Quantum states with minimum phase uncertainty

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The problem of the quantum states of the harmonic oscillator is solved, yielding the minimum value of the uncertainty in the phase $\Delta\hat{\phi}$ of the oscillations for specified values of the uncertainty $\Delta\hat{N}$ in the number of quanta and the average number $\langle\hat{N}\rangle$ of quanta. It is shown that the "standard" form of the uncertainty principle relating the number of quanta and the phase, $\Delta N\Delta\hat{\phi}\geqslant 1/2$, breaks down both for small values in the uncertainty of the number of quanta, $\Delta\hat{N}\lesssim 1$, and for sufficiently large values, $\Delta\hat{N}\gg\langle\hat{N}\rangle$. An explicit form is found for the quantum states of the oscillator which realizes the minimum value in the uncertainty of the oscillation phase for a given average number of quanta. It is shown that for $\langle\hat{N}\rangle\gg 1$ the phase uncertainty satisfies $\Delta\hat{\phi}\geqslant 1.376.../\langle\hat{N}\rangle$.

1. INTRODUCTION

The uncertainty principle in terms of the number of quanta and the phase of a harmonic oscillator continues to be the object of controversy at the present time. It is well known that the "standard" form

$$\Delta\hat{N}\Delta\hat{\phi}\geqslant 1/2 \tag{1}$$

(here $\Delta\hat{N}$ is the uncertainty in the number of quanta and $\Delta\hat{\phi}$ is the phase uncertainty) is incorrect, generally speaking, if only because the uncertainty in the phase is bounded by 2π and for $\Delta\hat{N}\to 0$ the inequality (1) manifestly fails.

Carruthers and Nieto[1] and Jackiw[2] have made substantial progress in understanding this problem. However, because of an inadvertent oversight the "states with minimum product of the uncertainties" obtained in these works (an experiment was recently proposed to achieve these states[3]) are not actually as claimed (see below, Sec. 6).

The question also remains open as to what the minimum value of the uncertainty in the phase of a harmonic oscillator can be for a given value of the average number of quanta, and what states achieve this minimum. This question has recently become pressing in connection with the development of laser gravitational antennas, for which it is necessary to find the smallest possible phase shift of an electromagnetic wave whose power is limited.

The purpose of the present work is to study the uncertainty principle in terms of the number of quanta and the phase, and also the oscillator states with minimum phase uncertainty.

2. THE OBSERVABLE PHASE

It is well known that the definition of the observable phase in quantum mechanics encounters great difficulty (see, e.g., Ref. 1). To overcome these difficulties we will use the method proposed in Refs. 4 and 5.

We define an operator $\hat{E}_-$ on a finite-dimensional space **D** with a number of measurements M as follows (here $|n\rangle$ is the eigenvector of the operator $\hat{N}$ for the number of quanta):

$$\hat{E}_-|n\rangle=\begin{cases}|n-1\rangle, & n>0,\\ |M-1\rangle, & n=0.\end{cases}$$

The Hermitian conjugate of the operator $\hat{E}_-$ is the operator $\hat{E}_+$, defined so that

$$\hat{E}_+|n\rangle=\begin{cases}|n+1\rangle, & n<M-1,\\ |0\rangle, & n=M-1,\end{cases}$$

so that we have $\hat{E}_-\hat{E}_+=\hat{E}_+\hat{E}_-=\hat{I}$ ($\hat{I}$ is the unit operator). Consequently, the operator $\hat{E}_-$ is unitary and the self-adjoint phase operator $\hat{\phi}_M$ on the finite-dimensional space **D** with M measurements can be defined as $\hat{E}_-=\exp[i\Phi_M]$ (compare Refs. 4, 5), where

$$\hat{\Phi}_M=\sum_{\{\theta_k\}}\theta_k|\theta_k\rangle\langle\theta_k|. \tag{2}$$

