac{i\mu_0}{2\gamma_0^2}(a_1b_1^*+b_1a_1^*)\right)a_1-\left(-\frac{\mu_0}{\gamma_0^2}b_2+\frac{3\mu_0^2}{8\gamma_0^6}b_1^2-a_1b_1\right)a_1^*-\frac{i\nu_0}{2\gamma_0^4}\varepsilon_1\varepsilon_2^*\times\exp(iW-i\eta_0\tau)\Bigg[2-\frac{3\mu_0}{4\gamma_0^3}(a_1b_1^*+b_1a_1^*)+\frac{3}{4}\frac{\mu_0}{\gamma_0^3}(|\varepsilon_1|^2-|\varepsilon_{10}|^2)-\frac{1}{4}|a_1|^2+\frac{\hat{\nu}}{\gamma_0^2}\varepsilon_1\varepsilon_2^*\exp(iW-i\eta_0\tau)\left(ia_1^*-\frac{5\mu_0}{2\gamma_0^3}b_1^*\right)+\frac{1}{\gamma_0^2}(\nu_1|\varepsilon_1|^2+\nu_2|\varepsilon_2|^2)\left(2-\frac{1}{4}|a_1|^2-\frac{5\mu_0}{4\gamma_0^3}(a_1b_1^*+a_1^*b_1)+\frac{5}{4}\frac{\mu_0}{\gamma_0^3}(|\varepsilon_1|^2-|\varepsilon_{10}|^2)\right)\Bigg]-\frac{i}{2}\frac{\nu_0}{\gamma_0^4}\varepsilon_1^*\varepsilon_2\exp(-iW+i\eta_0\tau)\Bigg[\frac{1}{\gamma_0}\left(-\frac{3\mu_0}{2\gamma_0^3}b_2+\frac{3\mu_0^2}{4\gamma_0^5}b_1^2+\frac{3\mu_0}{2\gamma_0^3}ia_1b_1\right)+(\nu_1|\varepsilon_1|^2+\nu_2|\varepsilon_2|^2)\left(1+\frac{5\mu_0}{2\gamma_0^3}\left(b_2-\frac{3\mu_0}{4\gamma_0^3}b_2-i\frac{i}{2}a_1b_1\right)\right)+(1+\nu_1|\varepsilon_1|^2+\nu_2|\varepsilon_2|^2)\left(ia_2+\frac{1}{4}a_1^2\right)\Bigg], \tag{54}$$

$$\frac{d^2a_2}{d\tau^2}=-\frac{i}{2}a_1^2+\frac{\mu_0}{2\gamma_0^3}b_1a_1+\frac{1}{2}\frac{\nu_0}{\gamma_0^4}\varepsilon_1\varepsilon_2{}^*\exp(iW-i\eta_0\tau)\times\Bigg[a_1(1+\nu_1|\varepsilon_1|^2+\nu_2|\varepsilon_2|^2)+i\frac{5\mu_0}{2\gamma_0^3}b_1\left(\frac{\nu_1}{\gamma_0}|\varepsilon_1|^2+\frac{\nu_2}{\gamma_0}|\varepsilon_2|^2+\frac{6}{5}\right)-2i\hat{\nu}\varepsilon_1\varepsilon_2^*\exp(iW-i\eta_0\tau)\Bigg],$$

