A complete system of estimates of the minimal Rosseland mean free path of photons on the basis of sum rules

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On the basis of quantum-mechanical sum rules for oscillator strengths we construct a complete system of estimates for the minimal Rosseland mean free path of photons l_R for a plasma, under conditions of local thermodynamic equilibrium at arbitrary ion distribution in degree of ionization and state of excitation. For the case of a lead plasma compare the system of minimal estimates with the estimates of real values of l_R.

INTRODUCTION

The correct treatment of radiative transfer in a plasma is of great importance in the hydrodynamics of high-temperature phenomena. For a plasma transparent to its own radiation (that is to say, for low densities) one must take into account bulk radiative energy losses. If the plasma is opaque or transparent in some range of radiation frequencies (the spectral path of the photons is less than the characteristic dimension of the plasma), the equations of thermodynamics should be supplemented by the equations describing transfer of radiant energy. This type of problem arises, for example, in the description of targets for inertially confined thermonuclear fusion.[1] Under the typical conditions needed to initiate thermonuclear fusion, the plasma density in the target shell must be hundreds of times higher than normal, and the temperatures in the central regions of a hydrogen fuel of the order of 1 keV. For such conditions we need to know the spectral path in an inhomogeneous dense plasma, which in general is not an ideal gas. In such a plasma conditions of local thermodynamic equilibrium (LTE) are realized between matter and the proper radiation, and therefore the transfer of radiant energy can be viewed in the approximation of radiant heat conductivity. For the description of the interaction between matter and radiation one needs here only the Rosseland mean free path of the photons, which is defined as the integral of the inverse photon absorption cross section weighted with the Rosseland function.[2]

To calculate the spectral dependence of the photon absorption coefficient in a plasma of multiply charged ions of heavy elements, it is necessary to take into account the distribution of the ions in various degrees of ionization, to sum an enormous number of transitions between quantum states, and, finally, to take into account line broadening due to the thermal motion of the ions. For a non-ideal plasma one should, strictly speaking, pass to a description of quantum states not of individual ions but of a group of interacting neighboring ions. For a known degree of ionization of the plasma, one can find with sufficient precision the contributions of bremsstrahlung absorption (free–free transitions) and scattering on free electrons. However, obtaining the contributions from free–bound and bound–bound transitions, including the effects of line broadening

and ion interactions, is still an extremely complex problem (see, for example, Ref. 3, where calculations were performed of the spectral coefficients in the framework of the modified Hartree–Fock–Slater model, with account taken of both the real structure of the excited states of the ions and the line broadening). Along with this it was noted that allowance for the effect of absorption in the lines (bound–bound transitions) is important, since it can result in a significant change of the Rosseland path calculated for an ideal plasma.[4]

A simple method for obtaining estimates of the minimal Rosseland path of photons was proposed in Ref. 5 starting from general principles, without the need for solving the complex spectral problem. On the basis of the "golden sum rule" of Thomas–Reiche–Kuhn for oscillator strengths, we have formulated and solved a variational problem for the determination of the minimal path as a function of just the temperature T and the plasma density ρ.

In this paper we shall use the remaining additional sum rules and obtain for the Rosseland path a complete system of estimates, which gives more realistic values for the minimal Rosseland mean free path.

1. SUM RULES FOR CROSS SECTIONS OF PHOTO PROCESSES

We give here without derivation for the oscillator strengths a sum rule which will be needed in the following (to avoid extra factors in the formulas we make use of the atomic system of units):

$$\int_0^\infty \sigma(\omega)\omega^S d\omega = \frac{2\pi^2}{c} Q_S, \quad S=2,1,0,-1,-2,\dots \quad (1)$$

As is well known, certain sum rules are expressed in terms of averages of quantum mechanical operators in the ground state of the quantum system.[6] The sum Q_0 is given by the number of electrons of the atomic system N: $Q_0 = N$, and the sum Q_{-1} is expressed through the mean square radius of the atom in the ground state:

$$Q_{-1} = (2/3)\overline{r^2}, \quad \text{where} \quad \overline{r^2} = 3\sum_{k=1}^{N}\langle 0|z_k^2|0\rangle,$$

the sum Q_{+1} is expressed through the mean square momentum of the electrons in the ground state:

$$Q_{+1}=(2/3)\overline{p^2}, \quad \overline{p^2}=3\sum_{k=1}^{N}\langle 0|p_{zk}^2|0\rangle.$$

