Nonadiabatic neutrino oscillations in inhomogeneous media

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We propose a group-theoretical method of analyzing the dynamics of N-neutrino oscillations that makes it possible to calculate analytically the probability of flavor transformation and survival in media with some variable density $\rho(x)$. The evolution of two flavors can be described by a simple equation of motion for the $SU(2)$ group parameter, which then admits of exact solutions for a family of profiles $\rho(x)$ with no further constraints on the neutrino parameters. For the three-flavor case, we use perturbation theory to calculate the nonadiabatic evolution of state vectors near resonance, in which the $SU(2)$ evolution matrix is the unperturbed solution.

1. INTRODUCTION

Neutrino mixing and oscillations[1,2] lead to far-reaching consequences in weak-interaction theory, neutrino astrophysics, and cosmology. The medium influences neutrino oscillations[3] in a manner reminiscent of resonance phenomena.[4] The abrupt increase in the depth of oscillations for certain densities of the medium or neutrino energies E provides a natural explanation for the observed long-term deficit of solar neutrinos,[5] even for the small vacuum mixing angle Θ_v found in nature.

The properties of neutrino oscillations are dictated by the way in which the density of matter varies. On its way from the sun to the earth, a neutrino flux with a broad energy spectrum will pass through a number of stages of oscillation:[6] adiabatic ($d\rho/dx$ small), nonadiabatic (near resonances and density discontinuities), and oscillatory in a medium at constant density. If the neutrino equation of motion in the latter case admits of an exact solution,[3] then various approximate methods[4,6,7] will be available for analyzing oscillations of two flavors even in a variable-density medium; those methods will apply over a variety of neutrino parameter ranges.

In the present paper, we propose a new method for analyzing the dynamics of neutrino oscillations in variable-density media, based on a group-theoretical solution of the evolution equation (Sec. 2). The method is a universal one, in the sense that it provides a unified description of various oscillation regimes for both two and three flavors. In two-flavor systems, we are able to find solutions for a number of functions $\rho(x)$ modeling the density of matter along the neutrino propagation path. The explicit expressions obtained for flavor survival hold for all values of Δm^2, Θ_v, and E, and all oscillation regimes (Sec. 3). A combination of the group-theoretical approach with perturbation theory enables us to describe nonadiabatic oscillations of three flavors near resonance analytically, to any desired accuracy in a particular small parameter (Sec. 4).

2. GENERAL FORMALISM

Ultrarelativistic neutrino motion in a medium of arbitrary composition and with an arbitrary density distribution is governed by the Schrödinger equation,

$$i\frac{d}{dx}\,|\nu(x)\rangle=\mathscr{M}[\rho(x)]\,|\nu(x)\rangle,\quad c=1,\quad \hbar=1. \tag{1}$$

When there are N flavors, $|\nu(x)\rangle$ is an N-component state vector for the neutrino system at a point x, and $\mathscr{M}$ is an $N\times N$ matrix that takes mixing and neutrino interaction with the medium into account. We single out a particular point x_0 and define the coordinate translation matrix via $|\nu(x)\rangle=U(x,x_0)\,|\nu(x_0)\rangle$. We then expand the matrix $\mathscr{M}$ in the basis $M_1,\ldots,M_n$ for an N-dimensional representation of some n-dimensional Lie algebra L,

$$\mathscr{M}=\sum_{j=1}^{n} h_j(x)M_j,$$

where some of the coefficients $h_j(x)$ may vanish. U then belongs to the N-dimensional Lie group matrix representation of the Lie algebra L; it translates a solution of Eq. (1) from the point x_0 to the point x, and it satisfies the equation

$$i\frac{d}{dx}\,U(x,x_0)=\mathscr{M}[\rho(x)]\,U(x,x_0),\quad U(x_0,x_0)=I_N, \tag{2}$$

where I_N is the $N\times N$ unit matrix.

