Kinetics of the onset of long-range order during Bose condensation in an interacting gas

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The kinetics of the onset of long-range order during very nonequilibrium Bose condensation in an interacting Bose gas is analyzed. The nonequilibrium state which arises after the establishment of short-range order is distinguished by both the presence of a vortex structure—a clump of vortex filaments—and the presence of anomalously large fluctuations in the regular part of the phase of the order parameter. The latter can be expressed in terms of a highly nonequilibrium distribution of long-wave phonons. The rise time of the topological long-range order (and thus the rise time of true superfluidity) associated with the annihilation of the vortex structure is found to be macroscopically long (and to depend on the size of the system). The time required for the appearance of a genuine condensate, for which relaxation of long-wave phonons is also required, is also found to be macroscopically long. Depending on the conditions, this time may turn out to be longer than the relaxation time of the vortex structure.

1. INTRODUCTION

The interesting problem of the kinetics of the formation of a Bose condensate has recently taken on particular importance in connection with attempts to experimentally observe Bose–Einstein condensation and superfluidity in nonequilibrium systems with a finite lifetime. The systems involved here are systems such as spin-polarized atomic hydrogen (Refs. 1–3 for example), a gas of excitons[4–6] or biexcitons (Ref. 7, for example) created in a pulsed fashion, and atomic systems at ultralow temperatures reached by laser cooling (Ref. 8, for example). In all these cases we are dealing with gaseous systems of low density, i.e., with weakly interacting Bose gases. A study of the kinetics of Bose–Einstein condensation in this case can reveal the basic features of this phenomenon in an arbitrary Bose system, while retaining the possibility of a quantitative comparison of theoretical and experimental results.

Since there is no Bose condensate at the initial time, while at equilibrium the system must have acquired a condensate with a macroscopic number of particles—a number comparable to the total number of particles—the kinetic problem is a problem of the time evolution of a highly nonequilibrium system. There is no single relaxation time under these conditions, as was discussed in Refs. 9 and 10. This time depends on the phenomenon under study or, more precisely, on the correlation properties which determine this phenomenon. The time scales for the formation of the various correlation properties may be radically different. On the other hand, the methods actually used to experimentally observe Bose–Einstein condensation are based on the implicit assumption that the corresponding correlation properties are established over times shorter than the lifetime of the system.

The onset of a hierarchy of times is associated with that step of the evolution in which most of the particles which eventually form the equilibrium condensate are in the low-energy interval $\varepsilon < n_0\tilde{U}$, where n_0 is the equilibrium density of the condensate, $\tilde{U}=4\pi\hbar^2 a/m$ is the effective vertex of the two-particle interaction, a is the s-wave scattering length, and m is the mass of the particles. In this so-called coherent region, the kinetic energy of the particles is lower than the potential energy, and the particle modes cease to be independent. The strong interaction between modes leads to the progressive formation of coherent correlation properties. In this case, in view of the large occupation numbers $n_{\mathbf{k}} \gg 1$, we know that the system can be described adequately in the coherent region by a classical c-number field $\psi(\mathbf{r},t)$ whose evolution obeys the equation ($\hbar=1$)

$$i\frac{\partial\psi}{\partial t}=-\frac{\nabla^2}{2m}\psi+\tilde{U}|\psi|^2\psi. \tag{1}$$

In the limit $t\to\infty$ the solution of Eq. (1) corresponds to a complex order parameter with certain values of the absolute value and phase characterizing the equilibrium state of the interacting Bose gas for $T<T_c$. In the course of the evolution, the magnitude and phase of the field ψ fluctuate wildly. The coherent correlation properties arise only after these fluctuations relax.

The first and fastest stage of the relaxation corresponds to a substantial suppression of nonequilibrium fluctuations of the magnitude of ψ, i.e., of the density.[9] The corresponding time scale is equal to the correlation time

$$\tau_c=1/n_0\tilde{U}. \tag{2}$$

This step of the relaxation leads to the appearance of a short-range order in regions with a typical size

$$r_c=(2mn_0\tilde{U})^{-1/2}, \tag{3}$$

with correlation properties which are approximately the same as those characteristic of systems with an intrinsic condensate. All processes for which the dependence on the

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presence of a Bose condensate stems from the correlation properties at distances $r \lesssim r_c$ behave as if the formation of the Bose condensate had gone to completion.[9] In particular, this is true of the decrease in the probability for inelastic processes[11,12] and of the nonlinear increase in reaction rates[13] with increasing density which accompanies Bose–Einstein condensation in an external field.[14] The state which forms in this step thus contains a quasicondensate, a characteristic feature of which is the presence of small fluctuations in the field amplitude (i.e., in the density of the quasicondensate), while the large fluctuations in the phase generally persist. The latter circumstance predetermines the absence of a long-range order. In the momentum representation of the particles, there is of course no genuine δ-function peak in this case, as would be characteristic of a Bose–Einstein condensate for $T < T_c$.