Knowing the kinetic energy of the electrons of the atom $K_0=\overline{p^2}/2$ one can define the sum Q_{+1} as $Q_{+1}=(4/3)K_0$. Using the virial theorem $(E_0=-K_0)$ we express this sum rule through the total energy of the atom E_0:

$$Q_{+1}=-(4/3)E_0.$$

The last nontrivial sum rule with a positive index Q_{+2} is expressed in terms of the density of the electrons at the nucleus $\rho(0)$:

$$Q_{+2}=\frac{4\pi}{3}\sum_{k=1}^{N}\langle 0|\delta(r_k)|0\rangle Z,$$

where Z is the charge of the nucleus. The next positive sum Q_{+3} becomes infinite, which is due to the fact that in the region of high frequencies the photoionization cross section decreases like $\omega^{(-3+\eta)}$, $0<\eta<1$. Similarly all the expressions Q_{+n} with $n>3$ diverge. The sums Q_{-n} for $n\geqslant 2$ are defined by convergent integrals, however there are no simple expressions for them in terms of ground state averages, as was the case for the other moments. However there are *a priori* constraints. Since $\sigma(\omega)$ is by definition a positive function we have the following inequality for the ratio

$$\frac{Q_{S+1}}{Q_S}\bigg/\frac{Q_S}{Q_{S-1}}=\frac{Q_{S+1}Q_{S-1}}{Q_S^2}\geqslant 1.$$

In the following we shall use just the one inequality:

$$Q_{-2}\geqslant\frac{Q_{-1}^2}{Q_0}. \tag{2}$$

For the simplest atom—the hydrogen atom—there exist exact values, which we shall use to normalize the model of the atom considered below:

$$Q_{-2}^H=\frac{9}{2}, \quad Q_{-1}^H=2, \quad Q_0^H=1, \quad Q_{+1}^H=\frac{2}{3}, \quad Q_{+2}^H=\frac{4}{3}.$$

(To convert Q_S from the atomic system of units it is necessary to multiply the result by ϵ_0^S, $\epsilon_0=me^4/\hbar^2$ is the atomic unit of energy.)

It should be recalled that the sum rules that we are using refer to the long wavelength-dipole approximation, which is applicable in the frequency range given by the inequalities:

$$Z^2<\omega\leqslant Z\alpha^{-1},$$

where $\alpha=e^2/\hbar c$ is the fine structure constant. Based on these constraints one can, generally speaking, also indicate the range of temperatures for which the dipole approximation is valid.

2. THE SYSTEM OF ESTIMATES OF THE MINIMAL ROSSELAND MEAN

The Rosseland mean l_R is defined as the integral of the inverse spectral absorption coefficient with weight function $R(u)$:

$$l_R=\int_0^\infty\frac{R(u)du}{k(u)(1-e^{-u})}, \tag{3}$$

where $R(u)=(15/4\pi^4)u^4e^{-u}(1-e^{-u})^{-2}$ is the Rosseland function, $k(u)=\sigma(u)\,n[cm^{-1}]$ is the absorption coefficient per unit volume (the cross section, averaged over the mixture of various atoms and multiplied by the density of the atoms $n[cm^{-3}]$), and $u=\hbar\omega/\theta$ is the dimensionless frequency, $\theta=kT$. The factor $1-e^{-u}$ in the denominator is due to the condition of LTE and is introduced to correct for forced emission. Accordingly we define a modified Rosseland function $\overset{*}{R}(u)$:

$$\overset{*}{R}(u)=R(u)(1-e^{-u})^{-1}$$

and rewrite formula (3) in the form

$$\overset{*}{l_R}=\int_0^\infty\frac{\overset{*}{R}(u)du}{k(u)}. \tag{4}$$

Besides photo-ionization processes (σ_{Comp}) one should take into account in the absorption coefficient processes of Compton scattering of photons by electrons (σ_{ph}). It was shown in Ref. 5 that the corresponding modification of the estimate for the Rosseland mean, in the range of temperatures and densities of interest, is insignificant. Below we shall show that the effective Compton cross section can be taken into account by recalling that in the important for applications region of temperatures and densities one has $\sigma_{ph}\gg\sigma_{Comp}$.

We pose now the following variational problem: let $k(u)$ be an unknown function of frequency determined from the extremum condition of the functional (4) with the imposition of some additional Sth condition (1). In Ref. 5 was obtained the minimal estimate for the "golden sum rule" $(S=0)$, which leads to $l_R^{(0)}\propto\theta$ growing linearly with the temperature. Including the remaining acceptable sum rules we obtain additional powers of the temperature in the estimate of the photon path. And the resultant minimal mean value in the low and high temperature regions turns out to be closer to the real values.