For a finite-dimensional L, the solution of Eq. (2) can be represented as a product of exponentials,[8]

$$U(x,x_0)=\prod_{k=1}^{n}\exp\,(g_k(x)M_k). \tag{3}$$

The g here satisfy a set of n ordinary first-order differential equations

$$h_k(x)=a_{kl}\frac{dg_l}{dx},\quad k,l=1,\ldots,n, \tag{4}$$

where the a_{kl} are analytic functions of the g, and can be found by substituting (3) into (2). The form taken by Eqs. (4) depends on the parametrization of the dynamic symmetry group, and not on the representation of L. There are various ways[9] to parametrize U; among these, a multiplicative approach is the most convenient for calculating transition probabilities. In general, a solution of Eq. (2) in the

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form (3) will hold only locally, but for the $SU(2)$ dynamic symmetry that we consider here, it also holds globally.[8] Having solved Eqs. (4), one can easily find an explicit form for the matrix (3), which then completely determines the quantum-mechanical evolution of the system, and in particular, the evolution of the flavor survival and transformation probability

$$P(\nu_\eta;x|\nu_\xi;x_0) = \left| \left\langle \nu_\eta(x) \right| \prod_{k=1}^{n} \exp\left(g_k(x)M_k\right) \left| \nu_\xi(x_0) \right\rangle \right|^2,$$

where subscripts ξ and η specify the flavors e, μ, τ,... .

3. EXACT SOLUTIONS FOR TWO-NEUTRINO OSCILLATIONS IN INHOMOGENEOUS MEDIA

In the two-flavor case, the matrix $\mathcal{M}$ in the basis of weak-interaction eigenstates is a linear combination of the Pauli spin matrices,

$$\mathcal{M}[\rho(x)] = \tfrac{1}{2} h_0(x)\sigma_0 + h(\sigma_- + \sigma_+),$$

with coefficients[6,7]

$$h = \frac{\pi}{l_v} \sin 2\Theta_v, \quad h_0(x) = \frac{\sqrt{2}YG\rho(x)}{m} - \frac{2\pi}{l_v} \cos 2\Theta_v,$$

where l_v is the oscillation length in vacuum, Y is the number of electrons in the medium per nucleon of mass m, and G is the Fermi constant. The dynamic symmetry group of the 2ν system is $SU(2)$, which we parametrize in the form

$$U = \exp[(g_0 - is)\sigma_0] \exp(g_-\sigma_-) \exp g_+\sigma_+, \tag{5}$$

where $ds/dx \equiv h_0(x)/2$. By virtue of Eq. (5), the equations for the three parameters of $SU(2)$ can lead to a single relatively simple second-order differential equation for $g \equiv \exp g_0$,

$$\frac{d^2g}{dx^2} - ih_0(x)\frac{dg}{dx} + h^2 g = 0, \quad g(x_0) = 1,$$

$$\frac{dg}{dx}(x = x_0) = 0. \tag{6}$$

The functions $g_\mp$ can be expressed in terms of g.[9] In terms of these parameters, the evolution matrix of the two-neutrino system takes the form

$$U = \begin{bmatrix} ge^{-is} & gg_+e^{-is} \\ g_-e^{is}/g & (1+g_-g_+)e^{is}/g \end{bmatrix}. \tag{7}$$

The flavor survival or transformation probability is given by the absolute square of the corresponding matrix element.[7] Thus, the flavor survival probability is simply

$$P(\nu_\eta;x|\nu_\eta;x_0) = |g(x,x_0)|^2. \tag{8}$$

The evolution of flavors in a medium with arbitrary density distribution, according to (7) and (8), is thus completely and accurately determined by (6). The evolution equation is simpler than the exact equations for the wave functions or transition probabilities given by (6). Note that when the latter are solved in the adiabatic approximation, off-diagonal elements of the matrix $\mathcal{M}$ are neglected. Corrections to the adiabatic approximation are given by the Landau–Zener equation, which holds for a linearly varying density.[6,7] It follows directly from the form of Eq. (6) that the solution for linear and exponential density profiles can be expressed in terms of Whittaker functions.

