Fluctuation conductivity in strong-coupling superconductors

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The effect of fluctuations on the properties of a conventional superconductor with a strong electron–phonon interaction and no impurities is considered. The correction to the conductivity of the metal above the critical temperature is calculated and the role of phonon damping in fluctuation effects is estimated.

1. INTRODUCTION

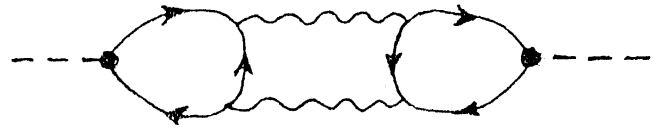
The effect of fluctuations on the properties of superconductors near the transition point have been the subject of intensive studies since the early 1960's. Since the characteristic correlation length in a conventional superconductor is rather large, thermodynamic fluctuations in it are small, and before the problem of fluctuations came under study, the critical temperature had been frequently viewed as just "marking" the sharp boundary between the regions of normal and superconducting behavior. Although this view is quite adequate for most practical applications, extensive studies of the vicinity of the phase transition have been performed and now fairly a complete picture of the fluctuation effects exists.

The most interesting fluctuation-related phenomena usually occur above the transition temperature, where fluctuations precede the onset of superconductivity by making a small but experimentally measurable contribution of specifically superconducting nature to the observables of the system. Although electron pairing is energetically unfavorable above the critical temperature, the system always contains fluctuation pairs which contribute to the conductivity of the metal. The number of such pairs and the magnitude of the conductivity increase as the critical temperature is approached. The excess conductivity resulting from the acceleration of the pairs (or *paraconductivity*) was first calculated microscopically by Aslamazov and Larkin.[1] Their results are in good agreement with the experimental data of Glover.[2]

Later experiments[3–5] have shown, however, that conductivity may exhibit an anomalous temperature behavior inconsistent with the universal prediction of Ref. 1. This anomalous effect was first described by Maki[6] for three-dimensional systems. Thompson[7] observed that this correction diverges in lower dimensions and was the first to introduce the concept of pair breaking to obtain a finite result. Qualitatively, the effect is as follows. Instead of forming a Cooper pair, two electrons may fluctuate into coherent small-total-momentum states in which, owing to the time-reversal symmetry, they remain even after being scattered by a diamagnetic impurity. Such electrons are accelerated as if they were a fluctuation pair. The above picture depends on the coherence of the electron states and the time-reversal symmetry, so pair-breaking mechanisms naturally include magnetic fields, paramagnetic impurities, and those types of retarded interactions which lead to inelastic collisions.

In all of the above works microscopic treatments are performed within the BCS model and the resistance mechanism is invariably impurity scattering. A question which remains unsettled, however, is the influence of the strong electron–phonon interaction on fluctuation effects. The correction to the specific heat in the framework of Eliashberg theory was calculated by Bulaevskiĭ and Dolgov,[8] but their method is difficult to apply in order to obtain corrections to kinetic coefficients, which is a problem of considerable current interest. This interest is in part stimulated by the advent of high-temperature superconductivity (HTSC) which cannot be described by BCS theory. The strong-coupling theory, in contrast, does not restrict the value of the critical temperature and so may be of use for the qualitative description of superconductor properties in the absence of a consistent microscopic HTSC theory. But high critical temperatures correspond to large coupling constants, and it is this correlation which stimulates, to a certain extent, the calculation of fluctuation corrections within the Eliashberg theory.

The present analysis gives the fluctuation correction to the conductivity. As indicated above, a retarded interaction leading to inelastic collisions suppresses the Maki–Thompson correction, and because of this the Aslamazov–Larkin diagram is considered (see Fig. 1). The static conductivity σ is obtained at low frequencies from the expression $\mathbf{j}_\omega = i\omega\sigma\mathbf{A}_\omega$, where $\mathbf{j}$ is the current density and $\mathbf{A}$ a small uniform field.