Thus, using in the sum rules (1) the averaging over dimensionless frequency we obtain the following additional conditions:

$$\int_0^\infty k(u)u^S du=\frac{2\pi^2}{c}\frac{Q_S}{\theta^{S+1}}n. \tag{5}$$

The variational problems (4) of interest, after imposition of the sequence of conditions (5), take the form:

$$\overset{*}{l_R^{(S)}}=\int_0^\infty\frac{\overset{*}{R}(u)du}{k_S(u)}+\lambda_S^2\left(\int_0^\infty k_S(u)u^S du-\frac{2\pi^2}{c}\frac{Q_S}{\theta^{S+1}}n\right). \tag{6}$$

TABLE I. The system of roots (8) for Pb, giving the temperatures at which the minimal mean free paths calculated from the sum rules using the atom model (at) and the ion model (ion) become equal, $na_0^3 = 1$, θ[eV], l_R^{KS}[cm] (br—bremsstrahlung).

S	−1	0	1	2	br
$\theta^{(-2,S)}_{\text{at}}$	$2,208\cdot10^2$	$1,820\cdot10^2$	$3,481\cdot10^2$	$2,884\cdot10^2$	$5,773\cdot10^1$
$l_R^{(-2,S)}$	$6,750\cdot10^{-5}$	$8,188\cdot10^{-5}$	$4,283\cdot10^{-5}$	$5,168\cdot10^{-5}$	$2,582\cdot10^{-4}$
$\theta^{(-2,S)}_{\text{at}}$	$3,874\cdot10^4$	$3,193\cdot10^4$	$2,984\cdot10^4$	$3,820\cdot10^3$	$1,528\cdot10^3$
$l_R^{(-2,S)}$	$9,710\cdot10^{-1}$	$1,178$	$1,260$	$9,846$	$2,462\cdot10^1$
—	$\theta^{(-1,S)}_{\text{at}}$	$1,501\cdot10^2$	$4,370\cdot10^2$	$3,153\cdot10^2$	$3,935\cdot10^1$
—	$l_R^{(-1,S)}$	$6,750\cdot10^{-5}$	$6,750\cdot10^{-5}$	$6,750\cdot10^{-5}$	$6,750\cdot10^{-5}$
—	$\theta^{(-1,S)}_{\text{ion}}$	$2,633\cdot10^4$	$2,619\cdot10^4$	$1,765\cdot10^3$	$6,066\cdot10^2$
—	$l_R^{(-1,S)}$	$9,710\cdot10^{-1}$	$9,710\cdot10^{-1}$	$9,710\cdot10^{-1}$	$9,710\cdot10^{-1}$
—	—	$\theta^{(0,S)}_{\text{at}}$	$1,272\cdot10^3$	$4,569\cdot10^2$	$2,303\cdot10^1$
—	—	$l_R^{(0,S)}$	$5,723\cdot10^{-4}$	$2,055\cdot10^{-4}$	$1,036\cdot10^{-5}$
—	—	$\theta^{(0,S)}_{\text{ion}}$	$2,606\cdot10^4$	$4,569\cdot10^2$	$1,342\cdot10^2$
—	—	$l_R^{(0,S)}$	$9,614\cdot10^{-1}$	$1,685\cdot10^{-2}$	$4,952\cdot10^{-3}$
—	—	—	$\theta^{(1,S)}_{\text{at}}$	$1,641\cdot10^2$	$1,588$
—	—	—	$l_R^{(1,S)}$	$9,520\cdot10^{-6}$	$8,917\cdot10^{-10}$
—	—	—	$\theta^{(1,S)}_{\text{ion}}$	$8,011$	$4,004$
—	—	—	$l_R^{(1,S)}$	$9,081\cdot10^{-8}$	$2,268\cdot10^{-8}$
—	—	—	—	$\theta^{(2,S)}_{\text{at}}$	$1,488\cdot10^{-4}$
—	—	—	—	$l_R^{(2,S)}$	$7,091\cdot10^{-24}$
—	—	—	—	$\theta^{(2,S)}_{\text{ion}}$	$1,000$
—	—	—	—	$l_R^{(2,S)}$	$1,768\cdot10^{-10}$

Here S enumerates the permissible values, $S \in -\overline{2,2}$ (see below), and λ_S^2 denotes the Lagrange multiplier for the corresponding variational problem.