Here θ_k are a uniform covering of the integral $[-\pi,\pi)$ with lengths $2\pi/M$; $\{\theta_k\}$ signifies that the summation in (2) is over all θ_k, and $|\theta_k\rangle$ is given by

$$|\theta_k\rangle=\frac{1}{\sqrt{M}}\sum_{n=0}^{M-1}(e^{-i\theta_k})^n|n\rangle,\quad \langle\theta_n|\theta_m\rangle=\delta_{nm}. \tag{3}$$

The operator $\hat{\Phi}_M$ is convenient to use as a phase operator, as was done in Ref. 5, since

1) The operator $\hat{\Phi}_M$ is self-adjoint;

2) the energy of an arbitrary real physical system is always bounded (e.g., the total energy of the universe). We can therefore always describe such a system by a vector in the finite-dimensional space **D** with a large but finite number of measurements M. In our case, when the system is the harmonic oscillator, it suffices to require $M\gg\langle\hat{N}\rangle+\Delta\hat{N}$, where $\langle\hat{N}\rangle$ is the average number of quanta and $\Delta\hat{N}=[\langle N^2\rangle,\langle\hat{N}\rangle^2]^{1/2}$. Moreover, the number M can generally be omitted from the final formulas, since for an arbitrary kth power of the operator $\hat{\Phi}_M$ there exists a weak limit as $M\to\infty$, which we will denote by $\hat{\Phi}^k$. This means that

$$\lim_{M\to\infty}\langle f|\hat{\Phi}_M^k|g\rangle=\langle f|\hat{\Phi}^k|g\rangle$$

for any two vectors $|f\rangle$ and $|g\rangle$ (see Appendix A).

3. THE CONCEPT OF A "STATE WITH MINIMUM UNCERTAINTY"

The commonly employed definition is something like this: "States with minimum uncertainty are those in which the product $\Delta\hat{A}\Delta\hat{B}$ of the uncertainties of two noncommuting observables $\hat{A}$ and $\hat{B}$ is a minimum." Obviously the desire to minimize specifically the product of the uncertainties is a reflection of the Heisenberg uncertainty principle (if $[\hat{A},\hat{B}]=i\hat{C}$, then $\Delta\hat{A}\Delta\hat{B}\geqslant|\langle\hat{C}\rangle|/2$). A different formulation of the problem appears physically more natural: for a given quantity $\Delta\hat{B}$ (e.g., imposed by the possibilities of a device) it is required to find the minimum value of $\Delta\hat{A}$. Or, in a more general sense, states with minimum uncertainty are those in which the quantity $\Delta\hat{A}$ is a minimum for given values of certain other parameters characterizing our system (for example, $\langle\hat{B}\rangle,\Delta\hat{B},\langle\hat{A}\rangle$). As shown by Kulaga,[6] in minimizing $\Delta\hat{A}$ for different prescribed values of $\Delta\hat{B}$ we are required to find solutions that minimize the product $\Delta\hat{A}\Delta\hat{B}$ of the uncertainties. When we do this it is also possible to obtain solutions that cannot be obtained by minimizing the product of the uncertainties directly. In the present work we will therefore take states with minimum uncertainty to be those which have a minimum phase uncertainty for given values of the average number $\langle\hat{N}\rangle$ of quanta and the uncertainty $\Delta\hat{N}$ of the number of quanta.

4. UNCERTAINTY RELATION IN TERMS OF THE PHASE AND THE NUMBER OF QUANTA

It is always possible to choose the zero of phase so that $\langle\hat{\Phi}\rangle$ is equal to zero. We will therefore look for the states in which the quantity $\langle\hat{\Phi}^2\rangle$ is a minimum for given $\langle\hat{N}\rangle$ and $\Delta\hat{N}$ (i.e., for given values of $\langle\hat{N}^2\rangle$ and $\langle\hat{N}\rangle$ plus the normalization condition $\langle\Psi|\Psi\rangle=1$). Using the method of Lagrange multipliers, we will seek the minimum of the functional

$$U=\langle\Psi|\hat{\Phi}^2|\Psi\rangle+\lambda_2\langle\Psi|\hat{N}^2|\Psi\rangle+\lambda_1\langle\Psi|\hat{N}|\Psi\rangle +\lambda_0\langle\Psi|\Psi\rangle\cdot$$

Here $\lambda_0,\lambda_1,\lambda_2$ are Lagrange multipliers.