2. VERTEX FUNCTION EQUATION

Within the Eliashberg–Migdal theory[9,10] the vertex function for the electron–phonon interaction obeys the following integral equation:

$$\Gamma(p,p';k) = D(p-p') + T\sum_n \int \frac{d^3q}{(2\pi)^3}$$

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$$\times D(p-q)G(q)G(k-q)\Gamma(q,p',k). \quad (1)$$

The electron Green's function is given by

$$G(q)=[\xi_q-i\varepsilon_n-\Sigma(i\varepsilon_n)]^{-1}, \quad (2a)$$

where

$$\Sigma(i\varepsilon_n)=T\sum_n \int \frac{d^3q'}{(2\pi)^3} D(-q')G(q-q') \quad (2b)$$

is the self-energy and the $\varepsilon_n=(2n+1)\pi T$ are the frequencies. After averaging over the Fermi surface, the phonon Green's function $D(p')$ yields a function λ defined by

$$\lambda(i\varepsilon_n)=\int_0^\infty \frac{2\Omega d\Omega}{\varepsilon_n^2+\Omega^2}\alpha^2(\Omega)F(\Omega). \quad (3)$$

The spectral function $\alpha^2(\Omega)F(\Omega)$ is extracted directly from experiment.[11]

As is known, Cooper pairs are mainly formed by electrons located in a thin crust near the Fermi surface. Accordingly, the momenta $\mathbf{p}$ and $\mathbf{p}'$ are set equal to the Fermi momentum in the argument of the function $\Gamma(p,p',k)$.

The vertex function has a pole near the transition point as ω, $k\to 0$. The singular nature of Γ allows to retain only one term in the eigenfunction expansion near T_c, which we assume to be of the form

$$\Gamma(p,p',k)=\Delta(i\varepsilon)\Delta(i\varepsilon')L(i\omega,k^2,T-T_c), \quad (4)$$

where the pole factor L is

$$L=\frac{a_0}{a_1k^2+a_2\dfrac{T-T_c}{T_c}-i\omega a_3}. \quad (5)$$

The functions $\Delta(i\varepsilon)$ are precisely the eigenfunctions of the corresponding homogeneous equation for $T=T_c$.

The homogeneous equation for the functions Δ is the linearized transition-point equation for T_c, which is well studied below the critical temperature[12] and has the form

$$\Delta(i\varepsilon)=T_c\sum_n \int \frac{d^3q}{(2\pi)^3} D(i\varepsilon-i\varepsilon_n)G(q)G(-q)\Delta(i\varepsilon_n). \quad (6)$$

The functions Δ (related to the system's energy gap at $T<T_c$) again play the role of "wave functions," but this time (i.e., above T_c) these are the wave functions of fluctuation pairs (assuming a proper renormalization).

In the following the analytic functions Δ^R and Δ^A will be needed. The corresponding equations are obtained by analytically continuing Eqs. (6) using the Eliashberg method:[13]

$$\Delta^R(\varepsilon)=\frac{mp_F}{16\pi^2}\int d\cos\theta\int_{-\infty}^{\infty}dx \times\left[\tanh\left(\frac{x}{2T_c}\right)D^R(\varepsilon-x)2\,\mathrm{Re}\left(\frac{\Delta^R(x)}{x+i\gamma}\right) -\coth\left(\frac{x-\varepsilon}{2T_c}\right)\frac{\Delta^R(x)}{(x+i\gamma)}2\,\mathrm{Im}\,D^R(\varepsilon-x)\right], \quad (7)$$

where γ is the phonon damping. The interaction is written in a form convenient for a comparison with the corresponding BCS formulas. For numerical calculations it is convenient to first integrate over the angle, with a view to obtaining the interaction in the form (3). The function Δ^A obeys an equation which is complex conjugate to Eq. (7).

3. EIGENFUNCTION PROPERTIES

Let us now examine some of the properties of the functions Δ. Integrating over ξ and the angle θ on the right-hand side of Eq. (6) we find

$$\Delta(i\varepsilon_n)=-\frac{iT_c}{2\pi}\sum_m \lambda(m-n)\frac{\Delta(i\varepsilon_m)}{i\varepsilon_m+\Sigma(i\varepsilon_m)}. \quad (8)$$

The interaction enters in the form of (3). Remembering that we are interested in the electrons near the Fermi surface, in the argument of Σ we set the momentum variable equal to its Fermi value. After integrating in Eq. (2b) we have

$$\Sigma(i\varepsilon_n)=-\frac{iT_c}{2\pi}\left(\lambda(0)+2\sum_{l=1}^{n}\lambda(l)\right). \quad (9)$$

The continuation of the imaginary part of Σ to the real axis determines the phonon damping γ.