The extremum conditions give rise to the following equalities:

$$k_S(u) = \overset{*}{R}{}^{1/2}(u) u^{-S/2}/\lambda_S,$$

$$\lambda_S = \frac{\overset{*}{f}_S \theta^{S+1}}{2\pi^2 Q_S n/c}, \qquad (7)$$

$$\overset{*}{l}_R^{(S)} = \frac{\overset{*}{f}_S^2 \theta^{S+1}}{2\pi^2 Q_S n/c},$$

where

$$\overset{*}{f}_S = \int_0^\infty \overset{*}{R}{}^{1/2}(u) u^{S/2} du.$$

For convenience in going over to other systems of units we give the explicit expression for $\overset{*}{l}_R^{(S)}$:

$$\overset{*}{l}_R^{(S)} = \frac{a_0^{-2}}{2\pi^2\alpha} \frac{\theta^{S+1}}{\epsilon_0^{S+1}} \frac{\overset{*}{f}_S^2}{Q_S n},$$

where $a_0 = \hbar^2/(me^2)$ is the atomic unit of length.

We see from the definition of the coefficient $\overset{*}{f}_S$ that the integral exists only for $S \geqslant -2$. Combining this with the restriction on S that follows from the sum rules, $S < 3$, we find that the system of estimates for the minimal Rosseland mean is the only one possible for those sum rules which are expressible in terms of averages over the ground state of the quantum system. We can obtain closed-form expressions for the integrals f_S:

$$f_S = \int_0^\infty R^{1/2}(u) u^{S/2} du$$

$$= \frac{\sqrt{15}}{2\pi^2} \Gamma\left(3 + \frac{S}{2}\right) \zeta\left(3 + \frac{S}{2}\right) (2^{3+S/2} - 1),$$

where ζ is the Riemann function, Γ the gamma function, and the coefficients $\overset{*}{f}_S$ have to be found numerically.

S	-2	-1	0	1	2
$\overset{*}{f}_S$	1.17	1.76	3.42	7.70	19.27
f_S	0.96	1.63	3.30	7.58	19.11 .

Thus we have obtained a system of estimates for the minimal Rosseland photon mean path, which can be called complete, since we have used all allowed discrete values of the parameter S belonging to the interval $[-2,3)$. Moreover the quantum-mechanical sum rules Q_S are expressible in terms of expectation values of the quantum mechanical operators in the ground state of the quantum system.
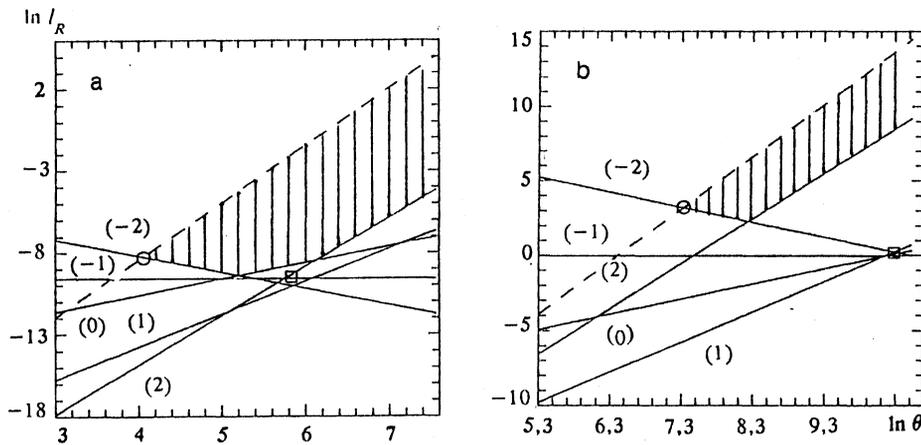
FIG. 1. The Rosseland mean free path in a lead plasma: sum rules calculated for the atom model (a) and ion model (b). The shaded region indicates allowed values of the photon mean free path. The left-most point, from which the maximal degree of ionization can be deduced (see text), and the point at which several criteria of minimality are simultaneously satisfied, are marked.

3. APPLICATION OF THE SYSTEM OF ESTIMATES

For each given temperature θ of the plasma we have a certain series of values for the minimal Rosseland mean path $\overset{*}{l}_R^{(S)}(\theta)$, $S \in -\overline{2,2}$. By choosing the largest among them we obtain the best possible estimate of the minimal path. Exact values for the sums of oscillator strengths are available only for the hydrogen atom and it is possible to indicate the range of the $\theta(S)$ interval in which the corresponding estimate is applicable. For the remaining elements we shall make use of approximate expressions.