Starting from the condition $\delta U=0$, after varying the vector $|\Psi\rangle$ (that is, $|\Psi\rangle\to|\Psi\rangle+|\delta\Psi\rangle$) and separating the terms that are linear in $|\delta\Psi\rangle$, we find the equation

$$[\hat{\Phi}^2+\lambda_2\hat{N}^2+\lambda_1\hat{N}+\lambda_0]|\Psi\rangle=0. \quad (4)$$

Unfortunately, Eq. (4) cannot be solved in general. However, we can get an overview by considering two cases:

1) the states with minimum uncertainty have an average number of quanta of order unity;

2) the average number of quanta and the uncertainty in the number of quanta are much greater than unity.

4.1. Average Number of Quanta of Order Unity

In this case the parameters of states with minimum uncertainty were found by solving Eq. (4) numerically. The results are shown in Fig. 1. From the figure we see that the relation $\Delta\hat{N}\Delta\hat{\Phi}=1/2$ holds for states with minimum uncertainty only in the range $0.5<\Delta\hat{N}<\langle\hat{N}\rangle$. In the limit $\Delta\hat{N}\to0$ the dispersion in the phase approaches the asymp-

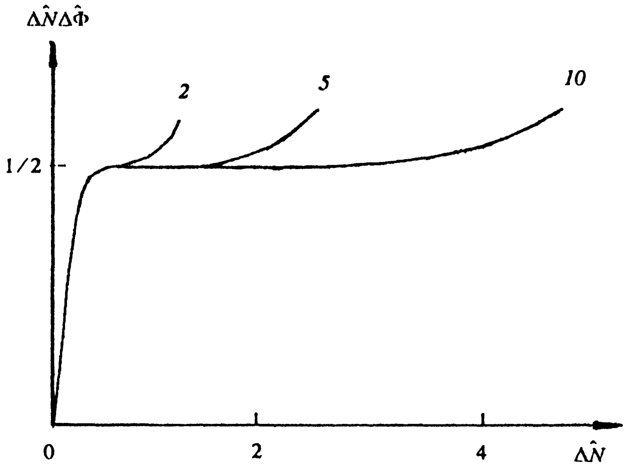

FIG. 1. The product $\Delta\hat{N}\Delta\hat{\Phi}$ for the states with minimum uncertainty as a function of the average number of quanta $\langle\hat{N}\rangle$ and the uncertain $\Delta\hat{N}$ in the number of quanta. The numbers at the top refer to the average number of quanta for each curve.

totic value $\pi^2/3$, which corresponds to the dispersion of a random variable uniformly distributed in the interval $(-\pi,\pi)$. Then the uncertainty relation assumes the form

$$\Delta\hat{N}\Delta\hat{\Phi}=\frac{\pi}{\sqrt{3}}\Delta\hat{N}$$

(the initial linear portion of the plot in Fig. 1).

For large values of $\Delta\hat{N}$, comparable with $\langle\hat{N}\rangle$, the minimum value of the product of the uncertainties becomes larger than 1/2 (see Sec. 4.2 for further details). For every specified $\langle\hat{N}\rangle$ there exists a maximum value $\Delta\hat{N}$ for which Eq. (4) still has a solution (the rightmost extremities of the curves in Fig. 1). These solutions determine the states with minimum phase uncertainty for given $\langle\hat{N}\rangle$ (see Sec. 5).