Let us introduce a function $\bar{\Delta}$ such that $\bar{\Delta}(i\varepsilon)\eta(i\varepsilon)\varepsilon=\Delta(i\varepsilon)$, where

$$\eta(i\varepsilon)=1+\frac{\Sigma(i\varepsilon)}{i\varepsilon}.$$

Then

$$\bar{\Delta}(i\varepsilon_n)\left[\varepsilon_n-\frac{T_c}{2\pi}\left(\lambda(0)+2\sum_{l=1}^{n}\lambda(l)\right)\right] =-\frac{T_c}{2\pi}\sum_m \lambda(m-n)\bar{\Delta}(i\varepsilon_m).$$

Here we have substituted (9) for Σ. It can be seen that the term accounting for the elastic phonon scattering has canceled, and for the function $\bar{\Delta}$ we obtain the equation

$$\bar{\Delta}(i\varepsilon_n)\left[\varepsilon_n-\frac{T_c}{\pi}\sum_{l=1}^{n}\lambda(l)\right]=-\frac{T_c}{2\pi}\sum_{m\neq n}\lambda(m-n)\bar{\Delta}(i\varepsilon_m). \quad (10)$$

Now let us consider the limit $\lambda\to\infty$ which, although generally unrealizable in nature, helps to understand some of the physical features of the problem. In this limit (see Ref. 12, for example) $T_c\sim\sqrt{\lambda}$ and the quantity ω_D/T_c is small. Therefore, looking back at Eq. (3), in which the effective frequencies Ω are less than, or of the order of, the Debye frequency, we neglect Ω compared with T_c in the denominator and approximate λ by

$$\lambda(l)\cong\frac{\rho\pi^2}{l^2}, \quad (11)$$

where

$$\rho=\frac{1}{4\pi^4}\frac{\lambda\langle\omega^2\rangle}{T_c^2}. \tag{12}$$

Substituting the frequencies ε_m explicitly into Eq. (10) we have

$$\bar{\Delta}(i\varepsilon_n)[2n+1-\rho a_n]=-\frac{1}{2}\rho\sum_{m\neq n}\frac{\bar{\Delta}(i\varepsilon_m)}{(m-n)^2}, \tag{13}$$

where the numbers $a_n=\Sigma_{l=1}^{n}1/l^2$ are of order unity ($a_1=1$, $a_\infty=\pi^2/6$).

The eigenvalues of Eq. (13) determine the number ρ and thus T_c:

$$T_c=\alpha\sqrt{\lambda\langle\omega^2\rangle}, \tag{14}$$

where α is a number. The same result is obtained using a similar procedure for $T<T_c$. The quantity α as found in the work of Allen and Dynes[14] is $\alpha=0.183\approx 1/\pi\sqrt{2}$. Thus the parameter ρ turns out to be numerically small.

We are interested in the properties of the eigenfunctions. As discussed in Ref. 14, the iterative procedure for Eq. (14) converges fairly rapidly because of the factor $1/(m-n)^2$ on the right-hand side. Let us try to find an approximate analytical structure of the solution by neglecting terms with $(n-m)\geqslant 2$. This reduces the equation to the recursive formula

$$\alpha_m\bar{\Delta}_m+\bar{\Delta}_{m-1}+\bar{\Delta}_{m+1}=0, \tag{15}$$

where $\alpha_m=2[2n+1-\rho a_m]/\rho$. We will consider Eq. (15) as an infinite system of algebraic equations and solve the system by iterations remembering that, in our approximation, only the $\bar{\Delta}_m$ with close values of m correlate. The matrix of the system has α_m's along the principal diagonal, units along the two nearest diagonals, and zeros everywhere else.

