Shielding of a moving charge in a Maxwellian plasma

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The dynamical shielding of a free charge moving with constant velocity in a classical isothermal plasma is studied. Assuming that the response is linear, an expression is derived for the potential of the longitudinal part of the electric field in terms of the velocity and the spatial variables, using the nonrelativistic approach. The condition for its applicability are discussed. On the basis of numerical calculations the variation in the screening behavior is examined as a function of the charge speed and the direction of the point of observation. It is shown that the transition that takes place inside the polarization cloud from the Debye potential to the Coulomb potential is stretched out to a few dozen times the thermal velocity. In the back hemisphere relative to the direction of motion the effective charge changes sign and can exceed the original charge in magnitude. The potential of a plasma particle, averaged over the Maxwellian distribution, is calculated. It is concluded that the shielded electron potential in a hydrogen plasma falls off as a function of distance considerably more slowly than the Debye potential on account of the thermal motion. The results of T. Peter [J. Plasma Phys. **44**, Part 2, 269 (1990)] and of C.-L. Wang and G. Joyce [J. Plasma Phys. **25**, Part 2, 225 (1981)] are extended to larger values of the particle velocity.

1. INTRODUCTION

It is well known that a free charge at rest in a classical isothermal plasma produces a Debye-shielded potential there. The shielding decreases as the particle begins to move. Moreover, when the charge moves the angular distribution of its potential becomes anisotropic; of course, this disappears both in the case of velocities which are small in comparison with the plasma electron thermal velocity and when the velocity of the charge is large (here we assume that all velocities are small in comparison with that of light, so that the motion is nonrelativistic). As shown by Montgomery *et al.*,[1] when the system consisting of a free charge and the polarization cloud it produces moves the lowest-order multipole moment which appears is quadrupole, and it vanishes only when the velocity of the free charge equals zero. Outside of the polarization cloud the potential of the system falls off in inverse proportion to the cube of the distance.

The problem treated in the present work is to investigate the behavior of the dynamical screening for both the particles of the equilibrium plasma itself and the free charges moving in it. The most suitable quantity for describing this situation is the effective charge of a particle. Numerical calculations reveal that dynamical shielding differs qualitatively from static shielding; this is seen in the transition that takes place at the extremities of the polarization cloud as the velocity of the charge increases, when the Debye potential turns into the Coulomb potential (this is extended out to several dozen or even several hundred times the electron thermal velocity), and also in a change in the sign of the effective charge, whose magnitude can exceed that of the original charge.

Although general expressions for the time-independent electric field of a point charge moving in an isotropic plasma with constant velocity are well known,[2–4] until recently numerical calculations have been performed only with a model dispersion relation.[5,6] Peter[7] has recently shown that this approach is unsatisfactory for describing the far structure of the wake field of a charge by using the exact dispersion relation for a Maxwellian plasma. There was an earlier numerical study[8] with an exact treatment of the dispersion which, however, essentially lacked a clear physical interpretation of the resulting dynamical screening effects. There it was also incorrectly asserted that the change in the sign of the potential in the rear hemisphere can be interpreted as being due to the emission of Langmuir waves. In fact, it is a consequence of the inertia of the polarization cloud, and wave emission does not enter into the steady formulation of the problem.

The present work supplements Refs. 7 and 8, extending them to larger velocities, and represents a continuation of previous work[9] in which we investigated how the Debye shielding potential is established as a function of time for a stationary charge placed in the plasma.

In Sec. 2 we derive general expressions for the potential of a moving charge in a Maxwellian plasma. Section 3 is devoted to a discussion of the conditions under which these expressions are applicable. An analytical treatment of the nonexponential decrease in the potential as a function of distance is given in Sec. 4. The results of the numerical calculations are presented in Sec. 5. Finally, in Sec. 6 the potential of the plasma's own charges, averaged over a Maxwellian distribution, is treated and evaluated for the first time. Here the average is performed over both the velocities of the plasma particles and their direction in space.