The obtained estimates (7) can be applied to both the atomic system, without separating the electrons into bound and free, as well as to the average ion with degree of ionization $\bar{i}(\rho,\theta)$, which is a function of the plasma temperature and density that can be calculated on the basis of various models, for example, Saha–Raĭzer or Thomas–Fermi. In the case of the ion the contribution of the free electrons to the absorption coefficient is found relatively simply. However closed analytic expressions cannot be obtained in this way. Instead we determine the temperature intervals for two extreme cases: either for a non-ionized atom (this provides the lowest possible estimate) or for an ion with maximum degree of ionization (absorption by the K shell). The true value of the root $\theta(S)$ for the given temperature and density is contained somewhere within this interval (see Table I and Fig. 1).

When considering an average ion we determine the average dimensions, the electron kinetic energy and the density by starting from the ionization potentials. We normalize here the sum rules in the simplest way, such that in the passage to the case of the hydrogen atom the corresponding sum rules go over into the known exact values $Q_{(S)}^{H}$.. For each element with charge Z we designate the first and last ionization potentials as $I(1)$ and $I(Z)$. The ionization potential of the hydrogen atom will be denoted by $I(H)$.

The system of sums for the atom takes then the form:

$$Q_{-2} \geqslant \frac{9}{2} \frac{I(H)^2}{I(1)^2} Z^{-1}, \quad Q_{-1} = 2 \frac{I(H)}{I(1)},$$

$$Q_0 = Z, \quad Q_{+1} = \frac{2}{3} Z^{7/3}, \quad Q_{+2} = \frac{4}{3} Z^3;$$

for a fully ionized atom we have:

$$Q_{-2} \geqslant \frac{9}{2} \frac{I(H)^2}{I(Z)^2}, \quad Q_{-1} = 2 \frac{I(H)}{I(Z)}, \quad Q_0 = 1,$$

$$Q_{+1} = \frac{2}{3} \frac{I(Z)}{I(H)}, \quad Q_{+2} = \frac{4}{3} Z^2.$$

The sums Q_{+2} are determined by the powers Z^3 and Z^2, since the probability density of the wave function at zero, $\Psi(0)$, for "heavy" elements is proportional to $\sqrt{Z}$ (Ref. 7, §71). The sum Q_{+1} for the atom is given in terms of the energy of the atom in the Thomas–Fermi model: $E = 16 Z^{7/3}$ [eV].[7] We take the dimensions of the atomic system to be proportional to the inverse of the ionization potential; this leads to the factors $I(Z)/I(H)$ and $I(H)/I(1)$ in the expressions for the sums.

Sorting out pairwise all possible sum rules we determine the temperatures $\theta^{S,K}$, for which the paths determined from different sum rules become equal:

$$\overset{*}{l}_R^{(S)}(\theta) = \overset{*}{l}_R^{(K)}(\theta),$$

$$S < K, \quad S,K \in -\overline{2,2}.$$

(8)

For the two cases of evaluation of the sum rules considered, we give in Table I the corresponding system of roots (8) for Pb. Of course, for heavy elements the scatter in the values of the roots is very large and a universal prescription for the determination of the minimal Rosseland mean cannot be given. Nevertheless, as can be seen from the examples below, for a non-ideal plasma the table for the atoms of lead can give path values differing from the exact one by no more than a factor of two. A qualitative picture of the location of the roots is given by Fig. 1, which shows $\ln l_R$ as a function of $\ln \theta$ for a neutral and for a fully ionized lead atom. To each estimate of the minimal mean value there corresponds the straight line with the appropriate slope.

It is interesting to note that for the unionized atomic system the estimate of $\overset{*}{l}_R^{(-1)}$ contains the square of the size of the atomic system $\bar{r}^2$, which, naturally, is independent of the degree of ionization and, consequently, of the temperature. This is also in agreement with the behavior $\overset{*}{l}_R^{(-1)} \propto \theta^0$ found for this estimate from the variational problem.

The estimated $\overset{*}{l}_R^{(-1)}$ is represented in Fig. 1 by a straight

TABLE II. Comparison of the Rosseland mean free paths for a lead plasma at $\theta = 300$ eV.

