The two-body problem in a Maxwell–Chern–Simons theory

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We consider a system of two particles interacting through a gauge field with a Lagrangian which is the sum of a Maxwell term and a Chern–Simons term. We show that if the magnetic flux associated with the particles is sufficiently large and the temperature sufficiently low, the system may possess (unstable) states with a negative compressibility such as can occur in a van der Waals gas.

1. Leinaas and Myrheim have shown in Ref. 1 that there may exist particles (later called anyons[2]) in two-dimensional space which satisfy an intermediate statistic. Under an exchange of two anyons, the wavefunction acquires a phase factor $\exp(i\pi\Delta)$ where the quantity Δ, the so-called statistical parameter, can be any real number. The anyon problem is of interest in its own right (and is rather complicated), and at the same time the problem of their possible physical nature is very important. Essentially the only known model in which intermediate statistics arise effectively is a model of particles interacting with a Chern–Simons gauge field. (We note that it is the Chern–Simons field that appears in the theory of the quantum Hall effect.[3]) Considering the Lagrangian

$$L=\tfrac{1}{2}\,\alpha\varepsilon^{\mu\nu\lambda}A_\mu\partial_\nu A_\lambda-A_\mu j^\mu \tag{1}$$

with conserved current j^μ, and writing down the corresponding field equations, we note that the charge e effectively becomes "a magnetic flux tube", $\Phi=e/\alpha$; in other words, it generates a vector potential

$$\mathbf{A}(\mathbf{r})=\frac{\Phi}{2\pi}\frac{[\mathbf{e}_z,\mathbf{r}]}{r^2}=\frac{\Phi}{2\pi}\nabla\varphi, \tag{2}$$

where φ is the polar angle (the charge is at the origin). The potential is clearly a pure gauge potential, and can be removed through a gauge transformation; however, such a transformation multiplies the wave function by a phase factor, as a result of which the new wave function satisfies the commutation relation

$$P_{jk}\Psi=\exp(i\pi\Delta)\Psi, \tag{3}$$

where

$$\Delta=\frac{e^2}{2\pi\alpha}, \tag{4}$$

and this means the effective appearance of anyons. (Here and henceforth we assume the "bare" particles to be bosons.) Moreover, it has been established[4,5] that a Chern–Simons field can be induced by fermions interacting with a gauge field. In fact, if we introduce into the theory a term

$$\Delta L=\bar\psi(i\gamma\partial+\gamma\mathbf{A})\psi+\bar\psi\gamma_0(i\partial_0+\mu+A_0)\psi+m\psi\bar\psi, \tag{5}$$

then after integration over ψ, the Chern–Simons term that enters into (1) appears in the effective action for A. At a finite temperature and density,[6] the coefficient α is given by

$$\alpha=\frac{m}{|m|}\frac{1}{4\pi}\frac{\mathrm{sh}(\beta|m|)}{\mathrm{ch}(\beta|m|)+\mathrm{ch}(\beta\mu)}, \tag{6}$$

where β is the inverse temperature and μ the chemical potential. Another interesting possibility for the appearance of a Chern–Simons term was considered in Ref. 7. It was shown that the classical dynamics of a liquid in a plane can be considered to be a gauge theory. It turns out that the field theory describing small oscillations of the liquid relative to the minimum of some (density-dependent) potential is the usual electrodynamics, and if the liquid is charged and a magnetic field is applied to it, there will effectively be a Chern–Simons term, and the vortex excitations acquire intermediate statistics.

2. In connection with the possibility of inducing a Chern–Simons term, it is of interest to study a theory in which both the latter and a Maxwell term are present. Consider, for instance, the Lagrangian

$$L=-\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}+\tfrac{1}{2}\,\alpha\varepsilon^{\mu\nu\lambda}A_\mu\partial_\nu A_\lambda-A_\mu j^\mu. \tag{7}$$

The corresponding field equations have the form

$$\partial_\nu F^{\mu\nu}+\frac{1}{2}\alpha\varepsilon^{\mu\nu\lambda}F_{\nu\lambda}=j^\mu, \tag{8}$$

or, in terms of the field strengths,

$$\partial_k E^k=j^0+\alpha B, \tag{9}$$

$$\varepsilon^{kl}(\partial_l B+\alpha E_l)=j^k+\dot E^k. \tag{10}$$

The solution of these equations for a point charge ($j^0=e\delta^2(\mathbf{r})$, $j^k=0$), is[8]