$$\begin{aligned}\frac{d^2W}{d\tau^2}=\frac{\nu_0}{4\gamma_0^4}\Bigg\{&\varepsilon_1\varepsilon_2\Bigg[\left(1+\frac{\nu_1}{\gamma_0}|\varepsilon_1|^2+\frac{\nu_2}{\gamma_0}|\varepsilon_2|^2\right)\Bigg(\Bigg(1\\&-\frac{1}{8}|a_1|^2\Bigg)a_1-\frac{i}{2}a_1^*a_2\Bigg)+2\frac{\hat{\nu}}{\gamma_0}\varepsilon_1\varepsilon_2^*\Bigg(a_2^*\\&+\frac{i}{2}a_1^{*2}\Bigg)\exp(iW-i\eta_0\tau)\Bigg]\exp(iW-i\eta_0\tau)\\&+\frac{3}{2}i\frac{\mu_0}{\gamma_0^3}b_1^*\varepsilon_1\varepsilon_2^*\exp(iW-i\eta_0\tau)\\&+i\frac{5}{2}\frac{\mu_0}{\gamma_0^4}\hat{\nu}b_2^*\varepsilon_1^2\varepsilon_2^{*2}\exp(2iW-2i\eta_0\tau)\\&+i\frac{5}{2}\frac{\mu_0}{\gamma_0^3}b_1^*\varepsilon_1\varepsilon_2^*\exp(iW-i\eta_0\tau)(\nu_1|\varepsilon_1|^2\\&+\nu_2|\varepsilon_2|^2)+\text{c.c.}\Bigg\},\end{aligned}$$

$$\begin{aligned}\frac{db_1}{d\tau}&=-\gamma_0^3\left(a_1-\frac{5}{8}|a_1|^2a_1-\frac{3}{2}ia_1^*a_2\right)-i\frac{\nu_0}{\gamma_0}\varepsilon_1\varepsilon_2^*\\&\quad\times\exp(-i\eta_0\tau+iW),\\\frac{db_2}{d\tau}&=\frac{i}{2}\gamma_0^3a_1^2,\\\langle p\rangle&=p_0-\frac{1}{4}(|\varepsilon_1|^2-|\varepsilon_{10}|^2),\quad p_0=\frac{2\gamma_0^3}{\mu_0}.\end{aligned}$$

Here

$$f_1=\left(1-\frac{1}{8}|a_1|^2\right)a_1-\frac{i}{2}a_1^*a_2-i\frac{\mu_0}{2\gamma_0^3}b_1,\tag{55}$$

$$f_2=f_1-i\frac{\mu_0}{\gamma_0^3}b_1.$$

The system (54) is quite complicated and requires simplification. We consider therefore only limiting cases in accordance with the linear classification.

Consider the case of weakly relativistic beams, when $\mu_0\ll 1$. In this case $u\gg v_{\text{ph}}\sim c$, and $\hat{\nu}|\varepsilon_1\varepsilon_2^*|\sim\nu_1|\varepsilon_1|^2\sim\nu_2|\varepsilon_2|^2\ll 1$. Assuming further that

$$a_1=a_1'(\tau)\exp[i(\tau+W)],\quad a_2=a_2'(\tau)\exp[2i(\tau+W)],\tag{56}$$

$$b_1=b_1'(\tau)\exp[i(\tau+W)],\quad b_2=b_2'(\tau)\exp[2i(\tau+W)],$$

where a_1', a_2', b_1', and b_2' are slowly varying functions of the time (the apostrophes will henceforth be omitted), and the detuning η_0 is set equal to -1, the system (54) reduces to

$$\begin{aligned}\frac{d\varepsilon_1}{d\tau}&=-\frac{i}{8\gamma_0^3}(|\varepsilon_1|^2-|\varepsilon_{10}|^2)\varepsilon_1-\frac{\nu_0}{\gamma_0}\varepsilon_2a_1,\\\frac{d\varepsilon_2}{d\tau}&=-\frac{i}{8\gamma_0^3}(|\varepsilon_2|^2-|\varepsilon_{20}|^2)\varepsilon_2+\frac{\nu_0}{\gamma_0}\varepsilon_1a_1^*,\\\frac{da_1}{d\tau}&=-\frac{i}{16}\left(3+4\mu_0+\frac{3}{2}\mu_0^3\right)|a_1|^2a_1-i\frac{\mu_0}{8\gamma_0^3}\\&\quad\times(|\varepsilon_1|^2-|\varepsilon_{10}|^2)a_1-\frac{\nu_0}{2\gamma_0^4}\varepsilon_1\varepsilon_2^*.\end{aligned}\tag{57}$$