In order to extend the class of functions $\rho(x)$ that admit of exact solutions of (6), we introduce the new independent variable

$$z = \int_{x_0}^{x} f(r)dr, \tag{9}$$

and rewrite (6):

$$f^2 g'' - \left[ih_0(x)f - \frac{df}{dx}\right]g' + h^2 g = 0. \tag{10}$$

Here f is an arbitrary function that is integrable over the interval (x_0,x), and the prime denotes differentiation with respect to z. Although Eq. (10) is more complicated than the original Eq. (6), we now have the possibility—having specified a density distribution function $h_0[\rho(x)]$—of selecting the function f so as to transform Eq. (10) into a standard second-order equation with known solutions.

We now give two examples of the formalism, applying it to the analytic description of the dynamics of two-neutrino oscillations in matter with a nontrivial density distribution $\rho(x)$ along the neutrino propagation path in a natural medium.

We first consider the symmetric profile

$$h_0[\rho(x)] = a\,\mathrm{sech}(x-b), \tag{11}$$

which approximately describes the density of terrestrial matter along the path of proposed "geophysical" experiments using neutrino beams from high-energy accelerators.[10] The coefficients a and b in (11) are best-fit parameters. Comparing the ansatz

$$(z^2+1)g'' + (z-ia)g' + h^2 g = 0$$

with Eq. (10), we obtain the substitutions $z = \mathrm{sech}(x-b)$ and $2y = 1 - iz$, which enable us to transform Eq. (6) into a hypergeometric equation of the form

$$y(y-1)\frac{d^2g}{dy^2} + \left[y - \frac{1}{2}(1+a)\right]\frac{dg}{dy} + h^2 g = 0. \tag{12}$$

If the accelerator generates a beam of ν_μ that propagates along a chord through the earth with the density profile (11), then according to (8) and (12), the probability of detecting a muon neutrino at distance x from the accelerator can be written exactly in terms of hypergeometric functions:

$$P(\nu_\mu;x|\nu_\mu;0) = |C_1F(\alpha,-\alpha,\gamma;y) + C_2y^{1-\gamma}F \times (1+\alpha-\gamma, 1-\alpha-\gamma, 2-\gamma;y)|^2, \tag{13}$$

where $\alpha \equiv ih$, $2\gamma \equiv 1-a$, $2y = 1 - i\,\mathrm{sech}(x-b)$, and $C_{1,2}$ are constants of integration. Calculating the zeroes of $P(x)$, we find the points along the path where muon neutrinos are completely transformed into electron neutrinos. The probability of detecting a ν_e will be greatest at those points.

If we begin with a different ansatz,

$$(z^2+\delta)g''+(1-ia)zg'+h^2g=0, \tag{14}$$

it is not difficult to show that the density distribution

$$h_0[\rho(x)]=\begin{cases} a\,\mathrm{cth}(x-b), & \delta=-1, \\ a\,\mathrm{th}(x-b), & \delta=1 \end{cases} \tag{15}$$

admits of exact solutions of Eq. (6) if we make the substitution $z=\cosh(x-b)$ for $\delta=-1$ and $z=\mathrm{sech}(x-b)$ for $\delta=1$. The replacement $y=1+\delta x^2$ then transforms Eq. (14) into the standard Gauss form of the hypergeometric equation,

$$y(y-1)\frac{d^2g}{dy^2}+[(\alpha+\beta+1)y-\gamma]\frac{dg}{dy}+\alpha\beta g=0, \tag{16}$$

where $2\gamma\equiv 1-ia$, $4\alpha\equiv -i(a\pm\sqrt{4h^2+a^2})$, and $4\alpha\beta=h^2$. In media with the density profile (15), the probability of neutrino flavor survival can thus be expressed in terms of the solution of Eq. (16) with no approximations whatsoever, and for all values of the neutrino parameters. The function (15) has been used as an improvement upon an exponential radial density distribution for electrons in the sun.[7,11] It is well known[12] that the earth can resonantly enhance the oscillations of neutrinos traversing it. Certain portions of the density variation—at the core-mantle transition, for example—can be modeled by the function $a\tanh(x-b)$ with appropriate values of a and b.