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2. THE POTENTIAL OF A MOVING CHARGE IN A PLASMA

Here we consider a fully ionized Maxwellian plasma consisting of singly charged ions and electrons (although in fact all the calculations can be extended to other types of plasma after simple modifications). Assume that a unit free charge is moving in the plasma with some velocity u. The Maxwell equation for the electric displacement $\mathbf{D}(\mathbf{r},t)$ takes the form

$$\operatorname{div}\mathbf{D}(\mathbf{r},t)=4\pi\rho(\mathbf{r},t)=4\pi\delta(\mathbf{r}-\mathbf{u}t), \quad (1)$$

where $\rho(\mathbf{r},t)$ is the density of the free electric charge. Fourier transformation of Eq. (1) leads to a simple algebraic relation between the Fourier components $\mathbf{D}(\mathbf{k},\omega)$ of the displacement field and the charge density $\rho(\mathbf{k},\omega)$:

$$i\mathbf{k}\mathbf{D}(\mathbf{k},\omega)=4\pi\rho(\mathbf{k},\omega)=8\pi^2\delta(\omega-\mathbf{k}\mathbf{u}). \quad (2)$$

The first thing to notice is that the external sources have been introduced in a consistent fashion. Specifically, the continuity equation

$$\partial\rho(\mathbf{r},t)/\partial t+\operatorname{div}\mathbf{j}(\mathbf{r},t)=0, \quad (3)$$

where $\mathbf{j}(\mathbf{r},t)$ is the current density of the free charge, after Fourier transformation assumes the form

$$-i\omega\rho(\mathbf{k},\omega)+i\mathbf{k}\mathbf{j}(\mathbf{k},\omega)=0. \quad (4)$$

Since the current density is equal to

$$\mathbf{j}(\mathbf{r},t)=\mathbf{u}\delta(\mathbf{r}-\mathbf{u}t), \quad (5)$$

its Fourier transform is

$$\mathbf{j}(\mathbf{k},\omega)=2\pi\mathbf{u}\delta(\omega-\mathbf{k}\mathbf{u}). \quad (6)$$

Substituting (6) and (2) into (4) we find the relation

$$(\omega-\mathbf{k}\mathbf{u})\delta(\omega-\mathbf{k}\mathbf{u})=0, \quad (7)$$

which is satisfied identically.

Let us find the relation between the electrical displacement $\mathbf{D}(\mathbf{k},\omega)$ and the electric field $\mathbf{E}(\mathbf{k},\omega)$:

$$D_j(\mathbf{k},\omega)=\varepsilon_{ji}(\mathbf{k},\omega)E_i(\mathbf{k},\omega). \quad (8)$$

Here $\varepsilon_{ji}(\mathbf{k},\omega)$ is the dielectric tensor, whose longitudinal part (the only one that matters under the present conditions) is equal to

$$\varepsilon_{ji}(\mathbf{k},\omega)=\varepsilon_l(\mathbf{k},\omega)k_jk_i/k^2. \quad (9)$$

The longitudinal part $\varepsilon_l(\mathbf{k},\omega)$ of the permittivity takes the form (cf. Ref. 10, Sec. 31):

$$\varepsilon_l(\mathbf{k},\omega)=1+(ka)^{-2}\left[1+F\left(\frac{\omega}{\sqrt{2}kv}\right)\right]. \quad (10)$$

Here v is the average electron thermal velocity

$$v=(T/m)^{1/2}, \quad (11)$$

where T is the plasma temperature and m is the electron mass. The quantity a is the Debye radius, which is given in terms of the electron density N by

$$a=(T/4\pi Ne^2)^{1/2}. \quad (12)$$

Next, the function $F(x)$ is determined from the integral

$$F(x)=\frac{x}{\pi^{1/2}}\int_{-\infty}^{\infty}\frac{\exp(-z^2)dz}{z-x}+i\pi^{1/2}x\exp(-x^2), \quad (13)$$

where the first term is defined in the sense of a principal value.