As a first approximation we consider a 3×3 system with $\bar{\Delta}_{m+1}$ and $\bar{\Delta}_{m-1}$ only. This system is of the form

$$\begin{pmatrix}\alpha_{m-1} & 1 & 0\\ 1 & \alpha_m & 1\\ 0 & 1 & \alpha_{m+1}\end{pmatrix}\begin{pmatrix}\bar{\Delta}_{m-1}\\ \bar{\Delta}_m\\ \bar{\Delta}_{m+1}\end{pmatrix}=0.$$

Within the normalization factor the eigenfunctions are

$$\begin{pmatrix}\dfrac{-1}{\alpha_{m-1}}\\ 1\\ \dfrac{-1}{\alpha_{m+1}}\end{pmatrix}.$$

In the next approximation, a 5×5 system must be considered. For the quantity $\bar{\Delta}_{m-1}$ as an example, using the same normalization, we have

$$\bar{\Delta}_{m-1}=-\frac{1}{\alpha_{m-1}-(1/\alpha_{m-2})}. \tag{16}$$

The correction is of order $(\alpha_{m-1}\alpha_{m-2})^{-1}\sim\rho^2\ll 1$ and may be neglected because we are neglecting 1/4 compared with 1 in writing Eq. (15).

Thus, to within the normalization factor, the solution is

$$\bar{\Delta}_m\sim\frac{1}{2m+1-\rho a_m}. \tag{17}$$

The analytically continued function behaves like

$$\bar{\Delta}^R(\varepsilon)\sim\frac{1}{\varepsilon+i\gamma'(\varepsilon)}, \tag{18}$$

where $\gamma'\sim\pi\lambda\langle\omega^2\rangle/T_c^2\Sigma(\varepsilon)-\Sigma(0)/T$ is that part of the phonon damping corresponding to the purely inelastic phonon scattering. Note that $\gamma'(\varepsilon)\to 0$ as $\varepsilon\to 0$.

The function Δ^R thus involves the renormalization

$$\Delta^R(\varepsilon)=\frac{\varepsilon+i\gamma}{\varepsilon+i\gamma'}\tilde{\Delta}(\varepsilon), \tag{19}$$

where γ is the total phonon damping. The nonsingular function $\tilde{\Delta}(\varepsilon)$ depends only on the inelastic part of the damping and is related to the wave function of the fluctuation-induced pair.

4. VERTEX FUNCTION CALCULATION

We now turn to the calculation of the vertex function. Let the functions Δ be normalized to unity. In this case all the normalization of the vertex function resides in L and is determined by the free term of Eq. (1). From Eq. (1), substituting Eq. (4) for the vertex function, multiplicating both sides by $\Delta(i\varepsilon)\Delta(i\varepsilon')$ and summing over ε,ε', gives the equation for L

$$T_c^2L=T_c\sum_{\varepsilon}\Delta(i\varepsilon)T_c\sum_{\varepsilon'}\Delta(i\varepsilon')D(i\varepsilon-i\varepsilon')$$
$$+T_c^2\sum_{\varepsilon}\Delta(i\varepsilon)T\sum_{\varepsilon_n}\Delta(i\varepsilon_n)L\int\frac{d^3q}{(2\pi)^3}$$
$$\times D(i\varepsilon-i\varepsilon_n)G(q)G(k-q). \tag{20}$$

This equation turns out to be an algebraic one and has the solution

$$L=\frac{a_0}{1-A(i\omega,k^2,T-T_c)}, \tag{21}$$

where

$$A=T\sum_{\varepsilon}\Delta(i\varepsilon)\sum_{\varepsilon_n}\Delta(i\varepsilon_n)\int\frac{d^3q}{(2\pi)^3}$$
$$\times D(i\varepsilon-i\varepsilon_n)G(q)G(k-q), \tag{21a}$$

and the normalization factor is given by

$$a_0=\sum_{\varepsilon}\Delta(i\varepsilon)\sum_{\varepsilon'}\Delta(i\varepsilon')D(i\varepsilon-i\varepsilon'). \tag{21b}$$

Using (6) we readily find that at the point $k^2=0$, $\omega=0$, $T=T_c$, the function A takes the value $A(0)=1$.

In the following, the analytically continued functions L^R and L^A will be needed. With this in view, let us continue the function $L(i\omega)$ to the upper ω half-plane. The continued form of Eq. (21) is $L^R(\omega)=a_0[1-A^R(\omega)]^{-1}$,

and in order to obtain a singular term of the type (5), we must expand $A^R(\omega)$ in terms of the small values of k^2, ω, and $T-T_c$.