Choosing various forms of the replacement (9) will produce a set of different functions $\rho(x)$ that admit exact solutions of the equation of motion (6) in terms of standard special functions.

4. NONADIABATIC THREE-NEUTRINO OSCILLATIONS IN INHOMOGENEOUS MEDIA

We now present the equation of motion of a three-neutrino system in a medium with arbitrary density distribution. The basis consists of states of definite mass, and we write the equation with Planck's constant appearing explicitly:

$$i\frac{d}{dx}|\nu\rangle=(\mathcal{M}_d+\mathcal{M}_{nd})|\nu\rangle, \quad |\nu(0)\rangle=|\nu_0\rangle, \tag{17}$$

where $\mathcal{M}_d=(1/2E)\mathrm{diag}(\lambda_1,\lambda_2,\lambda_3)$, and the nonvanishing elements of the hermitian matrix $\mathcal{M}_{nd}$ are $\mathcal{M}^{21}_{nd}=-\mathcal{M}^{12}_{nd}=i(d\omega/dx)$, $\mathcal{M}^{31}_{nd}=-\mathcal{M}^{13}_{nd}=i(d\varphi/dx)\times\cos\omega$, and $\mathcal{M}^{23}_{nd}=-\mathcal{M}^{32}_{nd}=i(d\varphi/dx)\sin\omega$. The mass hierarchy and the parametrization of the mixing matrix are normally chosen so that the effect of the medium on the oscillations can be described solely in terms of the two mixing angles ω and φ, which depend on ρ.[7] The eigenvalues λ_i of the mass matrix[13] also depend on ρ, and are the energy levels in the 3ν system; for $m_3^2>m_2^2>m_1^2$ and small ω and φ, crossing those levels defines two resonances.

In general, it is extremely difficult to solve Eq. (17) with variable coefficients. When the density of the medium varies slowly enough (more specific criteria for the three-flavor case can be found, for example, in Ref. 6), the adiabatic approximation will suffice.[6,7] In that approximation, we neglect the matrix $\mathcal{M}_{nd}$ in (17), thereby ignoring transitions between states of definite mass in the medium. The adiabaticity conditions are primarily violated near resonances, where the most significant changes in the dynamics of the neutrino system take place.

In this section, we propose a way to find analytic solutions of the equation of motion (17) for a 3ν system in a medium with arbitrary density distribution when the evolution near one of those resonances is not adiabatic. Essentially, we employ perturbation theory, with the unperturbed solution being the $SU(2)$ evolution matrix.

We can find a solution of Eq. (17),

$$|\nu\rangle=U|\nu_0\rangle,$$

using an evolution matrix that satisfies

$$i\frac{dU}{dx}=(\mathcal{M}_d+\mathcal{M}_{nd})U, \quad U(0)=I_3.$$

Expressing the solution in the form

$$U=U_d\tilde{U}, \tag{18}$$

where

$$U_d=\exp\left[-i\int_0^x \mathcal{M}_d(r)dr\right],$$

we obtain

$$i\frac{d\tilde{U}}{dx}=\tilde{\mathcal{M}}\tilde{U}, \quad \tilde{U}(0)=I_3,$$

where

$$\tilde{\mathcal{M}}=U_d^+\mathcal{M}_{nd}U_d. \tag{19}$$

The evolution of the state vector of the 3ν system can now be represented in the form

$$|\nu\rangle=U_d|\tilde{\nu}\rangle,$$

where in the interaction picture, $|\tilde{\nu}\rangle$ in turn evolves according to

$$|\tilde{\nu}\rangle=\tilde{U}|\nu_0\rangle,$$

and

$$i\frac{d}{dx}|\tilde{\nu}\rangle=\tilde{\mathcal{M}}|\tilde{\nu}\rangle, \quad |\tilde{\nu}(0)\rangle=|\nu_0\rangle. \tag{20}$$