The standard solution of the system (57) is expressed in terms of elliptic functions, but the equations are quite unwieldy. We shall dwell therefore on the case of adiabatic application of the field when for $\tau=0$ we have $|\varepsilon_{10}|=0$ and $|\varepsilon_{20}|\neq 0$; the solution is then

$$|a_1|^2=\frac{|a_1|^2_{\max}}{\text{ch}(\sqrt{2}\,\nu_0|\varepsilon_{20}|\tau/\gamma_0^{5/2})},\tag{58}$$

$$|\varepsilon_1|^2=2\gamma_0^2|a_1|^2,\quad|\varepsilon_2|^2=|\varepsilon_{20}|^2-2\gamma_0^2|a_1|^2,$$

where

$$\begin{aligned}|a_1|^2_{\max}&=32\sqrt{2}\nu_0|\varepsilon_{20}|\gamma_0^{-5/2},\\|\varepsilon_1|^2_{\max}&=64\sqrt{2}\nu_0|\varepsilon_{20}|\gamma_0^{1/2},\\|\varepsilon_2|^2_{\min}&=|\varepsilon_{20}|^2\left(1-64\sqrt{2}\frac{\nu_0\gamma_0^{1/2}}{|\varepsilon_{20}|}\right).\end{aligned}\tag{59}$$

The maximum effectiveness of the process, defined above by Eq. (37), is given by

$$K_{\max}=8\sqrt{2}\mu_0\nu_0|\varepsilon_{20}|\gamma_0^{-5/2},\tag{60}$$

i.e., $K_{\max}\sim J_b^{1/4}$, where J_b is the beam current.

When $\mu_0\gg 1$ and $u<v_{\text{ph}}\sim c$, we have, as before, $\hat{\nu}|\varepsilon_1\varepsilon_2^*|\sim\nu_1|\varepsilon_1|^2\sim\nu_2|\varepsilon_2|^2\ll 1$. The system (54), with account taken of (56), reduces under these conditions to

$$\begin{aligned}\frac{d\varepsilon_1}{d\tau}+\frac{3}{8}i\frac{1}{\gamma_0^3}(|\varepsilon_1|^2-|\varepsilon_{10}|^2)\varepsilon_1&=-\frac{\nu_0\mu_0}{2\gamma_0}\varepsilon_2a_1,\\\frac{d\varepsilon_2}{d\tau}+\frac{3}{8}i\frac{1}{\gamma_0^3}(|\varepsilon_2|^2-|\varepsilon_{20}|^2)\varepsilon_2&=\frac{\nu_0\mu_0}{2\gamma_0}\varepsilon_1a_1^*,\\\frac{da_1}{d\tau}+\frac{3}{16}i\mu_0^2|a_1|^2a_1&=-\frac{\nu_0}{2\gamma_0^4}\varepsilon_1\varepsilon_2^*,\end{aligned}\tag{61}$$

and has for an adiabatically turned-on field the solutions

$$\begin{aligned}|a_1|^2&=\frac{|a_1|^2_{\max}}{\text{ch}(\nu_0|\varepsilon_{20}|\mu_0^{1/2}\gamma_0^{-5/2}\tau)},\\|\varepsilon_1|^2&=\mu_0\gamma_0^3|a_1|^2,\quad|\varepsilon_2|^2=|\varepsilon_{20}|^2-\mu_0\gamma_0^3|a_1|^2,\end{aligned}\tag{62}$$

where

$$\begin{aligned}|a_1|^2_{\max}&=\frac{32}{3}\frac{\nu_0|\varepsilon_{20}|}{\mu_0^{3/2}\gamma_0^{5/2}},\\|\varepsilon_1|^2_{\max}&=\frac{32}{3}\frac{\nu_0|\varepsilon_{20}|\gamma_0^{1/2}}{\mu_0^{1/2}},\\|\varepsilon_2|^2_{\min}&=|\varepsilon_{20}|^2\left(1-\frac{32}{3}\frac{\nu_0\gamma_0^{1/2}}{|\varepsilon_{20}|\mu_0^{1/2}}\right).\end{aligned}\tag{63}$$

The maximum scattering efficiency is given then by

$$K_{\max}\approx\tfrac{4}{3}\mu_0^{1/2}\gamma_0^{-5/2}\nu_0|\varepsilon_{20}|,\tag{64}$$

and $K_{\max}$ is independent of the type of beam. The latter is obvious if one returns to the dimensionless variables (36).