$$E^k(\mathbf{r})=\frac{e\alpha}{2\pi}\frac{x^k}{r}K_1(\alpha r), \tag{11}$$

$$B(\mathbf{r})=-\frac{e\alpha}{2\pi}K_0(\alpha r), \tag{12}$$

($\mathbf{r}=x^1\mathbf{e}_1+x^2\mathbf{e}_2$, $r=|\mathbf{r}|$). The function $K_0(x)$ behaves as $-\ln(x/2)$ for $x\ll 1$ and as $\sqrt{\pi/2x}\cdot\exp(-x)$ for $x\gg 1$, while $K_1(x) = -K_0'(x)$. Hence, the electric field is a Coulomb field (proportional to $1/r$) near the charge ($\alpha r \ll 1$) and the magnetic field is logarithmic, whereas far from the charge ($\alpha r \gg 1$) both fields decrease in proportion to $\exp(-\alpha r)$. The vector potential corresponding to (11) and (12) can be chosen in the form

$$A^0(\mathbf{r}) = \frac{e}{2\pi} K_0(\alpha r), \tag{13}$$

$$A^k(\mathbf{r}) = \frac{e}{2\pi\alpha} \frac{\varepsilon^{kl} x^l}{r^2} f(\alpha r), \tag{14}$$

where

$$f(x) = 1 - x K_1(x). \tag{15}$$

The magnetic flux through an area S is

$$\Phi[S] = \int_S d^2x B = \oint_C dx^k A^k = -\frac{e}{2\pi\alpha} \oint_C d\varphi f(\alpha R), \tag{16}$$

where the contour C is the boundary of S and is parametrized by $R(\varphi)$. In particular, if S is a circle of radius ρ we have $\Phi[S] = -(2\pi/e) D(\rho)$, where

$$D(\rho) = f(\alpha\rho)\Delta, \tag{17}$$

and Δ is given by Eq. (4).

The problem of N noninteracting anyons is, according to what we have said above, equivalent to the problem of N particles interacting through the vector potential (2). In our case we have (13) and (14) instead of (2). As this is not a pure gauge potential, the particles are necessarily interacting. We restrict ourselves here to considering the $N=2$ case, which is the only one that permits of a more or less exact discussion. In that case, the motion of the center of mass can be separated as usual, leading to the problem of a single mass $m/2$. In order for the problem to have a discrete spectrum, we introduce an external harmonic potential. The Hamiltonian of the relative motion is

$$H_{\text{rel}} = -\frac{1}{m}\left[\frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} + \left(\frac{1}{\rho}\frac{\partial}{\partial\varphi} - A_\varphi(\rho)\right)^2\right] + \frac{m}{2}\frac{\omega^2\rho^2}{2}, \tag{18}$$

where, according to (14) and (17), we have

$$A_\varphi(\rho) = -\frac{D(\rho)}{e\rho}. \tag{19}$$

The commutation relations for the wave function have the standard form

$$P_{jk}\Psi = \exp(i\pi\kappa)\Psi, \tag{20}$$

where $\kappa = 0(1)$ for bosons (fermions). The angular part of the wave function can be separated as

$$\Psi = \chi \exp(iM\varphi); \tag{21}$$

M must have the same parity as κ. It is now necessary to find the energy eigenvalues from the equation

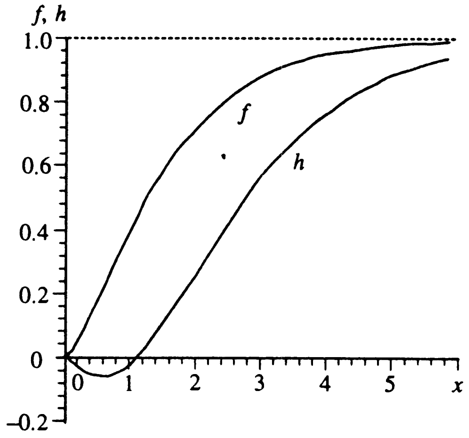

FIG. 1. The functions $f(x)$ and $h(x)$.

$$-\frac{1}{m}\left(\frac{d^2\chi}{d\rho^2} + \frac{1}{\rho}\frac{d\chi}{d\rho}\right) + V(\rho)\chi = E\chi, \tag{22}$$

where

$$V(\rho) = \frac{[M + D(\rho)]^2}{m\rho^2} + \frac{1}{4} m\omega^2\rho^2. \tag{23}$$

We have intentionally dropped the Coulomb interaction. Indeed, the scaling

$$e \to e/\gamma, \quad \alpha \to \alpha/\gamma^2, \quad m \to m/\gamma^4, \quad \rho \to \gamma^2\rho$$

leaves Eq. (22) unchanged, but transforms the Coulomb term $e^2 \ln \rho$ into

$$(e^2/\gamma^2)\ln\rho + (e^2/\gamma^2)\ln\gamma^2.$$

The Coulomb repulsion can thus be suppressed by appropriately choosing γ arbitrarily large.