4.2 Average Number of Quanta Much Greater than Unity

In this case Eq. (4) reduces to an ordinary differential equation. Let

$$|\Psi\rangle=\sum_{n=0}^{\infty}c_n|n\rangle.$$

For $\Delta\hat{N}\gg1$ we can assume that the distribution over the number of quanta is continuous, $c_n=c(n)$, where n is taken to vary continuously from 0 to $+\infty$. Then we obtain the following equation for the function $c(n)$ (see Appendix B):

$$-\frac{d^2}{dn^2}c(n)+(\lambda_2n^2+\lambda_1n+\lambda_0)c(n)=0,$$

with boundary conditions $c(0)=0$, $c(n)|\xrightarrow[n\to\infty]{}0$. After a change of variables,

$$n=\alpha(x-\beta),\quad c(n(x))=\psi(x),$$

where $\alpha=\lambda_2^{-1/4},\quad \beta=\frac{1}{2}\lambda_1\lambda_2^{-3/4},$

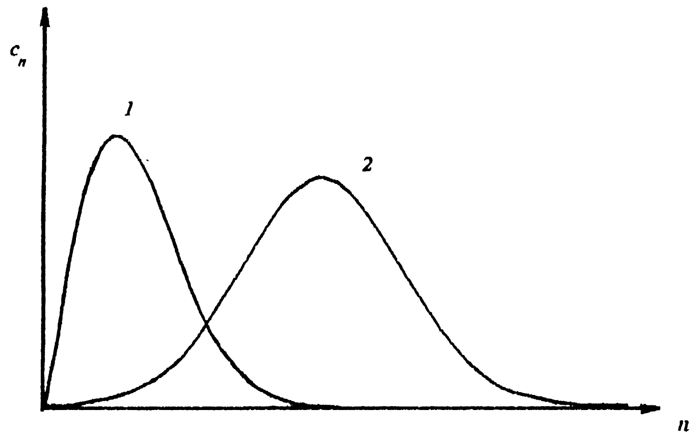

FIG. 2. The form of the distribution over the number of quanta for the states with minimum uncertainty: 1) $\Delta\hat{N}/\langle\hat{N}\rangle \approx 0.42$; 2) $\Delta\hat{N}/\langle\hat{N}\rangle \approx 0.2$. State 2 is essentially the same as a Gaussian distribution.

this equation is reduced to a form analogous to the Schrödinger equation for the harmonic oscillator:

$$\left[-\frac{d^2}{dx^2}+x^2\right]\psi(x)=2\varepsilon\psi(x) \tag{5}$$

with boundary conditions

$$\psi(\beta)=0, \quad \psi(x)\,|\xrightarrow[x\to\infty]{} 0. \tag{6}$$

Equation (5) has solutions for arbitrary ε.[7,8] The solution of Eq. (5) satisfying the boundary condition (6) has the form

$$\psi(x)=\psi_0 H_\nu(x)\exp(-x^2/2), \quad \nu=\varepsilon-1/2, \tag{7}$$

Here ψ_0 is a normalization factor, $H_\nu(x)$ is a Hermite function,[7] and the quantity ν is chosen so as to satisfy the boundary condition (6), $\psi(\beta)=0$. In this case, for $\nu=0,1,2,\ldots$, the Hermite function coincides with the Hermite polynomials $H_0=1$, $H_1=2x,\ldots$. When $\Delta\hat{N}/\langle\hat{N}\rangle \ll 1$ holds we have $\nu \ll 1$ and $H_\nu(x)$ is essentially equal to unity wherever $\exp(-x^2/2)$ is very different from zero. The wave function of the state with minimum uncertainty is therefore almost indistinguishable from a Gaussian distribution:

$$c(n)=\frac{1}{[2\pi(\Delta\hat{N})^2]^{1/4}}\exp\left[-\frac{(n-\langle\hat{N}\rangle)^2}{4(\Delta\hat{N})^2}\right]. \tag{8}$$

This state corresponds to a product of uncertainties given by $\Delta\hat{N}\Delta\hat{\Phi}=1/2$. When $\Delta\hat{N}/\langle\hat{N}\rangle \sim 1$ holds the function $H_\nu(x)$ differs considerably from unity and the wave function (7) is deformed (Fig. 2).

Figure 3 displays the quantity $\Delta\hat{N}\Delta\hat{\Phi}$ as a function of $\Delta\hat{N}/\langle\hat{N}\rangle$ for the state (7), found by numerical solution. It is clear that up to values $\Delta\hat{N}/\langle\hat{N}\rangle \approx 0.3$ the relation $\Delta\hat{N}\Delta\hat{\Phi}=1/2$ is essentially valid, and consequently, up to these values we can use the wave function (8) for the states with minimum uncertainty.