We consider now the case when the principal role is played at $\mu_0 \gg 1$ by processes of fourth order in the parameter $v_\perp/c$. Here $u \approx v_{\rm ph} \lesssim c$ and $\nu_2|\varepsilon_2|^2 \gg \nu_1|\varepsilon_1|^2$, $\hat{\nu}|\varepsilon_1\varepsilon_2^*|$ and furthermore $\nu_2|\varepsilon_2|^2\gamma_0^{-2} \gg 1$. The system of equations (54) is transformed then into

$$\frac{d\varepsilon_1}{d\tau} = -\frac{5i}{24\gamma_0^2}(|\varepsilon_1|^2 - |\varepsilon_{10}|^2)\varepsilon_1 - \frac{3\nu_0\mu_0\nu_2}{2\gamma_0^3}|\varepsilon_{20}|^2\varepsilon_2 a_1,$$

$$\frac{d\varepsilon_2}{d\tau} = -\frac{5i}{24\gamma_0^2}(|\varepsilon_2|^2 - |\varepsilon_{20}|^2)\varepsilon_2 + \frac{3\nu_0\mu_0\nu_2}{2\gamma_0^3}|\varepsilon_{20}|^2\varepsilon_1 a_1^*, \tag{65}$$

$$\frac{da_1}{d\tau} = -\frac{3}{16}i\mu_0^2 a_1 - \frac{\nu_0\nu_2}{2\gamma_0^6}\varepsilon_1\varepsilon_2^*|\varepsilon_{20}|^2.$$

The solutions of the system (65) are in structure to those of (62), and the expressions for $|a_1|_{\max}$, $|\varepsilon_1|_{\max}$ and $|\varepsilon_2|_{\min}$ are

$$|a_1|^2_{\max} = \frac{32\sqrt{3}}{3}\frac{\nu_0\nu_2|\varepsilon_{20}|^2}{\mu_0^{3/2}\gamma_0^{9/2}},$$

$$|\varepsilon_1|^2_{\max} = \frac{32\sqrt{3}}{3}\frac{\nu_0\nu_2|\varepsilon_{20}|^2}{\mu_0^{1/2}\gamma_0^{3/2}}, \tag{66}$$

$$|\varepsilon_2|^2_{\min} = |\varepsilon_{20}|^2\left(1 - \frac{32\sqrt{3}}{3}\frac{\nu_0\nu_2}{\mu_0^{1/2}\gamma_0^{3/2}}\right).$$

The maximum effectiveness of the scattering is

$$K_{\max} \approx \frac{4\sqrt{3}}{3}\mu_0^{1/2}\nu_0\nu_2\gamma_0^{-3/2}|\varepsilon_{20}|^2. \tag{67}$$

Note that the main analytic results above are listed in Table II.

If $\varepsilon_{10} \neq 0$ for $\tau = 0$, the solutions of the systems (57), (61), and (65) are expressed, as noted above, in terms of elliptic functions and have for $|\varepsilon_{10}| \ll |\varepsilon_1|_{\max}$ the structure

$$|a_1|^2 = |a_1|^2_{\max}\frac{\operatorname{sn}^2(z,r)}{1 + \dfrac{|\varepsilon_1|^2_{\max}}{|\varepsilon_{10}|^2}\operatorname{cn}^2(z,r)}. \tag{68}$$