ρ, g·cm^{-3}	$1\cdot10^{-5}$	$1\cdot10^{-2}$	$1\cdot10^{-1}$	1	10	$1\cdot10^{2}$	$1\cdot10^{3}$
i	55,3	49,4	43,6	37	30,2	20	21,9
I_i/θ	17,0	10,1	7,9	5,8	3,7	1,8	2,1
l_R^{min}, cm	38	$3,2\cdot10^{-2}$	$2,7\cdot10^{-3}$	$2,3\cdot10^{-4}$	$2,0\cdot10^{-5}$	$1,7\cdot10^{-6}$	$1,7\cdot10^{-7}$
l_R^{dp}, cm	$5,6\cdot10^{2}$	$2,2\cdot10^{-1}$	$1,4\cdot10^{-2}$	$8,3\cdot10^{-4}$	$5,2\cdot10^{-5}$	$3,6\cdot10^{-6}$	$3,8\cdot10^{-7}$
l_R^{ZR}, cm	$2,8\cdot10^{7}$	36	$5,1\cdot10^{-1}$	$8,5\cdot10^{-3}$	$1,5\cdot10^{-4}$	$5,0\cdot10^{-6}$	$5,6\cdot10^{-7}$
$l_R^{(-2)}$, cm	$8,9\cdot10^{5}$	$3,8\cdot10^{2}$	$2,7\cdot10^{1}$	1,7	$8,1\cdot10^{-2}$	$2,3\cdot10^{-3}$	$3,0\cdot10^{-4}$
$l_R^{(-1)}$, cm	$5,0\cdot10^{3}$	3,0	$2,3\cdot10^{-1}$	$1,7\cdot10^{-2}$	$1,1\cdot10^{-3}$	$5,3\cdot10^{-5}$	$6,2\cdot10^{-6}$
$l_R^{(0)}$, cm	$4,1\cdot10^{1}$	$3,4\cdot10^{-2}$	$2,9\cdot10^{-3}$	$2,5\cdot10^{-4}$	$2,1\cdot10^{-5}$	$1,8\cdot10^{-6}$	$1,8\cdot10^{-7}$
$l_R^{(+1)}$, cm	$4,4\cdot10^{1}$	$2,7\cdot10^{-2}$	$1,9\cdot10^{-3}$	$1,3\cdot10^{-4}$	$9,3\cdot10^{-6}$	$6,1\cdot10^{-7}$	$6,6\cdot10^{-8}$
$l_R^{(+2)}$, cm	$1,8\cdot10^{1}$	$1,5\cdot10^{-2}$	$1,2\cdot10^{-3}$	$1,1\cdot10^{-4}$	$9,2\cdot10^{-6}$	$7,7\cdot10^{-7}$	$7,9\cdot10^{-8}$

line parallel to the ln θ axis. Since for the temperatures of interest in applications atomic-system dimensions exceed the characteristic dimensions of the Compton scattering process ($\bar{r}_{at} \gg \bar{r}_{Comp}$), we in effect take into account also the scattering processes by considering the region above the straight line $l_R^{*(-1)}$. That is to say, we need not complicate the formulation of the variational problem as was done in Ref. 5.

The dashed line in the figure shows the mean path for the inverse bremsstrahlung process (free–free transitions, quasiclassical or Kramers approximation) with maximal degree of ionization, for $na_0^3 = 1$:

$$l_{br}^* = \frac{3\sqrt{3}}{4(2\pi)^{5/2}} \frac{f_{br}^*}{a_0^5 n^2} \frac{\theta^{7/2}}{\epsilon_0^{7/2} Z^3}, \qquad (9)$$

where $f_{br}^* = \int_0^\infty R^{*1/2}(u) u^3 du$.

Since the atom density enters the formula (7) linearly and the formula (9) quadratically, they must be compared more accurately. Here we note only that the highest power of the temperature in the system of estimates (7) is θ^3, while the temperature enters formula (9) as $\theta^{3.5}$, therefore for any density ρ there exists a temperature starting with which the bremsstrahlung processes dominate.

The dashed line of the bremsstrahlung processes acts as an upper bound of the Rosseland mean, since in the absorption coefficient for free–free transitions only positive terms should be added, that is, the real value of l_R lies below the straight line of the bremsstrahlung processes. The region in which the Rosseland mean should lie is shown hatched in the figure.

If the atom is not fully ionized the dashed line will move upward from the straight line with maximal degree of ionization l^*(br), and the extreme left point in the region of allowed values of l_R will move to the left in the direction of lower temperatures. Conversely, under conditions of applicability of the dipole approximation one can, given the temperature, determine from a known l_{min} the maximal possible degree of ionization of the matter.