Substituting (8) and (9) in (2) we find

$$i\varepsilon_l(\mathbf{k},\omega)\mathbf{k}\mathbf{E}(\mathbf{k},\omega)=8\pi^2\delta(\omega-\mathbf{k}\mathbf{u}). \quad (14)$$

The electric field strength $\mathbf{E}(\mathbf{k},\omega)$ is conveniently expressed in terms of the scalar potential $\varphi(\mathbf{k},\omega)$:

$$\mathbf{E}(\mathbf{k},\omega)=-i\mathbf{k}\varphi(\mathbf{k},\omega). \quad (15)$$

Substituting (15) in (14) we find

$$\varphi(\mathbf{k},\omega)=(8\pi^2/k^2\varepsilon_l)\delta(\omega-\mathbf{k}\mathbf{u}). \quad (16)$$

After inverting the Fourier transform we finally obtain

$$\varphi(\mathbf{R})=\frac{1}{2\pi^2}\int\frac{d\mathbf{k}}{k^2}\frac{\exp(i\mathbf{k}\mathbf{R})}{\varepsilon_l(\mathbf{k}\mathbf{u},\mathbf{k})} \quad (17)$$

(cf. Refs. 2–4). Here we have introduced the notation $\mathbf{R}=\mathbf{r}-\mathbf{u}t$, so that in the coordinate system moving with the charge the electric field distribution is independent of time.

3. CONDITIONS FOR APPLICABILITY OF THE SOLUTION

The approximation in which the motion of the free charge is prescribed is only valid if we can neglect the reciprocal effect of the field on this charge. As is well known,[2] in an isotropic plasma longitudinal waves can have such small velocities that they can be produced by Cherenkov emission when the resonance condition $\omega=\mathbf{k}\cdot\mathbf{u}$ is satisfied. Hence expression (17) is actually valid for times Δt large enough for the polarization clouds to develop but small enough so that their energy losses through Cherenkov radiation can be regarded as small.

If we extrapolate the results of our previous work,[9] then the first condition becomes

$$\Delta t\gg 10\omega_p^{-1}, \quad (18)$$

where ω_p is the plasma frequency

$$\omega_p=v/a=(4\pi Ne^2/m)^{1/2}. \quad (19)$$

On the other hand, as noted above, this time must be short enough that the energy losses of the free charge through Cherenkov emission of plasma waves are small in comparison with its kinetic energy. The energy lost per unit time through radiation of plasma waves by a particle with velocity u which is much larger than the thermal velocity v was studied by Pines and Bohm.[11] Based on the results of that work the second condition becomes

$$\Delta t\ll\frac{Mu^3}{e_1^2\omega_p^2}\frac{1}{\ln(1+2u^2/v^2)}, \quad (20)$$

where M is the mass of the free charge and e_1 is its charge (taken to be unity in the present work). For an electron condition (20) in fact reduces to the condition for an ideal