Here

$$r = 1 - \frac{|\varepsilon_{10}|^2}{|\varepsilon_1|_{\max}}, \quad z = \operatorname{Im}\delta\tau. \tag{69}$$

$\operatorname{Im}\delta$ is the imaginary part of the growth rate for the corresponding limiting case and the time of nonlinear stabilization of the scattering process is

$$\tau_0 = (\operatorname{Im}\delta)^{-1}\ln\left(2^{3/2}\frac{|\varepsilon_1|_{\max}}{|\varepsilon_{10}|}\right). \tag{70}$$

In the general case the equations for $|a_1|_{\max}$ and $|\varepsilon_1|_{\max}$ in expressions (68)–(70) are very unwieldy. In the strong pumping approximation, however when $|\varepsilon_2| \approx |\varepsilon_{20}| \gg |\varepsilon_{10}|$, the expressions for $|\varepsilon_1|^2_{\max}$ and $|a_1|^2_{\max}$ coincide with those given above for the corresponding limiting cases.

7. EFFECT OF ENERGY BUNCHING

We consider energy bunching in the limit $\mu_0 \gg 1$, when the beam modulation in density is extremely small and it can be assumed that $y \sim y_0$ (and accordingly $\eta_0 \approx 0$ and $f_1 \approx 0$). We assume in addition, to simplify the analytic equations that $|\varepsilon_{10}| \ll |\varepsilon_{20}| \approx |\varepsilon_2|$. This assumption is, of course, not a constraint in principle, and is made only for the sake of clarity. Assuming for simplicity, as above, that the pump wave is fixed and $\nu_2|\varepsilon_{20}|^2 \gg \hat{\nu}|\varepsilon_1\varepsilon_2^*|$, and $\nu_1|\varepsilon_1|^2$, we reduce the system (35) to the form

$$\frac{d\varepsilon_1}{d\tau} = -i\nu_0\varepsilon_2\hat{\rho}_1 - i\nu_0\nu_2|\varepsilon_{20}|^2\varepsilon_{20}\hat{\hat{\rho}}_1,$$

$$\frac{d\gamma}{d\tau} = -\frac{i}{4}\frac{\mu_0\nu_0}{\gamma_0^2\gamma}(\varepsilon_1\varepsilon_{20}^*\exp(iy_0) - \text{c.c.})$$

$$-\frac{i}{4}\frac{\mu_0\nu_0\nu_2}{\gamma_0^2\gamma^3}(|\varepsilon_{20}|^2\varepsilon_1\varepsilon_{20}^*\exp(iy_0) - \text{c.c.}), \tag{71}$$

$$\hat{\rho}_1 = \frac{1}{\pi}\int_0^{2\pi}\gamma^{-1}\exp(-iy_0)dy_0,$$

$$\hat{\hat{\rho}}_1 = \frac{1}{\pi}\int_0^{2\pi}\gamma^{-3}\exp(-iy_0)dy_0.$$

We consider first the case $\nu_2|\varepsilon_{20}|^2 \ll 1$ (a process of second order in the parameter $v_\perp/c$). In this case the second terms in the right-hand sides of the equations for ε_1 and γ can be discarded. We introduce the new variables

$$\varepsilon = -i\varepsilon_1(\nu_0\varepsilon_{20})^{-1/3}, \quad \tau' = (\nu_0\varepsilon_{20})^{2/3}\tau,$$
$$\mu' = \mu_0\frac{(\nu_0\varepsilon_{20})^{2/3}}{\gamma_0^2} \tag{72}$$

and assuming furthermore that $\varepsilon = dS/d\tau'$, where S, without loss of generality, is a real function, we reduce the system (71) to

$$\frac{d^2S}{d\tau'^2} = -\frac{1}{\pi}\int_0^{2\pi}\frac{\cos y_0}{\gamma}dy_0,$$
$$\frac{d\gamma}{d\tau'} = \frac{\mu'}{2\gamma}\frac{dS}{d\tau'}\cos y_0. \tag{73}$$