Comparing the graphs for the neutral and the maximally ionized atom we see that the region of allowed values of the minimal Rosseland mean is shifted in the direction

of higher temperatures, and consequently in the direction of longer paths. In addition, it should be noted that there exists a temperature region, particularly noticeable for light elements and for maximally ionized atoms, where several estimates for the minimal Rosseland mean come close together. For this temperature region the minimal estimate is, apparently, close to the real value of the photon path. In this region the atom is, as it were, "blackest," since several criteria of minimality are satisfied at the same time.

The straight line corresponding to the sum Q_{-2} is determined by the inequality (2) and is therefore somewhat less rigorous. Unfortunately the sum Q_{-2} is bounded from below, that is, we cannot say that the real value l_R lies strictly above this line. In that sense the straight line l_R^{*-2} should be viewed as tentative. On the whole, however, the system of estimates correctly duplicates the behavior of the Rosseland mean as a function of temperature.

In conclusion we examine how our system of estimates relates to existing calculational data for the Rosseland mean. We turn to the example of a lead plasma, which was considered in Ref. 5, where a table was given comparing the Rosseland paths with calculations: a) by the Zel'dovich–Raĭzer formula,[8] valid for a rarefied plasma:

$$l_R^{ZR} = 2 \cdot 10^{-3} A(\theta^2[\text{keV}]/\rho[\text{g}\cdot\text{cm}^{-3}])(\bar{i}+1)^{-2} \exp(I_{\bar{i}}/\theta);$$

b) by an estimate formula for the real path [formula (32), Ref. 5]:

$$l_R^{dp} = 0.4 \ A(\theta^3[\text{keV}]/\rho[\text{g}\cdot\text{cm}^{-3}])$$
$$\times a^{-1}(210 + 5b(I_{\bar{i}}/\theta)^2 + c/4(I_{\bar{i}}/\theta)^4),$$

where a, b, and c are approximation coefficients,

$$a = \frac{16\pi}{3^{3/2}} Z^4, \quad c = b+3,$$

$$b = \frac{4}{3^{3/2}\pi} \frac{Z^4}{Z-\bar{i}} \left(\frac{\epsilon_0}{I_{\bar{i}}}\right)^2 \ln\left[\frac{Z^4}{Z-\bar{i}}\left(\frac{\epsilon_0}{I_{\bar{i}}}\right)^2\right]$$

TABLE III. The Rosseland mean free path for a lead plasma at a density $\rho = 11.34$ g·cm^{-3}.

θ, eV	100	316,2	750	1000
$\bar{\imath}$	18,57	34,26	47,55	53,65
I_t/θ	4,56	4,25	3,54	3,30
l_R^{BL}, cm	$1,15\cdot10^{-5}$	$5,20\cdot10^{-5}$	$1,84\cdot10^{-4}$	$3,85\cdot10^{-4}$
l_R^{dp}, cm	$1,73\cdot10^{-5}$	$5,17\cdot10^{-5}$	$1,22\cdot10^{-4}$	$1,39\cdot10^{-4}$
$l_R^{(0)}$, cm	$5,10\cdot10^{-6}$	$2,14\cdot10^{-5}$	$7,03\cdot10^{-5}$	$1,14\cdot10^{-4}$
$l_R^{(1)}$, cm	$5,64\cdot10^{-6}$	$1,09\cdot10^{-5}$	$1,32\cdot10^{-4}$	$3,69\cdot10^{-4}$
$l_R^{(2)}$, cm	$2,44\cdot10^{-7}$	$1,03\cdot10^{-5}$	$1,90\cdot10^{-4}$	$5,46\cdot10^{-4}$
l_R^{ZR}, cm	$9,12\cdot10^{-5}$	$2,07\cdot10^{-4}$	$3,10\cdot10^{-4}$	$3,33\cdot10^{-4}$

c) and, finally, by a formula for the minimal Rosseland path [formula (21), Ref. 5], in which in addition to the estimate of $l_R^{*(-1)}$ the electrons were separated into free and bound:

$$l_R^{\min} = 1.65\cdot10^{-4}\, A(\theta[\text{keV}]/\rho[\text{g}\cdot\text{cm}^{-3}])(Z-\bar{\imath})^{-1},$$

where A is the atomic number of the element.