5. STATES WITH MINIMUM PHASE UNCERTAINTY FOR A GIVEN AVERAGE NUMBER OF QUANTA

The equation for the state that minimizes the phase uncertainty for a given average number of quanta is obtained from Eq. (4) by dropping the term $\lambda_2\hat{N}^2$:

$$[\hat{\Phi}^2+\lambda_1\hat{N}+\lambda_0]\,|\Psi\rangle=0, \quad |\Psi\rangle=\sum_{n=0}^{\infty} c_n|n\rangle. \tag{9}$$

Figure 4 displays a plot of the product $\langle\hat{N}\rangle\Delta\hat{\Phi}$ as a function of $\langle\hat{N}\rangle$, obtained by solving this equation numerically. Here we can distinguish two characteristic regimes for the values of $\langle\hat{N}\rangle$:

1) for small values of $\langle\hat{N}\rangle$ the function is almost linear, i.e., the phase uncertainty is almost independent of the average number of quanta and is close to $\pi/\sqrt{3}$;

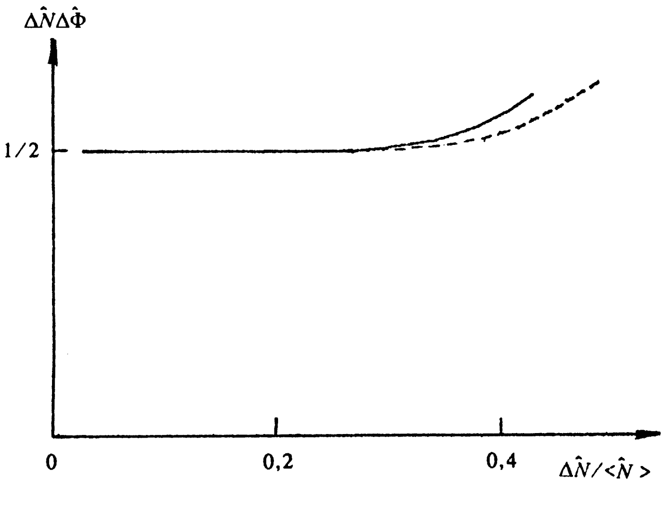

FIG. 3. The product $\Delta\hat{N}\Delta\hat{\Phi}$ of the uncertainties for the states $H_\nu(x)\exp(-x^2/2)$ as a function of the quantity $\Delta\hat{N}/\langle\hat{N}\rangle$. The broken trace represents this function for the states with $\langle\hat{N}\rangle=10$ (see Sec. 4.1).

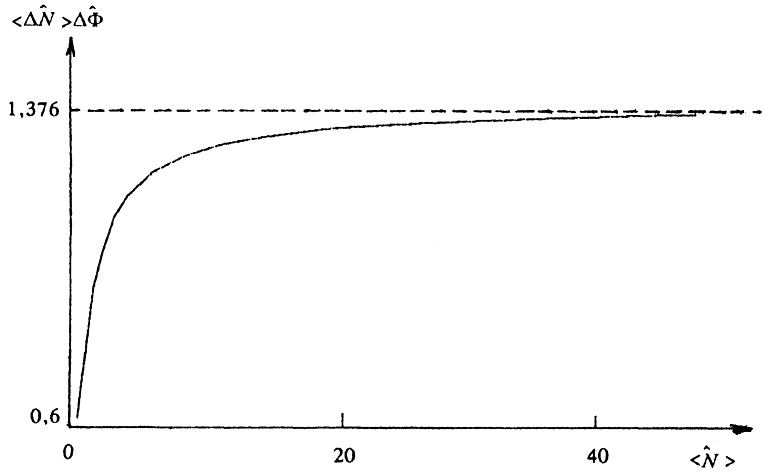

FIG. 4. The minimum possible product $\langle\hat{N}\rangle\Delta\hat{\Phi}$ for given $\langle\hat{N}\rangle$. It is clear that as $\langle\hat{N}\rangle$ increases the curve approaches the asymptote $\langle\hat{N}\rangle\Delta\hat{\Phi}\approx 1.376$.