For the function $A^R(\omega)$ we have

$$A^R(\omega)=\frac{1}{T}\frac{1}{(4\pi i)^2}\int_{-\infty}^{\infty}dx_1\int_{-\infty}^{\infty}dx_2\int\frac{d^3q}{(2\pi)^3}\times[G^R(\mathbf{q},x_1)G^A(\mathbf{k}-\mathbf{q},\omega-x_1)h_2(x_1,x_2,\omega)-G^A(\mathbf{q},x_1)G^R(\mathbf{k}-\mathbf{q},\omega-x_1)h_1(x_1,x_2)] \quad (22)$$

where we have defined

$$h_1(x_1,x_2)=\tanh\left(\frac{x_2}{2T}\right)\Delta^A(x_1)\left[\tanh\left(\frac{x_1}{2T}\right)2i\times\mathrm{Im}(\Delta^R(x_2)D^R(x_2-x_1))+\coth\left(\frac{x_1-x_2}{2T}\right)\times 2i\,\mathrm{Im}(D^R(x_2-x_1))\Delta^A(x_2)\right], \quad (23)$$

$$h_2(x_1,x_2,\omega)=\tanh\left(\frac{x_2-\omega}{2T}\right)\Delta^R(x_1)\Delta^R(x_2)2i\times\mathrm{Im}(D^R(x_2-x_1))\left(\tanh\left(\frac{x_1-\omega}{2T}\right)+\coth\left(\frac{x_1-x_2}{2T}\right)\right)+\Delta^R(x_1)\tanh\left(\frac{x_1-\omega}{2T}\right)\times\tanh\left(\frac{x_2}{2T}\right)2i\,\mathrm{Im}(\Delta^R(x_2))D^A(x_2-x_1). \quad (24)$$

The coefficients in the expansion

$$A^R(\omega)\approx 1-a_1k^2-a_2\frac{T-T_c}{T_c}+ia_3\omega \quad (25)$$

are readily found by differentiating A^R and are of the form

$$a_1=\frac{i}{4\pi T_c}\hat{C}\frac{v_F^2}{3}\frac{1}{2}\int_{-\infty}^{+\infty}dx_1\left[\frac{\tilde{h}_1(x_1)}{x_1-i\gamma}+\frac{\tilde{h}_2(x_1,0)}{x_1+i\gamma}\right], \quad (26a)$$

$$a_2=1+\frac{i}{4\pi}\hat{C}\int_{-\infty}^{+\infty}dx_1\left[\frac{1}{x_1-i\gamma}\frac{\partial\tilde{h}_1(x_1)}{\partial T}+\frac{1}{x_1+i\gamma}\frac{\partial\tilde{h}_2(x_1,0)}{\partial T}\right], \quad (26b)$$

$$ia_3=-\frac{i}{4\pi T_c}\hat{C}\int_{-\infty}^{+\infty}dx_1\frac{1}{x_1+i\gamma}\frac{\partial\tilde{h}_2(x_1,0)}{\partial\omega}, \quad (26c)$$

where

$$\hat{C}=\frac{mp_F}{16\pi^2}\int d\cos\theta. \quad (26d)$$

Here $\tilde{h}_i(x_1)$ denotes the integral of h_i over x_2. The normalization factor is determined by integrating the free term of Eq. (2). Represented in integral form, Eq. (21b) becomes

$$a_0=\frac{1}{(2\pi T_c)^2}\int_{-\infty}^{+\infty}dx_1\int_{-\infty}^{+\infty}dx_2\tanh\left(\frac{x_2}{2T}\right)\left[\tanh\left(\frac{x_1}{2T}\right)\times\mathrm{Im}\,\Delta^R(x_1)\mathrm{Im}(\Delta^R(x_2)D^R(x_2-x_1))+\coth\left(\frac{x_1}{2T}\right)\times\mathrm{Im}(D^R(x_2-x_1))\mathrm{Im}(\Delta^R(x_1)\Delta^R(x_1+x_2))\right]. \quad (27)$$

From (5), dividing the numerator and denominator by a_2 gives

$$L=-\frac{1}{N}\frac{1}{\frac{T-T_c}{T_c}+Dk^2-i\frac{\omega}{T_c}b}, \quad (28)$$

where

$$N=-\frac{a_2}{a_0},\quad D=\frac{a_1}{a_2},\quad b=\frac{a_3}{a_2}T_c. \quad (29)$$

Thus we have obtained the propagator L in its usual form. The quantities N and D play the roles of respectively the density of states at the Fermi surface and the diffusion coefficient, calculated by allowing for the electron–phonon interaction and near-critical fluctuations.