There are two aspects to the absence of a long-range order. The first stems from the unavoidable appearance of a large number of topological defects in the course of the evolution of the nonequilibrium system. These defects are vortex rings or a clump of vortex filaments, whose maximum size is limited only by the size of the system. In general, this situation leads to the absence of topological long-range order and thus to the absence of a true macroscopic superfluidity. The second aspect is associated with the anomalously large fluctuations in the nonsingular (smooth) part of the phase. These fluctuations disrupt the nondiagonal long-range order (i.e., prevent the formation of a true condensate with a δ-function peak in the density at $k=0$; more on this below). The time scales of the relaxation of the topological defects and of the nonequilibrium phase fluctuations turn out to depend on the geometric dimensions of the system. In this sense they are arbitrarily large in comparison with τ_c given by (2) and also in comparison with the time scale τ_{kin}, which corresponds to the evolution of the system before the particles of the future condensate reach the coherent energy interval. The time τ_{kin} is determined by the usual time scale for binary collisions of particles:

$$\tau_{\text{kin}} = 1/n\sigma v_T, \tag{4}$$

where σ is the scattering cross section, and v_T the thermal velocity of the particles.

Our purpose in the present paper is to analyze the kinetics of the onset of the long-range order, the onset of a macroscopic superfluidity, and the onset of a true condensate.

The appearance of a well-developed vortex structure during the evolution of a nonequilibrium system can be understood easily on the basis of the following simple considerations. We consider a time $t \gg \tau_c$, at which quasicondensate correlation properties have already arisen in regions of size $r_0(t) \gg r_c$. At this time there are obviously no phase correlations at distances $r > r_0(t)$. We imagine that this system is broken up into blocks of size $l > r_0(t)$. The absence of a coupling between the values of the phases in neighboring blocks predetermines the appearance of vortex rings with a size on the order of l. If we break up the space into blocks of large size l', then the same arguments lead to the conclusion that vortex rings of size l' arise. As a result we can assert that the number of vortex rings of size $\sim R$ in a unit volume has a dependence

$$W(R) \sim 1/R^3. \tag{5}$$

The primary relaxation channel for the vortex rings is the self-annihilation of these rings which results from a dissipative interaction with elementary excitations of the system (Ref. 15, for example). The small rings, with higher velocities, are the first to disappear. Let us assume that the minimum radius in the distribution (5) at the time t is $\bar{R}(t)$. It is easy to verify that the average ring radius is on the order of $\bar{R}(t)$ and that the total length L of a vortex filament in a unit volume is

$$L(t) \sim 1/\bar{R}^2(t). \tag{6}$$

This assertion means that the relaxation of the vortices at any time t is actually associated with the rings of a single size scale, $\bar{R}(t)$.

Note that the vortex structure which arises in the course of the evolution is apparently more complex, being somewhat more reminiscent of vortex clumps than of ensembles of vortex rings. In studying the kinetics of such a clump (Ref. 16, for example), one uses the average distance between the filaments as a basic length scale. The same length scale determines the average radius of curvature of a characteristic element of the vortex structure. Since this parameter is associated with the total filament length L by the same relation (6), the kinetics of these two models is actually the same.

Since the radius of the core of vortices is of order r_c [see (3)], we can ignore the core thickness for $t \gg \tau_c$, assuming $\bar{R}(t) \gg r_c$. In general, the phase Φ of the field ψ can be written as the sum

$$\Phi = \Phi_0 + \varphi. \tag{7}$$

Here Φ_0 is the value of the phase imposed by the configuration of the vortex structure, while φ is the regular part of the phase. To eliminate some ambiguity we must supplement the relation

$$\oint \nabla \Phi_0 d\mathbf{l} = \pm 2\pi$$

(for an arbitrary contour around the vortex filament) with the condition $\nabla^2 \Phi_0 = 0$ (everywhere except at the vortex filament itself).