We restrict our attention to the resonance range between $l_r-\delta l$ and $l_r+\delta l$; for definiteness, we take the lower resonance to be at $\lambda_1=\lambda_2=\lambda$, and we introduce the parameter

$$\varepsilon=\frac{1}{2E}\int_{l_r-\delta l}^{l_r+\delta l}|\lambda-\lambda_3|dx.$$

The analogous parameter of a 2ν system near resonance is of order $2\pi\delta l/l_m$,[6] where l_m is the oscillation length at resonance, and δl is the spatial halfwidth of the resonant layer. The condition $\delta l\ll l_m$ signals a large departure from the adiabatic oscillation regime.

We write the solution of Eq. (20) as an infinite series in the small parameter ε:

$$|\tilde{\nu}\rangle=\sum_{k=0}^{\infty}\varepsilon^k|\tilde{\nu}^{(k)}\rangle. \tag{21}$$

At the lower resonance, the matrix (19) is given by

$$\tilde{\mathcal{M}}=\mathcal{M}_{nd}+\mathcal{M}_a\sum_{k=0}^{\infty}\frac{(i\varepsilon)^{2k+1}}{(2k+1)!}+\mathcal{M}_h\sum_{k=1}^{\infty}\frac{(i\varepsilon)^{2k}}{(2k)!}. \tag{22}$$

The nonvanishing elements of the antihermitian matrix $\mathcal{M}_a$ and the hermitian matrix $\mathcal{M}_h$ are $\mathcal{M}_h^{13}=\mathcal{M}_a^{13}=-i(d\varphi/dx)\cos\omega$ and $\mathcal{M}_h^{23}=\mathcal{M}_a^{23}=i(d\varphi/dx)\sin\omega$, with $\mathcal{M}_a^{ij}=\mathcal{M}_a^{ji}$ and $\mathcal{M}_h^{ij}=-\mathcal{M}_h^{ji}$, $i\neq j$. Substituting (21) and (22) into Eq. (20), we obtain a set of linear differential equations for the terms in the series representation of (19), in which the $(k+1)$th equation takes the form

$$i\frac{d}{dx}|\tilde{\nu}^{(k)}\rangle=\mathcal{M}_{nd}|\tilde{\nu}^{(k)}\rangle+|\mu^{(k)}\rangle,$$

with initial conditions $|\tilde{\nu}^{(0)}(0)\rangle=|\nu_0\rangle$, $|\tilde{\nu}^{(j)}(0)\rangle=0$, $j=1,2,\ldots$. The recursion relations for the $|\mu^{(k)}\rangle$ with even and odd k are

$$|\mu^{(2k)}\rangle=\mathcal{M}_h\sum_{j=0}^{k-1}\frac{i^{2k-2j}|\tilde{\nu}^{(2j)}\rangle}{(2k-2j)!}+\mathcal{M}_a\sum_{j=0}^{k-1}\frac{i^{2k-2j-1}|\tilde{\nu}^{(2j+1)}\rangle}{(2k-2j-1)!},$$

$$|\mu^{(0)}\rangle=0,$$

$$|\mu^{(2k+1)}\rangle=\mathcal{M}_h\sum_{j=0}^{k-1}\frac{i^{2k-2j}|\tilde{\nu}^{(2j+1)}\rangle}{(2k-2j)!}+\mathcal{M}_a\sum_{j=0}^{k}\frac{i^{2k-2j+1}|\tilde{\nu}^{(2j)}\rangle}{(2k-2j+1)!}.$$