5. THE CONDUCTIVITY CORRECTION

Let us now calculate the correction to the normal-metal conductivity. With the fluctuation propagator (4), in the Matsubara technique we have (for the Aslamazov–Larkin diagram)

$$\mathbf{j}_\omega=4\frac{e^2}{m^2}T\sum_n\int\frac{d^3q}{(2\pi)^3}L(\omega_n,q^2)L(\omega_n-\omega,q^2)\mathbf{C}(\mathbf{A}_\omega\mathbf{C}), \quad (30)$$

where $\mathbf{C}$ denotes the integral

$$\mathbf{C}=T\sum_n\int\frac{d^3p}{(2\pi)^3}\mathbf{p}G(\mathbf{p},i\varepsilon)G(\mathbf{q}-\mathbf{p},i\omega_n-i\omega-i\varepsilon)\times G(\mathbf{p},i\varepsilon+i\omega)\Delta(i\varepsilon)\Delta(i\varepsilon+i\omega). \quad (31)$$

By symmetry arguments, the quantity $\mathbf{C}$ is represented in the form $\mathbf{C}=\mathbf{q}C(\omega,\omega_n)$. Since the whole of the critical point singularity is contained in L, we may treat $C(\omega,\omega_n)$ as a constant $C(0)$. For this constant we have a simple expression

$$C(0)=\frac{1}{6}\frac{p_F^3}{(2\pi)^2}\int_{-\infty}^{\infty}dx\tanh\left(\frac{x}{2T}\right)\times\left(\frac{\Delta^R(x)\Delta^R(x)}{(x+i\gamma)^3}+\frac{\Delta^A(x)\Delta^A(x)}{(x-i\gamma)^3}\right). \quad (32)$$

Equation (30) now becomes

$$\mathbf{j}_\omega=4\frac{e^2}{m^2}\mathbf{A}_\omega T\sum_n\int\frac{d^3q}{(2\pi)^3}\frac{q^2}{3}L(\omega_n,q^2)L(\omega_n-\omega,q^2). \quad (33)$$

Let us perform an analytical continuation to real ω's such that the resulting function has no singularities in the

upper ω half-plane. Then, substituting Eq. (28) for the fluctuation propagator and integrating over the energy variables we find, to first order in ω,

$$\mathbf{j}_\omega = \frac{e^2}{m^2}\mathbf{A}_\omega \int \frac{d^3q}{(2\pi)^3}\frac{q^2}{3}C(0)^2\frac{bT_c}{2N^2}\frac{i\omega}{\left(\frac{T-T_c}{T_c}+Dq^2\right)^3}. \tag{34}$$

Finally, the absolute value of the fluctuation correction to the conductivity is

$$\sigma' = \frac{1}{32\pi}\frac{e^2}{m^2}C(0)^2\frac{b}{N^2D^{5/2}}\sqrt{\frac{T_c}{T-T_c}}. \tag{35}$$

6. VERY STRONG COUPLING LIMIT

Let us examine the correction (35) in the limiting case $\lambda \to \infty$. Integrating the phonon Green's function over angle we obtain the averaged interaction (3),

$$\frac{1}{v_F}\int_{S_F}\frac{d^2p}{2\pi}D(\mathbf{p},\varepsilon) = \lambda(\varepsilon). \tag{36}$$

Note that $\lambda(\varepsilon)$ is a dimensionless quantity which in the $\lambda \to \infty$, large ε limit becomes

$$\lambda(\varepsilon) \to \frac{1}{\varepsilon^2}\int_0^{+\infty} 2\Omega d\Omega \alpha^2(\Omega)F(\Omega) = \frac{T_c^2}{\varepsilon^2}\frac{\lambda\langle\omega^2\rangle}{T_c^2}, \tag{37}$$

where λ is the coupling constant [cf. Eq. (11)]. The dimensionless complex $(\lambda\langle\omega^2\rangle)/T_c^2$ is nothing other than $1/\alpha^2$ [see Eq. (14)]. In the following analysis all quantities of interest will be expressed in terms of the critical temperature, so that the remaining dimensionless factors will be functions of $1/\alpha^2$ alone.