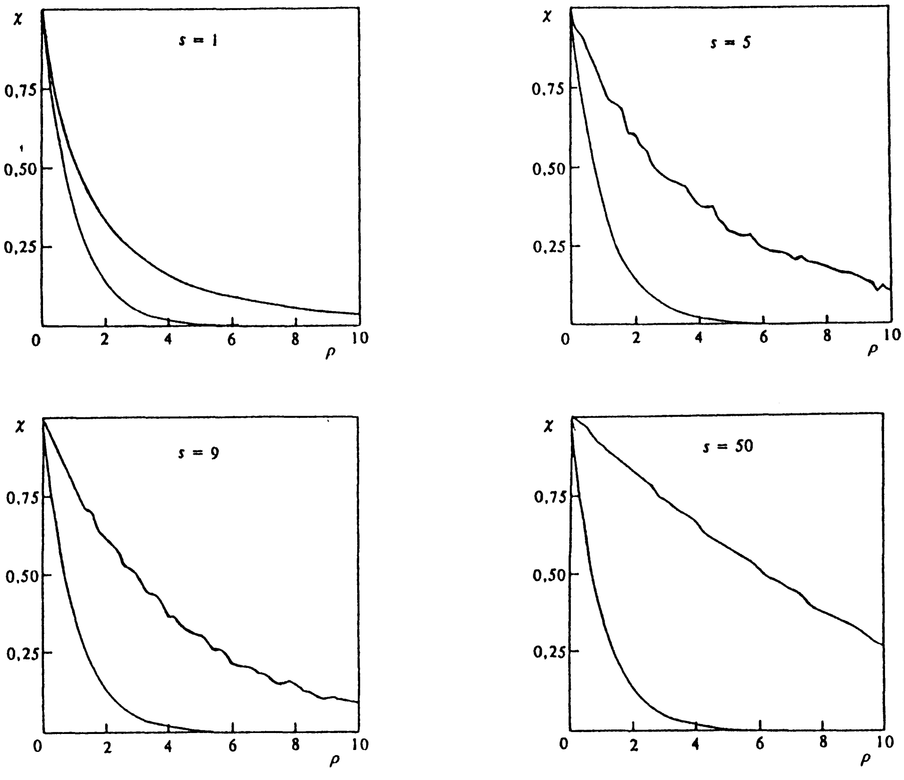

FIG. 1. Effective particle charge χ as a function of distance ρ, expressed in units of the Debye radius ($\rho = r/a$) in the direction $\theta = 0$ (ahead of the moving charge). Results are plotted for four values of the particle velocity, characterized by the dimensionless quantity $s = u/v\sqrt{2}$. The light trace shows the effective charge for the Debye potential $\exp(-\rho)$.

plasma if we combine it with the condition (18). As the mass of the free charge increases, condition (20) is satisfied more readily.

4. NONEXPONENTIAL DECAY OF THE POTENTIAL AS A FUNCTION OF DISTANCE

The shielding of a moving free charge by the plasma differs qualitatively from the Debye shielding of a charge at rest. This is seen in the nonexponential decay of the potential (17) as a function of distance R. It can be found analytically by considering slow motions which satisfy $u \ll v$. Expanding (17) to first order in the velocity u of the charge and assuming that from (13) we have

$$F(x) = -2x^2 + i\pi^{1/2}x, \quad x \ll 1, \tag{21}$$

we find the following result:

$$\varphi(R) = (1/R)\exp(-R/a) + 4(2\pi)^{-1/2}\frac{a^2(\mathbf{uR})}{vR^4},$$

$$R \gg a. \tag{22}$$

This was obtained first by Cooper[12] and agrees with the results of Ref. 1.

Thus, at distances large in comparison with the Debye radius the potential of a moving charge falls off as the inverse cube of the distance and, in addition, has a strong directional dependence. The first term in (22) can be neglected in comparison with the second. Then in the rear hemisphere relative to the direction of particle motion the effective charge of the moving particle is reversed, i.e., a moving electron repels electrons located ahead of it and drags electrons behind it. We shall see that the interaction between the moving charge and the plasma retains a similar character at large velocities as well.

5. RESULTS OF NUMERICAL CALCULATIONS

If we separate the Coulomb contribution in (17) and integrate with respect to one of the angular variables we can reduce the general expressions (17) to the following form:

$$\varphi = (1/R)\Big\{1 - (1/\pi)\int_0^\infty dz \int_{-1}^1 dx J_0 \times [z\sin\theta(1-x^2)^{1/2}]\exp(izx\cos\theta)R^2[1+F(sx)] \times [z^2a^2 + R^2 + R^2F(sx)]^{-1}\Big\}. \tag{23}$$

Here θ is the angle between the vectors $\mathbf{u}$ and $\mathbf{R}$ and we have written $s = u/v\sqrt{2}$.

For numerical calculations we rewrite expression (23) in an explicitly real form by introducing the effective charge function $\varphi = \chi/R$:

$$\chi = 1 - (2/\pi)\int_0^\infty dz \int_0^1 dz J_0 \times [z\sin\theta(1-x^2)^{1/2}]\Phi(z,x;\rho,s,\theta). \tag{24}$$