The relative motion is thus characterized by the potential $V(\rho)$. There are two characteristic quantities with dimensions of distance, α^{-1} and $(m\omega)^{-1/2}$, so that it makes sense to introduce the dimensionless parameter $\sigma = \alpha(m\omega)^{-1/2}$. For levels which do not lie too high ρ, takes on values of order $(m\omega)^{-1/2}$, so that the argument of f [see (17)] is of the same order as σ. We show the behavior of $f(x)$ in Fig. 1. Since the values of x for which f changes appreciably are of order unity, the spectrum depends in an essential way on how large σ is compared to unity.

3. We begin by considering limiting cases.

1) Let $\sigma \ll 1$, i.e., the particles are extremely close to one another. It follows from what we have said above that we can then neglect the quantity $D(\rho)$ in (23). Equation (22) is then the radial equation for the usual problem of a two-dimentional harmonic oscillator. The eigenvalues in that limit are

$$E_{Mn} = (2n + |M| + 1)\omega, \tag{24}$$

where $n = 0,1,2,...$ is the radial quantum number and M is even (odd) for bosons (fermions). At extremely small distances, the presence of the Chern–Simons term does not make any difference whatever.

2) Let $\sigma \gg 1$. We can then substitute for $D(\rho)$ in (23) its limiting value $D(\infty) = \Delta$. The equation has the same form as in the previous case, with $M+\Delta$ instead of M. Correspondingly, we have instead of (24)

$$E_{Mn} = (2n + |M+\Delta| + 1)\omega. \tag{25}$$

This is the form of the spectrum of the relative motion of two noninteracting anyons[1,2] with $\delta = \Delta(\Delta+1)$ for even (odd) M. As we mentioned earlier, this result occurs in the purely Chern–Simons theory. At large distances, the presence of the Maxwell term therefore has no effect whatever.[8] The relative part of the partition function has the form

$$\tilde{Z}(\delta,\omega,\beta) = \frac{\text{ch}\,\omega(1-\delta)}{2\,\text{sh}^2\,\omega\beta}, \tag{26}$$

and the second virial coefficient is[9]

$$a_2(\delta) = \tfrac{1}{4}\lambda^2[1 - 2(1-\delta)^2], \tag{27}$$

where $\lambda = (2\pi/mT)^{1/2}$ is the thermal wavelength.

A gas of particles interacting in the way considered here thus behaves like the usual Bose or Fermi gas at high densities, and like a gas of anyons at low densities. We now turn to a discussion of the intermediate case, in which we must analyze Eq. (22) in detail.

We denote the first and second terms in (23) by $V_1(\rho)$ and $V_2(\rho)$, respectively. It is clear from the definition of V_1, (1) that $V_1(\rho) > 0$ for all ρ, (2) that $V_1(\infty) = 0$, and (3) that $V_1(0) = \infty$ for $M \neq 0$ (centrifugal barrier) and $V_1(0) = 0$ for $M = 0$. Moreover, we have

$$V_1'(\rho) = -\mathscr{F}(\rho)[\nu + f(\alpha\rho)][\nu + h(\alpha\rho)], \tag{28}$$

where $\nu = M/\Delta$, $\mathscr{F}(\rho)$ is a positive function, and

$$h(x) = f(x) - xf'(x) = 1 - x^2K_0(x) - xK_1(x). \tag{29}$$

The behavior of $h(x)$ is shown in Fig. 1, together with that of $f(x)$. There is a minimum $h(0.59) = -0.062$, and a root $h(1.11) = 0$. For all $0 < x < \infty$, we have $h(x) < f(x)$. It follows from (28) that V_1 can behave in five different ways, depending on the value of ν. (a) $\nu < -1$; in that case V_1 decreases monotonically. (b) $-1 < \nu < 0$; in that case there is a minimum $V_1(\rho_1) = 0$, where ρ_1 is determined from the equation $f(\alpha\rho_1) = -\nu$, and a maximum at the point $\rho_2 > \rho_1$ where $h(\alpha\rho_2) = -\nu$. (c) $\nu = 0$, i.e., $M = 0$ (if the "bare" particles are bosons). In that case the minimum is shifted to the point $\rho_1 = 0$ and the maximum is positioned at $\alpha\rho_2 = 1.11$. (d) $0 < \nu < 0.062$. As in (b), there is a minimum at a point ρ_1 and a maximum at $\rho_2 > \rho_1$, but here ρ_1 and ρ_2 are two roots of the equation $h(\alpha\rho) = -\nu$, and $V_1(\rho_1)$ does not vanish. Finally, (e) $\nu > 0.062$ again leads to a monotonic decrease in V_1.