The analytically continued electron self-energy is given by

$$\Sigma^R = \frac{1}{4\pi i}\int\frac{d^3q}{(2\pi)^3}\int_{-\infty}^{+\infty}dx\left[\tanh\left(\frac{x}{2T}\right)D^R(\varepsilon-x)2i \times \mathrm{Im}G^R(x) - \coth\left(\frac{x}{2T}\right)2i[\mathrm{Im}\,D^R(x)]G^R(\varepsilon + x)\right]. \tag{38}$$

The integral above is dominated by the small values of the energy variables, so in the limit $\lambda \to \infty$ we find, in accord with Eq. (9), that

$$\Sigma^R \sim \lambda T_c \tag{39}$$

to within a factor which is a dimensionless function of the small parameter ε/T_c and tends to a finite value as $\varepsilon \to 0$. The phonon damping is determined by the imaginary part of the self-energy and has an estimate $\gamma \sim \lambda T_c$ [see Eq. (9)].

In a similar way, the coefficients entering the vertex function take the form

$$N \sim mp_F, \quad b \sim \frac{1}{\lambda}, \quad D \sim \frac{1}{\lambda^3}\frac{v_F^2}{T_c^2}, \quad C(0) \sim \frac{1}{\lambda^2}\frac{p_F^3}{T_c^2}. \tag{40}$$

In deriving this, Eq. (39) and the property (18) of the function Δ have been used.

The form of Eqs. (40) is quite understandable because in the large coupling-constant (i.e., large critical-temperature) limit, T_c is the only parameter with dimensions of energy to be involved in the phonon processes. An important point to note is the dependence of the coefficients on λ, something which is rather difficult to obtain without calculations.

Substituting the coefficients (40) into (35), we estimate the conductivity correction to within a numerical factor as

$$\sigma' \approx \lambda^{5/2}\frac{e^2n}{m}\frac{T_c}{\mu^2}\sqrt{\frac{T_c}{T-T_c}}, \tag{41}$$

where μ is the chemical potential and n the electron density.

The correction (41) holds for temperatures such that it is small compared to the normal-metal conductivity σ_n. In our case the conductivity is controlled by the phonon scattering time τ_{ph} and is given by

$$\sigma = \frac{e^2n}{m}\tau_{ph}. \tag{42}$$

The time τ_{ph} is a function of the temperature, related to the damping by $\tau_{ph} \sim 1/\gamma$. In the limit $\lambda \to \infty$ it can be estimated as

$$\tau_{ph} \sim \frac{1}{\lambda T_c}. \tag{43}$$

Hence for the ratio σ'/σ_n we have

$$\frac{\sigma'}{\sigma_n} \sim \lambda^{7/2}\left(\frac{T_c}{\mu}\right)^2\sqrt{\frac{T_c}{T-T_c}}. \tag{44}$$

It can be seen that, compared with the Aslamazov–Larkin results, we have a much wider fluctuation region for the case of an infinite coupling constant. This, of course, does not happen at realistic values of λ.

7. COMPARISON WITH BCS

Let us compare the above results with standard BCS formulas.[1] The change from Eliashberg to BCS theory is usually achieved[12] by making the replacements

$$\lambda(i\omega_m - i\omega_n) = \begin{cases}\lambda & \text{if} \quad |\omega_m|, |\omega_n| \leqslant \omega_D, \\ 0 & \text{elsewhere,}\end{cases} \tag{45a}$$

$$\Delta(i\omega_m) = \begin{cases}\Delta(T) & \text{if} \quad |\omega_m| \leqslant \omega_D, \\ 0 & \text{elsewhere,}\end{cases} \tag{45b}$$

where ω_D is the Debye frequency and the energy gap $\Delta(T)$ is no longer dependent on energy.