Here we have used the notation

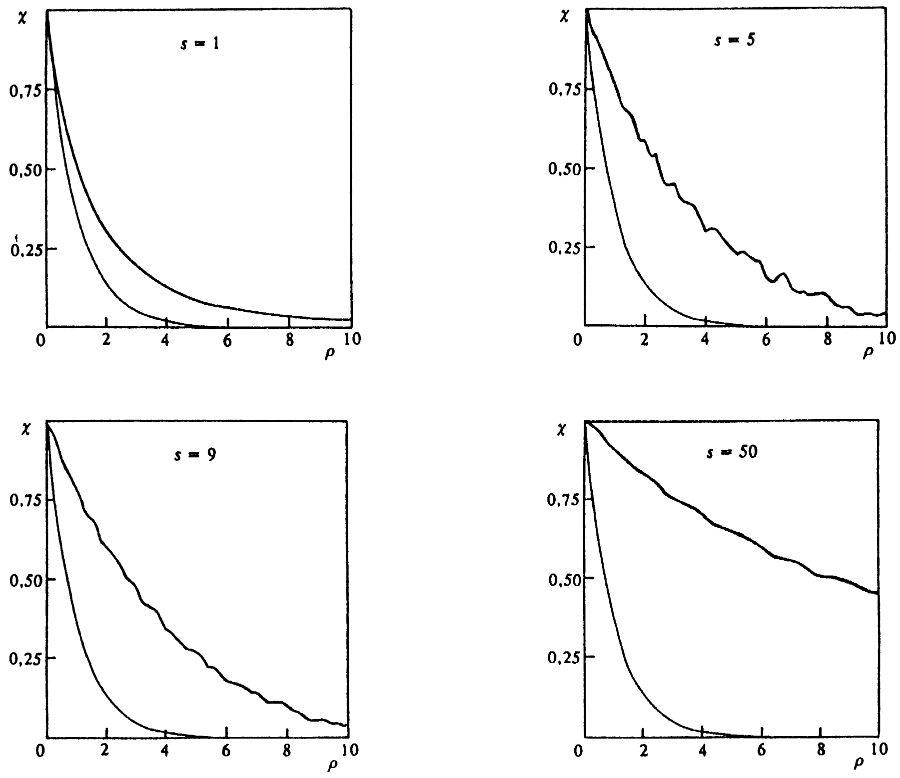

FIG. 2. As in Fig. 1, for $\theta=\pi/4$.

$$\Phi(z,x;\rho,s,\theta)=\{\{z^2\rho^2(1+F'(sx))+\rho^4[(1+F'(sx))^2+(F''(sx))^2]\}\cos(z\cos\theta x)-z^2\rho^2F''(sx)\sin(z\cos\theta x)\}\times\{[z^2+\rho^2(1+F'(sx))]^2+\rho^4[F''(sx)]^2\}^{-1},\quad \rho=R/a. \tag{25}$$

The function $\chi(\rho,\theta)$ has been evaluated from Eqs. (24)–(25) for angles θ between the direction of the radius vector **R** and the direction of the velocity u of the particle equal to 0, $\pi/4$, $\pi/2$, $3\pi/4$, and π. Figure 1 shows the results for $\theta=0$, i.e., in the direction ahead of the moving particle. It is clear that as the particle velocity increases the shielding becomes weaker, although even for velocity ratios $s=50$ the potential is significantly shielded (the complete disappearance of shielding would correspond to the horizontal straight line $\chi=1$). This is substantially in disagreement with Ref. 7, where it was asserted that for fast particles with $\rho\ll s$ the potential is essentially Coulombic.

The results in Fig. 2 for $\theta=\pi/4$ are similar. At very large velocities ($s=50$) the shielding is weaker than for the angle $\theta=0$, whereas for small velocities it is stronger.

Figure 3 shows χ as a function of ρ for different particle velocities at right angles to its motion. It is clear that for moderate velocities ($s=1$) there is still a small change in the effective charge of the moving particle. However, as the velocity increases the screening of the field of the particle becomes weaker, and at $s=50$ its potential differs from the Coulomb potential by less than a factor of two if we restrict ourselves to distances of less than ten Debye radii.

The change in the effective charge of the particle at small velocities is especially noticeable in the rear hemisphere (Fig. 4, $\theta=3\pi/4$, and Fig. 5, $\theta=\pi$). This is because the moving particle drags behind it a cloud of particles of the same species. However, when the velocity is large (see the case $s=50$ in Fig. 5), the particle is no longer able to entrain this cloud and the potential becomes Coulombic. It is precisely this which is the inertial behavior of the polarization referred to earlier.

Our results agree qualitatively with those of Refs. 7 and 8, in which the calculations were carried out to $s=10$ and $s=15$, respectively.

From Figs. 1–5 the general conclusion is that as the velocity of the moving particle in the plasma increases its potential becomes strongly anisotropic as the perturbation extends to larger and larger volumes of space. In the rear hemisphere the effective charge changes in sign and can be considerably larger in absolute value than the original charge.