The behavior of $V_1(\rho)$ for these cases is sketched in Figs. 2a to 2e.

Since V_1 tends to zero in all cases as $\rho \to \infty$, no bound states can exist without a harmonic attraction. The presence of V_2, on the other hand, guarantees the existence of such states. However, we shall see that a situation is possible in which their energy is hardly dependent on ω.

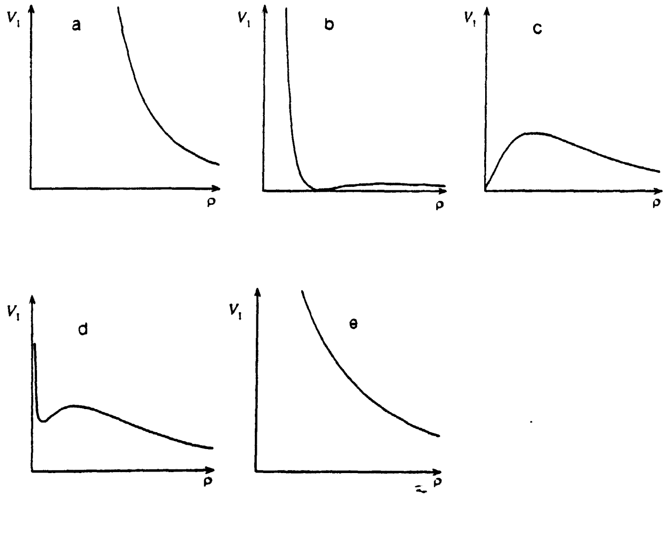

FIG. 2. The function $V_1(\rho)$ for various values of ν: (a) $\nu = -2$; (b) $\nu = -0.5$; (c) $\nu = 0$; (d) $\nu = 0.03$; (e) $\nu = 0.3$.

Of course, it is impossible to solve (22) exactly. We restrict ourselves here to a simple qualitative analysis which is fully sufficient to explain the basic properties of the system. We consider the case in which the "bare" particles are bosons, so that M must be even, and we pay attention to low-lying states—primarily to the ground state.

We explained that if ω is so small that $\sigma \gg 1$, the levels are determined by (25). We assume for definiteness that $\Delta > 0$. The lowest level will be the one corresponding to the even M which is closest to $-\Delta$. In particular, if Δ is an odd integer, the ground state will be doubly degenerate ($M = -\Delta - 1$ and $M = -\Delta + 1$, which corresponds to fermions).

The quantity $[M + D(\rho)]^2$ varies from M^2 for $\rho = 0$ to $[M+\Delta]^2$ for $\rho = \infty$; correspondingly, the ratio E_{Mn}/ω varies with increasing ω from $|M+\Delta| + 2n + 1$ for $\omega = 0$ to $|M| + 2n + 1$ for $\omega \to \infty$. For $\nu < -1$ ($\nu > 0.062$), $[M + D(\rho)]^2$ increases (decreases) monotonically. For $\nu < -1$ and $\nu > 0.062$, the function $V(\rho)$ has a single minimum and the ratio E_{Mn}/ω increases or decreases monotonically. For $0 < \nu < 0.062$ and moderate values of ω, $V(\rho)$ can have two minima, and the problem must be considered to be that of a double-well potential. In that case some singularities are possible in the behavior of E_{Mn}/ω, but as they do not affect the behavior of the ground state, and we shall not analyze them. (Note that as the minimum nonvanishing value of M is 2, states with such values of ν cannot exist at all for $\Delta < 33.3$.)

The most interesting case is $-1 < \nu < 0$; it is clear that the ground state always lies just in that region. We have

$$V_1(\rho) = \alpha^2 m^{-1} v(\alpha\rho), \tag{30}$$

where

$$v(x)=\frac{[M+f(x)\Delta]^2}{x^2} \tag{31}$$

is a dimensionless function, the behavior of which depends only on M and Δ; it behaves as shown in Fig. 2b for $-1<\nu<0$ and as shown in Fig. 2c for $\nu=0$. Thus, V_1 itself contains a potential well; its depth increases with Δ. For small Δ the well is too shallow to alter the spectrum appreciably, but for sufficiently large Δ, bound levels can appear in the well; it is just the behavior of such levels which is of most interest.