2) for large values of $\langle\hat{N}\rangle$ the product $\langle\hat{N}\rangle\Delta\hat{\Phi}$ approaches the asymptotic value 1.376... . This limiting case can be solved analytically.

Of course, if the average number of quanta is large enough, then Eq. (9) can be replaced by an ordinary differential equation for the function $c(n)$, similarly to the procedure followed in Sec. 4.2:

$$-\frac{d^2}{dn^2}c(n)+(\lambda_1 n+\lambda_0)c(n)=0 \tag{10}$$

with boundary conditions $c(0)=0$, $c(n)|\xrightarrow[n\to\infty]{} 0$.

The solution of this equation which is bounded as $n\to\infty$ takes the form

$$c(n)=C\,\mathrm{Ai}(\lambda_1^{1/3}n-\lambda_0\lambda_1^{2/3}), \tag{11}$$

where $\mathrm{Ai}(x)$ is the Airy function of the first kind and C is a normalization constant. The values of the parameters λ_0,λ_1 are determined by the boundary conditions and the prescribed average value $\langle\hat{N}\rangle$ of the number of quanta. Calculating them (using the formulas given in Ref. 10) and substituting in Eq. (11) we find

$$c(n)=\left(\frac{2|s_1|}{3\langle\hat{N}\rangle}\right)^{1/2}\frac{1}{\mathrm{Ai}'(s_1)}\mathrm{Ai}\left(|s_1|\left(\frac{2}{3}\frac{n}{\langle\hat{N}\rangle}-1\right)\right), \tag{12}$$

where $s_1=-2.338...$ is the first zero of the Airy function and $\mathrm{Ai}'(s_1)$ is the derivative of the Airy function at the point s_1.

The uncertainty in the number of quanta in the state (12) is equal to

$$\Delta\hat{N}=\left(\int_0^\infty n^2c^2(n)dn-\langle\hat{N}\rangle^2\right)^{1/2}=\frac{\langle\hat{N}\rangle}{\sqrt{5}},$$

and the phase uncertainty is

$$\Delta\hat{\Phi}=\left\{\int_0^\infty\left[\frac{dc(n)}{dn}\right]^2dn\right\}^{1/2}$$

$$=\frac{1}{\langle\hat{N}\rangle}\left(\frac{4|s_1|^3}{27}\right)^{1/2}\approx\frac{1.376...}{\langle\hat{N}\rangle}.$$

Consequently, the uncertainty relation for this class of quantum states takes the form

$$\Delta\hat{N}\Delta\hat{\Phi}=\left(\frac{4|s_1|^3}{135}\right)^{1/2}\approx 0.615... .$$

6. STATES WITH MINIMUM UNCERTAINTY OBTAINED IN REFS. 1 AND 9

Carruthers and Nieto[1] reported the following states with minimum uncertainty for the operators $\hat{S}$ and $\hat{N}$:

$$|\gamma,\lambda\rangle=\nu\sum_{n=0}^\infty c_n|n\rangle,\quad c_n=I_{n-\lambda}(\gamma) \tag{13}$$

with the boundary condition

$$c_{-1}=0,\quad (I_{-1-\lambda}(\gamma)=0). \tag{14}$$

Here ν is a normalization factor, we have written $\lambda=\langle\hat{N}\rangle$ and $\gamma=\Delta\hat{N}/\Delta\hat{S}$, and $I_\mu(\gamma)$ is a modified Bessel function of the first kind of order μ. As shown in Appendix C,

$$I_{n-\lambda}(\gamma)\approx\mathrm{const}\cdot\exp\left[-\frac{(n-\lambda)^2}{2\gamma}\right],$$

for $\gamma=\Delta\hat{N}/\Delta\hat{S}\gg 1$.