Using the same picture of uncorrelated blocks, we see that the difference between the values of the phase φ in neighboring blocks, smoothed over the size of a block, is on the order of π. Since the size $l > r_0(t)$ of a block is arbitrary, we have the following estimate of the Fourier components of the phase φ which correspond to $k \ll 1/r_0(t)$:

$$|\varphi_{\mathbf{k}}|^2 \sim 1/k^3. \tag{8}$$

After a quasicondensate forms, the spectrum of nonequilibrium elementary excitations becomes an acoustic spectrum.[9] Using the known relationship between $|\varphi_{\mathbf{k}}|^2$ and the phonon occupation numbers (Ref. 17, for example), we can reduce the problem of the relaxation of the fluctuations of the field φ to the problem of the relaxation of a nonequilibrium phonon distribution with occupation numbers

$$n_{\mathbf{k}} \sim \frac{n_0}{mc} \frac{1}{k^2}, \quad k \ll r_0^{-1}, \tag{9}$$

where c is the sound velocity. It is interesting to note that this distribution leads to a logarithmic divergence of the correlation function $\langle \varphi(\mathbf{r}) \varphi(0) \rangle$ at large distances, in close analogy with the corresponding result for the 2D case.

The kinetics of the formation of the long-range order thus reduces to the relaxation of a vortex structure whose total length depends on the leading scale in accordance with (6) and on the nonequilibrium phonon distribution in (9).

2. FRICTION FORCE EXERTED ON A VORTEX FILAMENT

In analyzing the relaxation of a vortex structure, we must first determine the frictional force f exerted on a unit length of a vortex filament as the latter moves through the interacting Bose gas. We are interested in the temperature range

$$n_0 \tilde{U} \ll T \ll T_c. \tag{10}$$

In this range, thermal excitations are essentially identical to free particles, and their wavelength satisfies the inequality

$$\lambda\!\!\!^{-} \ll r_c. \tag{11}$$

The wavelength of the particles is thus small in comparison with the core radius of the vortex filament, which is of order r_c, as we have already mentioned. We can thus treat the motion of the particles in the core region semiclassically. By virtue of (10), a particle is scattered only weakly by a vortex core, transferring only a small fraction of its momentum to the vortex. It is easy to show that the momentum transferred in the direction of motion in a single scattering event is

$$\Delta p_x \sim p_T (n_0 \tilde{U}/T)^2, \tag{12}$$

where p_T is the thermal momentum of the particles. In the coordinate system moving with a filament at a velocity v, the flux of normal excitations across a unit area is

$$j = \rho_n v/m. \tag{13}$$

Writing the longitudinal friction force in the form

$$f = Dv,$$

and using (12) and (13), we find

$$D \sim \frac{\rho_n}{m} r_c p_T \left(\frac{n_0 \tilde{U}}{T} \right)^2.$$

Using the temperature dependence $p_T \sim \sqrt{T}$ and $\rho_n = \rho (T/T_c)^{3/2}$, we can rewrite this expression as

$$D \approx \rho \Gamma_0 (n_0 \tilde{U}/T_c)^{3/2}, \tag{14}$$

where

$$\Gamma_0 = 2\pi\hbar/m \tag{15}$$

is the quantum of circulation. [A numerical factor has been omitted from (14).]

We have derived a very important result: Over the broad temperature range corresponding to (10) in a Bose gas, the frictional force exerted on a moving vortex filament is totally independent of the temperature. For $T \sim n_0 \tilde{U}$, i.e., at the lower limit of the interval (10), which coincides with the boundary of the acoustic region, expression (14) is literally the same as the result derived by Iordanskiĭ[18] for the frictional force exerted on phonons.