We shall utilize the data from Ref. 5 for the degree of ionization $\bar{\imath}$ of the lead plasma, at temperature $\theta = 300$ [eV], for a wide range of densities, and complement the table of Ref. 5 by the system of estimates (7), specifying the sums Q_S for an ion of multiplicity $\bar{\imath}$ by the following expressions:

$$Q_{-2} \geqslant \frac{9}{2}\frac{I(\text{H})^2}{I(\bar{\imath})^2}(Z-\bar{\imath})^{-1}, \quad Q_{-1} = 2\frac{I(\text{H})}{I(\bar{\imath})}, \quad Q_0 = Z-\bar{\imath},$$

$$Q_{+1} = \frac{2}{3}(Z-\bar{\imath})^{7/3}, \quad Q_{+2} = \frac{4}{3}(Z-\bar{\imath})Z^2,$$

(see Table II).

As is well known, in the case of a rarefied plasma the Zel'dovich–Raĭzer formula is applicable for l_R^{ZR}. The data of Table II show the closest to this estimate are: $l_R^{(-2)}(\rho = 10^{-5})$ and $l_R^{(-1)}(\rho = 10^{-2}, \rho = 10^{-1})$. Upon moving in the direction of higher densities the estimates of $l_R^{(-2)}$ and $l_R^{(-1)}$ no longer apply since the simple model for the calculation of the sum rules Q_{-2} and Q_{-1} is too crude. In a dense plasma, for which the criterion of applicability for the estimates of $l_R^{\min}$ and l_R^{dp} given in Ref. 5 is satisfied, we have:

$$I_{\bar{\imath}}/\theta \leqslant 5,$$

as can be seen from that same Table II, we can use the estimates $l_R^{(0)}$, $l_R^{(1)}$, and $l_R^{(2)}$.

We discuss one more example. In Ref. 9 were given results of the calculations of paths in a plasma of gold and lead, under conditions characteristic of compressed shells of targets for inertial thermonuclear fusion. In Tables III and IV these data are denoted by l_R^{BL}. Values of minimal mean values closest to the calculated ones are underlined. For finding the Rosseland mean in Table IV use was made of the simplest model for finding the sum of the oscillator strengths, in which the electrons are not separated into bound and free. As already noted above, the discrepancy from the "exact" value for the photon path does not exceed a factor of ~ 2. Calculations show that another model—the ion model—gives good results only for light elements.

The data obtained by us show that as one progresses to increasing densities, it is necessary each time to pass to the corresponding estimate dictated by the diagram of allowed values. Since the system of estimates reproduces qualitatively correctly the dependence of the absorption coefficient on the temperature in the entire range of temperatures, the system of estimates (7) can be effectively for calculations of radiative gasodynamics, using only the data on the degree of ionization $\bar{\imath}(\theta)$.

CONCLUSION

To supplement the estimate of the minimal Rosseland mean obtained from the "golden sum rule" (the sum Q_0 is a function growing linearly with the temperature),[5] we have obtained a complete system of estimates (the remaining sum rules give rise to divergent expressions), which reproduces qualitatively correctly the behavior of the spectral absorption curve in the whole range of temperatures, under the condition that the dipole approximation holds for the radiation.

TABLE IV. The Rosseland mean free path for a lead plasma at a density $\rho = 11.34$ g·cm^{-3} in the atom model.

θ, eV	100	316,2	750	1000
l_R^{BL}, cm	$1,15\cdot10^{-5}$	$5,20\cdot10^{-5}$	$1,84\cdot10^{-4}$	$3,85\cdot10^{-4}$
$l_R^{(-2)}$, cm	$1,31\cdot10^{-5}$	$4,13\cdot10^{-6}$	$1,74\cdot10^{-6}$	$1,31\cdot10^{-6}$
$l_R^{(-1)}$, cm	$5,91\cdot10^{-6}$	$5,91\cdot10^{-6}$	$5,91\cdot10^{-6}$	$5,91\cdot10^{-6}$
$l_R^{(0)}$, cm	$3,94\cdot10^{-6}$	$1,25\cdot10^{-5}$	$2,96\cdot10^{-5}$	$3,94\cdot10^{-5}$
$l_R^{(1)}$, cm	$1,14\cdot10^{-6}$	$3,60\cdot10^{-5}$	$4,80\cdot10^{-4}$	$1,14\cdot10^{-3}$
$l_R^{(2)}$, cm	$1,89\cdot10^{-7}$	$5,97\cdot10^{-6}$	$7,96\cdot10^{-5}$	$1,89\cdot10^{-4}$

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