In its simplest version BCS theory does not allow for pair damping. When we set γ equal to zero, Eq. (7) gives the standard equation for T_c:[15]

$$\frac{|\lambda|mp_F}{2\pi^2}\int_0^{\omega_D}\frac{d\xi}{\xi}\tanh\left(\frac{\xi}{2T}\right) = 1. \tag{46}$$

Equation (28) for the vertex function and Eq. (35) for the conductivity correction are identical in form to the results of Ref. 1, but the coefficients N, D, and $C(0)$ have assumed a different microscopic meaning. In our case these quantities have been calculated within a strong-coupling theory with inclusion of the fluctuations. It should be emphasized that a direct limiting process by which to retrieve the AK results is impossible in the final formulas of the form (29) because of the dramatic difference in the resistance mechanisms. We therefore turn to consider the limiting cases which may occur.

The clean material limit discussed in Ref. 1 implies, physically, that the electron mean free path is much larger than the characteristic pair dimension ξ_0. In this limit, formally, the dimensionless parameter $T_c\tau_{tr}$ (where τ_{tr} is the transport time for impurity scattering) tends to infinity or, equivalently the damping tends to zero. It is then clear that the change from the strong-coupling formulas to the pure-metal BCS limit requires that, apart from applying the procedure (45a)–(45b), we also let γ approach zero. Then all the necessary integrals are easily performed to yield the corresponding results of Ref. 1.

By the same reason as above, the inclusion of damping affords nothing more than a formal analogy. To see this, note that in BCS theory the introduction of impurities gives rise to the imaginary part $(1/2\tau_{tr})\mathrm{sgn}\omega$ in the denominator of the electron Green's function. Taking the average over the impurities of the quantities of the type $\langle GG\rangle$ then results in the renormalization of the vertices by the factor

$$\eta=1+\frac{1}{2\tau_{tr}|\omega|}. \tag{47}$$

This factor arises from using the ladder approximation to account for the impurities. In our case, the phonon ladder is taken into account in the vertex function calculation, and a similar renormalization is contained in the functions Δ [see Eq. (19)]. The imaginary part of the Green's function naturally arises from that of the self-energy. Thus it would appear that, formally, changing from the impurity to the phonon case amounts to simply replacing the damping parameters.

The analogy between the two treatments of the damping comes most clearly in the large coupling-constant limit. As is known,[16] at high temperatures the scattering of electrons by phonons is predominantly elastic and its inelastic part is often neglected altogether. Since $\gamma\to 0$ as $\varepsilon\to 0$, this assumption does not change the analytical structure of Eq. (19) and establishes its direct correspondence to Eq. (47).

Despite the above analogy, however, the final result (41) cannot be reduced to the dirty-metal Aslamazov-Larkin correction by a simple replacement of the damping parameters. The reason is that the effect of the phonon interaction is not limited to damping alone. Another important factor is the retardation effect, which makes the coefficients, say D and $C(0)$, depend differently on the coupling parameters, whereas in the impurity case they differed by just a factor. Hence an additional (as compared with the "dirty" case[1]) coupling-constant dependence arises in the $\lambda\to\infty$ limit.

8. CONCLUSION

In this paper we have studied fluctuation effects in a superconductor with a strong electron-phonon interaction within the framework of the Eliashberg theory. We have used the microscopic theory to obtain the vertex function for the electron–phonon interaction, and this enabled us to calculate the fluctuation correction to the conductivity of a normal metal near the superconducting transition. In addition, the properties of the eigenfunctions of the integral equation for the vertex function have been examined. These functions contain in themselves the renormalization of the vertices (analogous to that arising in the impurity case) as well as the fluctuation-pair "wave function" which turns out to depend only on the inelastic phonon scattering.

The final formulas of this work involve the coefficients of the fluctuation propagator and the constant $C(0)$ of the Aslamazov–Larkin diagram [see Eq. (35)], and are fully consistent with Ginzburg–Landau theory. The dependence on the coupling constant appears through these coefficients only. Note that in the work of Bulaevskiĭ and Dolgov,[8] the fluctuation correction to the specific heat contains an extra coupling-constant dependence not obtainable in Ginzburg–Landau theory. This discrepancy is attributed by the authors entirely to the retardation of the electron–phonon interaction. The results of the present study show that one cannot blame the retardation as such as that at least in the case of the kinetic coefficients its inclusion does not lead to any departure from Ginzburg–Landau theory.

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