Integrating the equation for γ and changing variables once more:

$$q = \frac{\mu' S}{\gamma_0^2}, \quad \xi = \sqrt{\frac{\mu'}{2\gamma_0^3}}\tau' \tag{74}$$

we reduce the system (73) to a single equation for q:

$$\frac{d^2q}{d\xi^2} = -\frac{2}{\pi}\int_0^{2\pi}\frac{\cos y_0 dy_0}{\sqrt{1 + q\cos y_0}}. \tag{75}$$

The scattering efficiency is in this case

$$K = \frac{1}{16}\left(\frac{dq}{d\xi}\right)^2. \tag{76}$$

To estimate the maximum efficiency we integrate (75) once more under adiabatic initial conditions:

TABLE II.

	$\nu_2\lvert\varepsilon_{20}\rvert^2 \ll 1,\ (v_\perp/c)^2$	$\nu_2\lvert\varepsilon_{20}\rvert^2 \gg 1,\ (v_\perp/c)^4$
	Collective scattering	
$\mu_0 \ll 1$	$\lvert\varepsilon_1\rvert_{max} = 8(\sqrt{2}\,\nu_0\lvert\varepsilon_{20}\rvert\gamma_0^{1/2})^{1/2}$ $\lvert a_1\rvert_{max} = 4(2\sqrt{2}\,\nu_0\lvert\varepsilon_{20}\rvert\gamma_0^{-5/2})^{1/2}$ $K_{max} = 8\sqrt{2}\,\mu_0\nu_0\lvert\varepsilon_{20}\rvert\gamma_0^{-5/2}$	
$\mu_0 \gg 1$	$\lvert\varepsilon_1\rvert_{max} = 4\left(\frac{2\nu_0\lvert\varepsilon_{20}\rvert\gamma_0^{1/2}}{3\mu_0^{1/2}}\right)^{1/2}$ $\lvert a_1\rvert_{max} = 4\left(\frac{2\nu_0\lvert\varepsilon_{20}\rvert}{3\mu_0^{3/2}\gamma_0^{5/2}}\right)^{1/2}$ $K_{max} = \frac{4}{3}\mu_0^{1/2}\gamma_0^{-5/2}\nu_0\lvert\varepsilon_{20}\rvert$	$\lvert\varepsilon_1\rvert_{max} = 4\left(\frac{2\sqrt{3}\,\nu_0\nu_2\lvert\varepsilon_{20}\rvert^2}{3\mu_0^{1/2}\gamma_0^{3/2}}\right)^{1/2}$ $\lvert a_1\rvert_{max} = 4\left(\frac{2\sqrt{3}\,\nu_0\nu_2\lvert\varepsilon_{20}\rvert^2}{3\mu_0^{3/2}\gamma_0^{9/2}}\right)^{1/2}$ $K_{max} = \frac{4\sqrt{3}}{3}\mu_0^{1/2}\nu_0\nu_2\gamma_0^{-3/2}\lvert\varepsilon_{20}\rvert^2$
	Energy bunching	
$\mu_0 \gg 1$	$K_{max} = 1 - \frac{1}{2\pi}\int_0^{2\pi}(1+\cos y_0)^{1/2}dy_0$	$K_{max} = 1 - \frac{1}{2\pi}\int_0^{2\pi}(1+\cos y_0)^{1/4}dy_0$

$$\left(\frac{dq}{d\xi}\right)^2 = 16\left(1 - \frac{1}{2\pi}\int_0^{2\pi}(1+q\cos y_0)^{1/2}dy_0\right). \tag{77}$$

Since $q_{\max}=1$, we have

$$K_{\max} = 1 - \frac{1}{2\pi}\int_0^{2\pi}(1+\cos y_0)^{1/2}dy_0 = 1 - \frac{2\sqrt{2}}{\pi} \approx 0.1. \tag{78}$$