6. SHIELDING OF THE FIELD OF THE PLASMA PARTICLES

Thus far we have been talking about an individual free charge moving in a Maxwellian plasma. In this section we turn to the aggregate of plasma charges. If we consider the simplest fully ionized hydrogenic plasma (which, to be sure, is what we were talking about previously), then we are concerned with the motion of the ions and electrons. As regards the ions in the isothermal case, because of their large mass their thermal velocity is much less than the electron thermal velocity (11), which is responsible for the

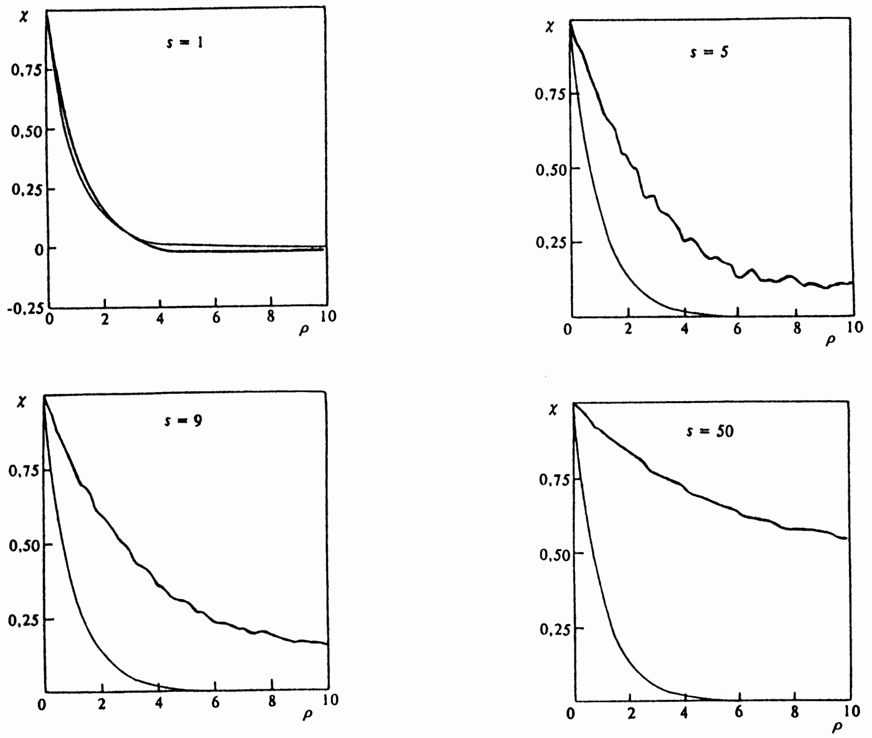

FIG. 3. As in Fig. 1, for $\theta=\pi/2$.

Debye shielding. Hence the potential produced by the ions in this case is almost purely Debye-like, since we can treat them as though they were at rest.

The situation is different, e.g., for the plasma electrons. In order to demonstrate this fact, let us find the mean potential for an ensemble of particles with thermal velocity v_1. For this purpose we will average the potential (23) found above with a Maxwellian distribution at time $t=0$. The average is carried out over both the velocity u of the ensemble particles and over the angle θ, on which the Maxwell distribution obviously does not depend.

It is more convenient to start by averaging expression (16) for the Fourier component of the potential of the moving charge over the Maxwellian distribution. We find

$$\langle\varphi(k,\omega)\rangle=[2^{5/2}\pi^{3/2}/v_1k^3]\varepsilon_l(k,\omega)\times\exp(-\omega^2/2k^2v_1^2). \qquad (26)$$