We first of all consider the $M=0$ case. We then have $v(x)=\Delta^2\eta(x)$, where $\eta(x)=[f(x)/x]^2$ (see Fig. 2c). For $\omega=0$, all states decay and there are no stationary ones. However, when we introduce the harmonic potential, the behavior of the levels is determined not only by this potential but also by the aforementioned well (and for sufficiently large Δ, mainly by the well). Let us consider the situation in more detail. When ω is so small that $\sigma\gg 1$, $V(\rho)$ has a minimum at a point where $1-f(\alpha\rho)\ll 1$, and the corresponding level lies far below where it would lie in the well considered. However, when ω increases, the amplitude of the wave function in the well also increases. For sufficiently large Δ, a state is ultimately formed in which essentially the whole wavefunction is concentrated in the well. For some value of ω, the right-hand minimum of $V(\rho)$ becomes higher than the level in the well so that outside the well there remains only the tail of the wave function, while for some larger ω, the right-hand minimum disappears altogether. For Δ not too large (of order 10^2 to 10^3, as shown by numerical estimates), V_2 is, in such a state in the classically allowed region, not much smaller than V_1, i.e., the harmonic potential makes an appreciable contribution to the ground state energy E_0. If, however, Δ is of order 10^5 to 10^6, the value of ρ corresponding to the right-hand classical turning point is so small that $V_2\ll V_1$ in the whole of the allowed region. This means that the energy of the ground state is essentially determined by the effective attraction caused by the presence of the Chern-Simons term and not by the harmonic attraction; the role of the latter reduces merely to guaranteeing the existence of a stationary state. Correspondingly, increasing ω in this case hardly affects the position of the level as long as σ does not become so small that $f(\alpha\rho)\ll 1$ at the right-hand turning point, which again means a transition to a regime of noninteracting bosons.

We now consider states with $-1<\nu<0$, i.e., $-\Delta<M<0$. They differ from the state with $M=0$ due to the presence of the centrifugal barrier, so that there exists a left-hand classical turning point. For extremely small ω, these states behave like the one considered above (in this case the larger $|M|$ within the indicated limits, the lower the energy). However, when ω increases the picture may be different. Since in the states considered, the average value of ρ is larger than in the one with $M=0$ (the particles are farther from one another), the frequency range between the "drop" in the level in the well and the start of a considerable growth of its energy due to the increase in ω is narrower. For sufficiently large $|M|$, this region (in which

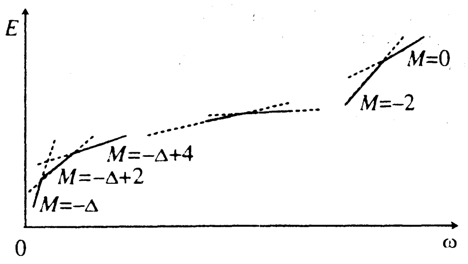

FIG. 3. Behavior of the ground state (solid line).

the energy of this level is practically constant) disappears. Of course, it does not have sharp boundaries, so that its disappearance is not a sudden process.

Levels with $-1<\nu<0$ thus behave with increasing ω qualitatively as follows: initially there is linear growth ($E\approx|M+\Delta|\omega$), then a region where the dependence is weak ($\partial E/\partial\omega$ is small), and finally again linear growth ($E\approx|M|\omega$); the transition from one regime to another is gradual (moreover, the width of the transition region may be of the same order or even larger than the width of any of the three regions mentioned). The larger $|M|$ is, the narrower the middle region.

It is interesting to trace the behavior of the ground state. We assume for definiteness that Δ is an even number. For extremely small ω there is then a single ground state with $M=-\Delta$ (we denote its energy by $E_{-\Delta,0}$). However, the quantity $E_{-\Delta,0}$ increases with ω faster than $E_{-\Delta+2,0}$, so that for some value of ω there is an intersection of levels, after which the state with $M=-\Delta+2$ becomes the ground state. For some yet larger value of ω, it is replaced in turn by the state with $M=-\Delta+4$, and so on, until $M=0$ (Fig. 3). For sufficiently large Δ, it turns out that over some range of ω values, some state is the ground state while its energy is weakly dependent on ω, as we described above. The pressure at $T=0$ (and at sufficiently low temperatures) is therefore rather low in that ω range, and in particular, may become less than the pressure at some lower values of ω. This indicates a situation analogous to the one occurring in a van der Waals gas, the pressure of which under well-known conditions decreases when the volume decreases, which indicates the formation of unstable states which are interpreted as belonging, in fact, to the phase transition region. We conclude that there are indications of the presence of a van der Waals type phase transition in the gas of particles which interact through a Maxwell–Chern–Simons field, caused by the effective attraction due to the presence of a Chern–Simons term.

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