Thus, in the case $\Delta\hat{N}\gg 1$ the states (13) coincide with the ordinary Gaussian distribution. However, Carruthers and Nieto[1] and Jackiw[2] erred by treating these states as exact states with minimum uncertainty. Regarding the states (13), Carruthers and Nieto said, "These are very complicated states, but they are indeed the states with minimum uncertainty." It turns out that these are not only very complicated states, but they are not the states with minimum uncertainty. Carruthers and Nieto simply repeated the mistake made by Jackiw in Ref. 2. The point here is that $I_\mu(x)>0$ holds for $x>0$ for any value of μ. Consequently, the boundary conditions (14) cannot be satisfied for any λ and μ. However, when $\Delta\hat{N}/\langle\hat{N}\rangle\ll 1$ holds the function $I_{n-\lambda}(\gamma)$ is very small for $n<0$. The states (13), like the usual Gaussian distribution, can therefore be used only in the limit $\Delta\hat{N}/\langle\hat{N}\rangle\ll 1$.

It should be noted that these results can be carried over without any change to the case of states with a minimum product of the uncertainty in position and momentum when the position is restricted to positive values, $0<x<+\infty$. The problem of finding the minimal states (i.e., the states with minimum dispersion of momentum for a given average value of position and dispersion in position) in the case $0<x<+\infty$ has already been treated.[9] However, in Ref. 9 it was erroneously assumed that Eq. (5) has solutions only for $\varepsilon=1/2,3/2,\ldots$. Consequently, instead of a family of states $H_\nu(x)\exp(-x^2/2)$, $\nu>0$, a single state $H_1(x)\exp(-x^2/2)$ was found for $\beta=0$, $\varepsilon=3/2$.

APPENDIX A

Consider the limit of the s-th power of the operator $\hat{\Phi}_M$ as $M\to\infty$. Let

$$|f\rangle=\sum_{n=0}^{M-1} f_n|n\rangle,\quad |g\rangle=\sum_{n=0}^{M-1} g_n|n\rangle$$

be two arbitrary vectors in a space of dimension M. Then

$$\langle f|\hat{\Phi}_M^s|g\rangle=\left\langle f\left|\sum_{k=0}^{M-1}\theta_k^s\right|\theta_k\right\rangle\langle\theta_k|g\rangle$$

$$=\sum_{k=0}^{M-1}\theta_k^s\langle f|\theta_k\rangle\langle\theta_k|g\rangle,$$

where (see Sec. 2)

$$\langle f|\theta_k\rangle=\frac{1}{\sqrt{M}}\sum_{n=0}^{M-1} f_n^*\exp(in\theta_k)=\frac{1}{\sqrt{M}}f^*(\theta_k),$$

$$\langle\theta_k|g\rangle=\frac{1}{\sqrt{M}}\sum_{n=0}^{M-1} g_n\exp(in\theta_k)=\frac{1}{\sqrt{M}}g(\theta_k),$$

$$f(\theta)=\sum_{n=0}^{M-1} f_ne^{i\theta n},\quad g(\theta)=\sum_{n=0}^{M-1} g_ne^{i\theta n}.$$

Thus,

$$\lim_{M\to\infty}\langle f|\hat{\Phi}_M^s|g\rangle=\lim_{M\to\infty}\frac{1}{M}\sum_{n=0}^{M-1}\theta_k^sf^*(\theta_k)g(\theta_k)$$

$$=\frac{1}{2\pi}\int_{-\pi}^{\pi}\theta^sf^*(\theta)g(\theta)d\theta\,.$$

Is simply the definition of an integral as the limit of a summation.