In an analysis of the evolution of a vortex structure, the magnitude of the dissipative interaction is characterized by the dimensionless parameter

$$\alpha = D/\rho_s \Gamma_0. \tag{16}$$

In the case at hand, this parameter becomes

$$\alpha \approx (n_0 \tilde{U}/T_c)^{3/2}. \tag{17}$$

In an interacting Bose gas, in the temperature interval (10), the parameter α is thus independent of T and is always small in comparison with unity. At lower T, in the phonon region, the parameter α begins to fall off sharply with decreasing temperature:[18]

$$\alpha \sim T^5. \tag{18}$$

3. RELAXATION OF A VORTEX STRUCTURE

A vortex ring of radius R (or an element of a vortex structure with a radius of curvature R), moving with a proper velocity $v_R \sim 1/R$ through a fluid at rest, is known to undergo a continual decrease in size because of energy dissipation in the course of the interaction with the excitations of the system. The lifetime of such a ring before complete self-annihilation is given by (Ref. 15, for example)

$$\tau_R \approx \frac{2\pi \rho_s R^2}{\gamma \ln(R/r_c)}, \tag{19}$$

where

$$\gamma = \frac{\gamma_0 \rho_s^2 \Gamma_0^2}{\gamma_0^2 + (\rho_s \Gamma_0 - \gamma_0')^2}. \tag{20}$$

In the case at hand, and in the temperature interval (10), we have $\gamma_0' \approx D' \approx -\Gamma_0 \rho_n$ (Ref. 19) and $\gamma_0 \approx D$. Hence

$$\gamma \approx \gamma_0 \approx D. \tag{21}$$

Using (16) and (17), we find the following expression for the time τ_R:

$$\tau_R \approx \frac{2\pi}{\alpha \Gamma_0} \frac{R^2}{\ln(R/r_c)}. \tag{22}$$

This time is of course also independent of the temperature.

We now wish to determine the distance l_V, over which a ring of radius R moves in a time τ_R. We can find l_V directly:[15]

$$l_V \approx \rho_s \Gamma_0 R/\gamma.$$

In the case at hand we find

$$l_V \approx R/\alpha. \tag{23}$$

An important point is that in a weakly interacting Bose gas the following condition holds in all cases:

$$l_V/R \gg 1. \tag{24}$$

This result means that in the structure which arises in the course of the evolution of a nonequilibrium Bose gas the vortex rings of any scale undergo multiple intersections before they collapse. That this is true can be seen even from the distribution (5), which predicts that a ring of radius R will encounter at least a ring of similar size over a distance $\sim R$. The same can be said of a vortex clump, for which there exists, at any time, a length scale $\bar{R}(t)$ which characterizes the average distance between filaments and the effective radius of curvature of linear elements in the clump.

Under these conditions, we are naturally led to ask about the probability for the reconnection of filaments when they intersect each other. If this reconnection probability is small, the inequality (24) plays no role. The relaxation of the vortices in this case would be determined at each instant by the minimum size of the unannealed vortex rings [which actually determines the total length of the vortex filaments, according to (6)] and by the corresponding value of τ_R from (22). In a clump a corresponding role is played by $\bar{R}(t)$ and thus $\tau_{R=\bar{R}(t)}$. Actually, however, there is no basis for assuming that the reconnection during the intersection of vortex filaments would be unlikely. Indeed, the opposite opinion has been expressed quite frequently (Ref. 16, for example). If this is the case, then the picture of the relaxation of a vortex structure in the limit $\alpha \ll 1$ will change substantially. The reason is that the reconnections should lead to the appearance (superposed the main size scale, dictated by the characteristic distance between filaments) of kinks of smaller size. A kink with a small radius of curvature r will rapidly "precess" around the main filament ($v_r \sim 1/r$), causing a relatively rapid relaxation of this small element of the vortex structure. In the limit $\alpha \ll 1$ [see (23)], a large number of such kinks should arise at the leading scale, and specifically these kinks can dominate factors in the relaxation of the vortex structure. In the limit $\alpha \ll 1$, and with a reconnection probability of order unity, a relaxing vortex structure will obviously constitute a clump of vortex filaments, not a hierarchy of vortex rings.

Milliken *et al.*[20] have suggested that the appearance of kinks due to reconnection should lead to a substantial increase in a dissipative interaction. They demonstrated that the incorporation of this circumstance leads to Vinen's empirical equation,[21] although this equation is not the same as when it was originally proposed. The suggestion by Milliken *et al.*[20] that the kinetics of the intersection of vortex lines is governed by the local kink velocity looks incorrect: It is quite obvious that this kinetics should be determined by the velocity of the elements of a filament of the main scale $\bar{R}(t)$. This circumstance changes the results considerably.