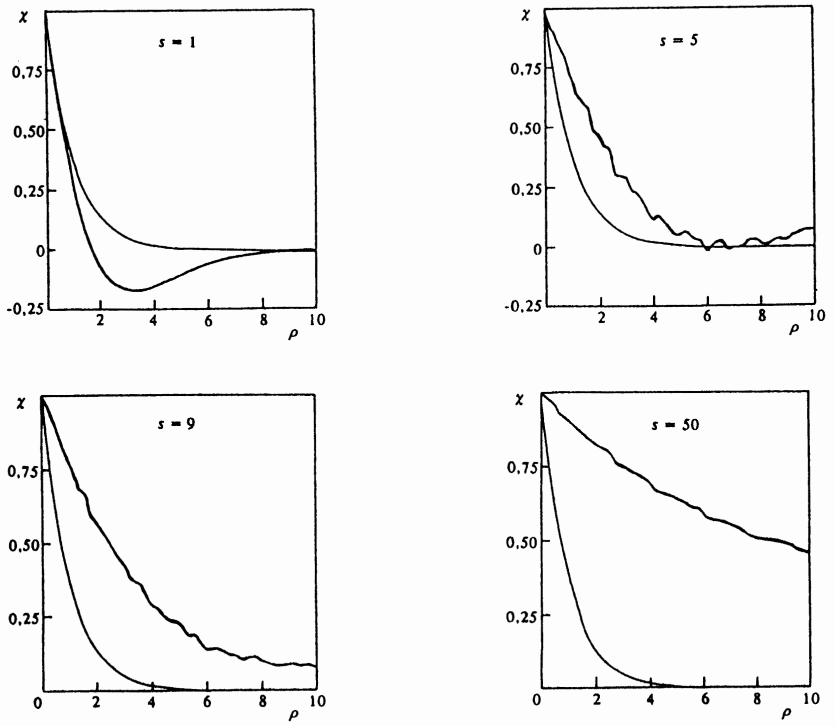

FIG. 4. As in Fig. 1, for $\theta=3\pi/4$.

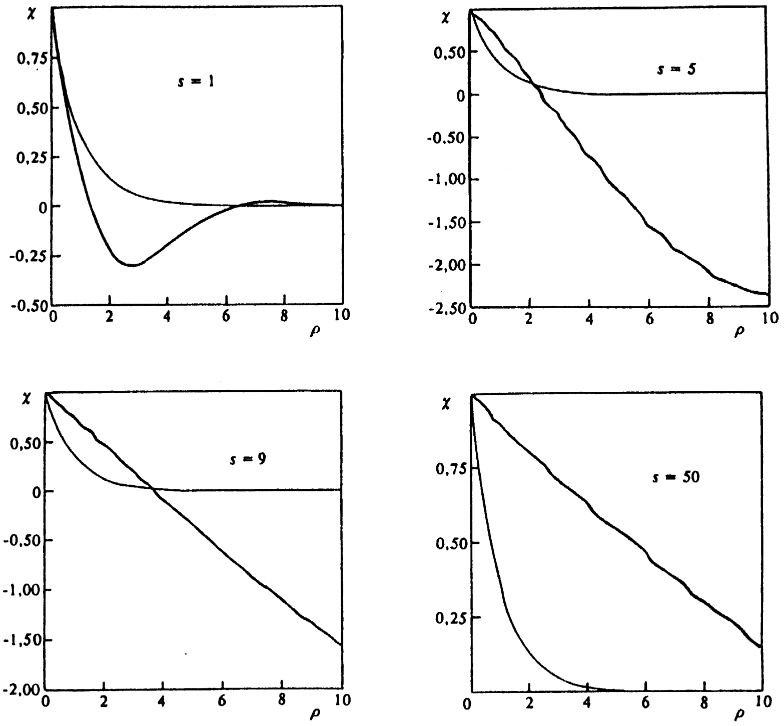

FIG. 5. As in Fig. 1, for $\theta=\pi$.

Inverting the Fourier transform and integrating over the angles of the vector **k** we find

$$\langle\varphi(r)\rangle=\langle\chi(r)\rangle/r; \tag{27}$$

$$\langle\chi(r)\rangle=2^{1/2}\pi^{-3/2}v_1^{-1}\int_{-\infty}^{\infty}d\omega\int_0^{\infty}dk$$

$$\times\frac{\sin(kr)}{k^2\varepsilon_l(k,\omega)}\exp\left[-\frac{\omega^2}{2k^2v_1^2}\right]. \tag{28}$$

Using expression (10) for the dielectric function and making a change of variables, we can perform the integral with respect to k. We finally obtain