We denote $\lim_{M\to\infty}\hat{\Phi}_M^s$ as the operator $\hat{\Phi}^s$, such that for any two vectors

$$|f\rangle=\sum_{n=0}^{\infty} f_n|n\rangle,\quad |g\rangle=\sum_{n=0}^{\infty} g_n|n\rangle,$$

we have

$$\langle f|\hat{\Phi}^s|g\rangle=\frac{1}{2\pi}\int_{-\pi}^{\pi}\theta^sf^*(\theta)g(\theta)d\theta, \tag{A1}$$

where

$$f(\theta)=\sum_{n=0}^{\infty} f_ne^{i\theta n},\quad g(\theta)=\sum_{n=0}^{\infty} g_ne^{i\theta n}.$$

APPENDIX B

Let

$$|\Psi\rangle=\sum_{n=0}^{\infty}c_n|n\rangle.$$

We multiply Eq. (4) by $\langle n|$ on the left and set the resulting numbers equal to zero:

$$\langle n|\hat{\Phi}^2|\Psi\rangle+(\lambda_2n^2+\lambda_1n+\lambda_0)c_n=0,\quad n=0,1,2\ldots\,.$$

Since

$$\langle n|\hat{\Phi}^2|\Psi\rangle=\frac{1}{2\pi}\int_{-\pi}^{\pi}e^{-i\theta n}\theta^2\Psi(\theta)d\theta,$$

$$\Psi(\theta)=\sum_{n=0}^{\infty}c_ne^{i\theta n}$$

[see Eq. (A1)], we have the system of equations

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}e^{-i\theta n}\theta^2\Psi(\theta)d\theta+(\lambda_2n^2+\lambda_1n+\lambda_0)c_n=0,$$

$$n=0,1,2\ldots\,. \tag{B1}$$

We define the function

$$c(\nu)=\frac{1}{2\pi}\int_{-\pi}^{\pi}e^{-i\nu\theta}\Psi(\theta)d\theta,\quad -\infty<\nu<\infty.$$

Since

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}e^{-i\theta n}e^{i\theta m}d\theta=\delta_{nm},$$

we have $c(\nu)=c_n$ for $\nu=n=0,1,2,\ldots$ and $c(\nu)=0$ for $\nu=-1,-2,-3,\ldots$. Since we have

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}e^{-i\theta n}\theta^2\Psi(\theta)d\theta=-\frac{d^2}{d\nu^2}c(\nu)$$

for $\nu=n=0,1,2\ldots$,

Eq. (B.1) can be rewritten in the form

$$-\frac{d^2}{d\nu^2}c(\nu)+(\lambda_2\nu^2+\lambda_1\nu+\lambda_0)c(\nu)=0.$$

This equation must be satisfied at the points $\nu=0,1,2,\ldots$.

Thus far we have made no approximations. Now for the case $\Delta\hat{N}\gg1$ we assume that n varies continuously between 0 and $+\infty$, $0<n<+\infty$. Then we have $c_n=c(n)$, $c(\nu)=c(n)$ and we obtain an equation for the distribution over the number of quanta in states with minimum phase uncertainty for a given average number of quanta and dispersion in the number of quanta:

$$-\frac{d^2}{dn^2}c(n)+(\lambda_2n^2+\lambda_1n+\lambda_0)c(n)=0. \tag{B2}$$

The function $c(n)$ is equal to zero for $n<0$. Consequently, the boundary condition $c(0)=0$ is imposed on the solutions of Eq. (B2).

APPENDIX C

The Bessel function of order n can be written in terms of the Sommerfeld representation[7]

$$J_n(z)=\frac{1}{2\pi}\int_{-\pi}^{\pi}\exp(iz\sin\phi-in\phi)d\phi.$$

Using the relation between the Bessel functions $J_n(z)$ and the modified Bessel functions of the first kind $I_n(z)$,

$$I_n(z)=e^{-in\pi/2}J_n(iz),$$

we find

$$I_n(\gamma)=\frac{e^{-in\pi/2}}{2\pi}\int_{-\pi}^{\pi}\exp(-\gamma\sin\phi-in\phi)d\phi. \quad \text{(C1)}$$

The integral of (C1) is evaluated by the method of steepest descent and we find that it depends on n as follows:

$$I_n(\gamma)=\text{const}\cdot\exp\left(-\frac{n^2}{2\gamma}\right)\left[1+O\left(\frac{1}{K}\right)\right],$$

where $K=\max(\gamma,n)$.

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