The proposed series expansion in ε makes an iterative solution of the equation of motion of the 3ν system possible. To $(k+1)$th order in ε, the state vector $|\tilde{\nu}^{(k)}\rangle$ in the interaction picture (18) can be calculated with the aid of the matrix U_0 for the first term of the series (21),

$$|\nu^{(k)}\rangle=U_0(|\nu_0\rangle+\int_0^x U_0^+|\mu^{(k)}\rangle dr),$$

where U_0 satisfies the evolution equation

$$i\frac{d}{dx}U_0=\mathcal{M}_{nd}U_0,\quad U_0(0)=I_3. \tag{23}$$

We expand the matrix $\mathcal{M}_{nd}$ using as a basis the three-dimensional representation of the $SU(2)$ algebra,

$$\mathcal{M}_{nd}=M_0\frac{d\omega}{dx}+\frac{i}{2}M_-e^{-i\omega}\frac{d\varphi}{dx}-\frac{i}{2}M_+e^{i\omega}\frac{d\varphi}{dx},$$

with basis matrices satisfying the standard commutation relations $[M_+,M_-]=2M_0$, $[M_0,M_\pm]=\pm M_\pm$, and having nonvanishing elements $M_0^{21}=-M_0^{12}=i$, $M_+^{31}=-M_+^{13}=1$, $M_+^{32}=-M_+^{23}=i$, $M_-^{13}=-M_-^{31}=1$, $M_-^{32}=-M_-^{23}=i$. Writing U_0 in product form,

$$U_0=\exp(i\omega M_0)\tilde{U}_0, \tag{24}$$

we obtain the evolution equation for the matrix $\tilde{U}_0$, which belongs to the three-dimensional representation of $SU(2)$:

$$i\frac{d}{dx}\tilde{U}_0=\left(2M_0\frac{d\omega}{dx}+\frac{i}{2}M_-\frac{d\varphi}{dx}-\frac{i}{2}M_+\frac{d\varphi}{dx}\right)\tilde{U}_0.$$

Using a parametrization similar to the one in Sec. 3,

$$\tilde{U}_0=\exp(\tilde{g}_0-2i\omega)M_0\exp\tilde{g}_-M_-\exp\tilde{g}_+M_+, \tag{25}$$

the evolution problem (23) can be fully dealt with after solving the differential equation for the parameter $\tilde{g}=\exp(\tilde{g}_0/2)$,

$$\frac{d^2\tilde{g}}{dx^2}-\left(\frac{d^2\varphi/dx^2}{d\varphi/dx}+2i\omega\right)\frac{d\tilde{g}}{dx}+\left(\frac{1}{2}\frac{d\varphi}{dx}\right)^2\tilde{g}=0, \tag{26}$$

where $\tilde{g}(0)=1$ and $d\tilde{g}/dx|_{x=0}=0$. An explicit form for the matrix U_0 can be found in terms of $\tilde{g}$ and $\tilde{g}_\pm$ by substituting M_0 and $M_\pm$ into Eqs. (24) and (25).

In a medium with variable density, $d\varphi/dx$ and $d\omega/dx$ are proportional to $d\rho/dx$. Assuming the density variation to be given approximately by kx, we obtain

$$\frac{d^2\tilde{g}}{dx^2}-2i\omega\frac{d\tilde{g}}{dx}+\left(\frac{k}{2}\right)\tilde{g}=0,$$

analogous to Eq. (6), which was considered in detail in Sec. 3. For nonlinear functions $\rho(x)$, Eq. (26) can be solved by invoking results obtained for other physical systems with $SU(2)$ or $SU(1,1)$ dynamic symmetries. Such systems have been studied in the theory of modulated laser-beam interaction with two- and three-level atoms,[14] and in the quantum theory of a phase-modulated and amplitude-pumped parametric amplifier and frequency converter.[15]

We conclude by pointing out the possibility of dealing with other interactions via the group-theoretical approach, such as the interaction of the neutrino magnetic moment with solar fields.[16]

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