In the opposite limit, when $\nu_2\lvert\varepsilon_{20}\rvert^2 \gg 1$, we can neglect the first terms in the right-hand sides of the equations for ε_1 and γ of the system (71). Just as above, we introduce the new variables

$$\varepsilon = -i\varepsilon_1(\nu_0\varepsilon_{20}^3\nu_2)^{-1/3}, \quad \tau' = (\nu_0\varepsilon_{20}^3\nu_2)^{2/3}\tau,$$
$$\mu' = \mu_0\gamma_0^{-2}(\nu_0\varepsilon_{20}^2\nu_2)^{2/3} \tag{79}$$

and, after substituting $\varepsilon = dS/d\tau'$, we reduce the system (71) to

$$\frac{d^2S}{d\tau'^2} = -\frac{1}{\pi}\int_0^{2\pi}\frac{\cos y_0}{\gamma^3}dy_0, \tag{80}$$

$$\frac{d\gamma}{d\tau'} = \frac{\mu'}{2}\frac{dS}{d\tau'}\frac{\cos y_0}{\gamma^3}.$$

The solution of (80) is similar to that of (73). We write therefore directly the expression for the maximum scattering efficiency

$$K_{\max} = 1 - \frac{1}{2\pi}\int_0^{2\pi}(1+\cos y_0)^{1/4}dy_0 \approx 0.16. \tag{81}$$

Thus, a scattering process of fourth order in the parameter $v_\perp/c$ is more effectively realized than a second-order process.

The expressions for the scattering efficiencies in the energy-bunching regime are listed in Table II.

8. CONCLUSION

We have thus classified all orders of scattering of linearly polarized waves by a nonmagnetized beam in terms of the current and density of the beam. We have shown that under certain conditions the fourth-order processes begin to predominate over second-order ones in collective-effect and energy-bunching regimes.

[1)] The functions $\exp(ik_1z)$ and $\exp(ik_2z)$ are assumed to be orthogonal on some spatial segment $z\in[0;L]$.

[1] T. Marshall, *Free-Electron Lasers*, Mir, Moscow (1987).
[2] *Free-Electron Coherent Emission Generators* [Russ. transl. anthology ed. by A. A. Rukhadze], Mir, Moscow (1983).
[3] V. L. Bratman, N. S. Ginzburg, and M. I. Petelin, Zh. Eksp. Teor. Fiz. **76**, 930 (1979) [Sov. Phys. JETP **49**, 469 (1979)].
[4] A. G. Litvak, V. I. Petrukhina, and V. Yu. Trakhtengerts, Pis'ma Zh. Eksp. Teor. Fiz. **18**, 190 (1973) [JETP Lett. **18**, 111 (1973)].
[5] V. V. Ognivenko, Radiotekh. Elektron. **27**, No. 9, 1818 (1982).
[6] V. A. Balakirev, Izv. Vyssh. Ucheb. Zaved. Radiofizika **25**, 1198 (1982).
[7] M. V. Kuzelev and V. A. Panin, *ibid.* **27**, 26 (1984).
[8] M. V. Kuzelev, V. A. Panin, A. P. Plotnikov, and A. A. Rukhadze, Zh. Eksp. Teor. Fiz. **95**, No. 9, 1625 (1989) [Sov. Phys. JETP **68**, 939 (1989)].
[9] M. V. Kuzelev and V. A. Panin, Izv. Vyssh. Ucheb. Zaved. Radiofizika **34**, 1191 (1991).
[10] M. V. Kuzelev and A. A. Rukhadze, *Electrodynamics of Dense Electron Beams in a Plasma* [in Russian], Mir, Moscow (1990).
[11] M. V. Kuzelev and A. A. Rukhadze, Usp. Fiz. Nauk **152**, 285 (1987) [Sov. Phys. Usp. **30**, 507 (1987)].
[12] J. Weiland, and H. Wilhelmsson *Coherent Nonlinear Interaction of Waves in Plasmas*, Pergamon (1976).

Translated by J. G. Adashko