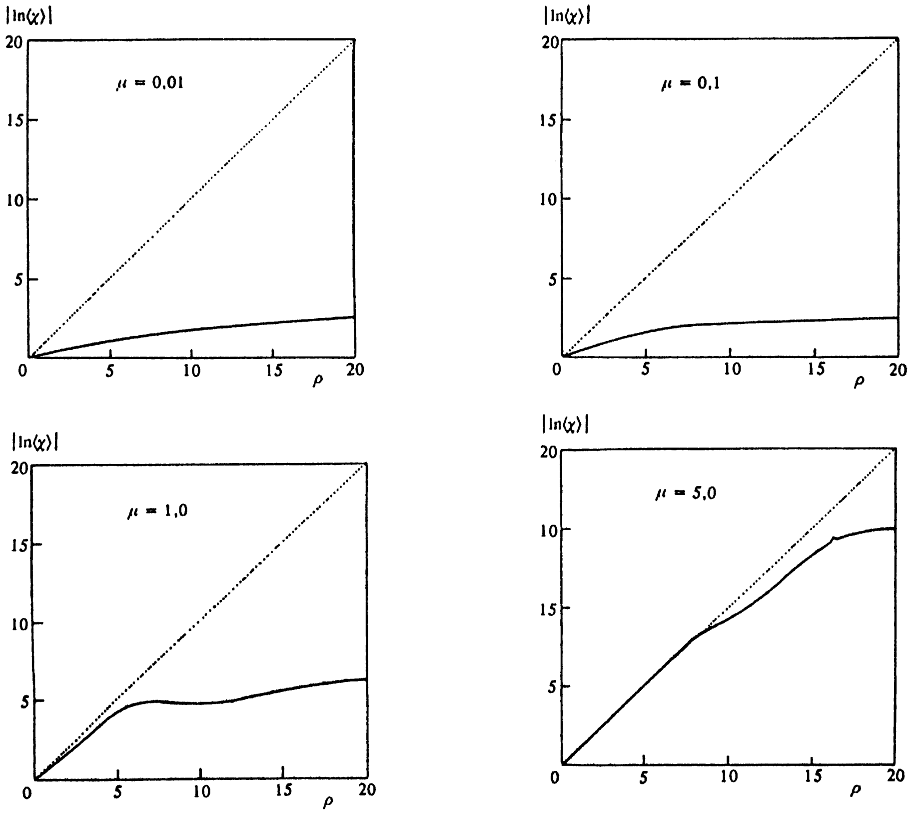

FIG. 6. The absolute value of the logarithm of the average effective charge of a moving particle in a plasma with different values of $\mu=v_1/v_3$ the thermal velocity scaled by the thermal velocity v of the plasma electrons, as a function of distance from the charge ρ. The broken trace corresponds to the Debye potential.

$$\langle\chi(\rho)\rangle=2(\mu/\pi)^{1/2}\int_0^\infty dz\cos[S_-(z)\rho]\times\exp[-\mu z^2-S_+(z)\rho]. \quad (29)$$

Here we have introduced the dimensionless variables

$$\mu=(v/v_1)^2, \quad \rho=r/a, \quad (30)$$

and the notation

$$S_\pm(z)=2^{-1/2}\{[(1+F'(z))^2+(F''(z))^2]^{1/2}\pm(1+F'(z))\}^{1/2}. \quad (31)$$

Figure 6 shows the results of the numerical calculation using Eq. (29) for different values of the parameter μ defined in Eq. (30). It is clear that the inclusion of electron motion ($\mu=1$) weakens the Debye shielding, as one would expect.

These results can also be applied to a weakly ionized plasma containing electrons and ions. In addition, the results can easily be generalized to the case of the ion thermal motion in electrolytic solutions. They are analogous to the so-called relaxation correction (see Ref. 10, Sec. 26). This arises because the ion motion in an external electric field distorts the distribution of charges in the polarization cloud, as a result of which an additional electric field acting on the ion in question develops. As is well known, this effect diminishes the ion mobility. Including the ion thermal motion effectively increases the Debye radius, thereby increasing the relaxation correction, i.e., still further reducing the ion mobility. A similar change occurs in the so-called electrophoretic correction, which results because the motion of the polarization cloud causes the fluid to move, which in turn produces "drift" of the ion in question. This also reduces the ion mobility.

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