With condition (24) in mind, we can suggest (and will demonstrate below) that for each value of α there is a certain corresponding characteristic number of kinks on an element of a vortex filament of size $\bar{R}(t)$. For simplicity we assume that all the kinks have the same radius of curvature r. It is natural to assume that in the course of the intrinsic evolution (in the absence of reconnections and in the absence of dissipative self-annihilation) the number of kinks would remain the same.

We denote by N the number of kinks per unit length of a vortex filament, and we write the balance equation for this quantity as follows:

$$\dot{N} = N_+ - N_-. \tag{25}$$

The increase in the number of kinks, N_+, is associated with the number of intersections accompanied by reconnection. Statistically, there are $1/\bar{R}$ large-scale elements of the half-ring type over a unit length of a vortex structure which undergo intersection in a time

$$\tau_{\text{int}} \sim \bar{R}/v_{\bar{R}},$$

where $v_{\bar{R}}$ is the velocity of an element of size $\bar{R}$. We know[22] that the intrinsic velocity of an arbitrary element of a vortex filament is actually determined by the local radius of curvature of this filament. For an element with kinks, however, the local radius of curvature corresponds to the velocity at which the kinks precess around the average position of the filament, not by the velocity of the element as a whole. It is reasonable to assume that the velocity of the latter is determined by the "global" radius of curvature of scale $\bar{R}$. We are then led to the estimate

$$v_{\bar{R}} \sim \frac{\Gamma_0}{4\pi} \frac{1}{\bar{R}} \ln \frac{\bar{R}}{r_c}. \tag{26}$$

Using the relation between $\bar{R}$ and L [see (6)], we then find

$$N_+ \approx \xi_1 \frac{\Gamma_0}{4\pi} L^{3/2} \ln \frac{\bar{R}}{r_c}. \tag{27}$$

The numerical coefficient ξ_1 also contains a reconnection probability factor.

The decrease in the number of kinks, N_-, is determined by their self-annihilation. Let us assume that we have a distribution of kinks with respect to radius r which is characterized by a function f_r. Expressed in terms of a unit length of the vortex filament, this function is normalized by the condition

$$\int f_r dr = N. \tag{28}$$

Under the assumption that the kinks do not overlap, we have $r_N \approx 1/N$ as an upper limit on this integral. If the size distribution of the kinks which are produced is comparatively uniform, then it is easy to see that small kinks manage to undergo "annealing" in the course of the evolution, and the function f_r shifts toward larger sizes. (The annealing of small kinks itself causes only a very slight decrease in the total length of the vortex filament, L.) Clearly, the self-annihilation of the bulk of the kinks will be characterized by a time determined by the length scale r_N. Correspondingly, using expression (22), we find the following expression for N_-:

$$N_- \approx \xi_2 \frac{\Gamma_0 \alpha}{2\pi} N^3 \ln \frac{r_N}{r_c}. \qquad (29)$$

Along with (25), we need to write an equation for L:

$$\dot{L} = L_+ - L_- . \qquad (30)$$

The "incoming" term L_+ is not present in the pure relaxation problem, since it exists only to the extent that the quantity $v_{\rm ns}$ (the velocity of the relative motion of the normal and superfluid components) is nonzero. We will nevertheless retain this term, keeping the more general problem in mind. As was found by Vinen,[21] we have

$$L_+ \approx \xi_3 \alpha v_{\rm ns} L^{3/2}. \qquad (31)$$

The disappearance of kinks described by expression (30) is accompanied by a decrease by $\sim N_- \pi r_N$ in the length of a vortex filament per unit time. This quantity, which is expressed per unit length must be multiplied by L in order to find the rate of decrease of the total length of the filaments in a unit volume. As a result we find the following expression for L_-:

$$L_- \approx \xi_4 \Gamma_0 \alpha N^2 L \ln \frac{r_N}{r_c}. \qquad (32)$$

In analyzing the question of the relaxation of topological defects, we need to solve Eqs. (25) and (30) simultaneously, setting $L_+ = 0$ in the latter. This nonlinear system of equations has the simple solution

$$L = A/t. \qquad (33)$$

From (30) and (32) we immediately find [ignoring the logarithmic N dependence in (32) and (29) and the L dependence in (27)]

$$N(t) = \left(\xi_4 \Gamma_0 \alpha \ln \frac{r_N}{r_c} \right)^{-1/2} \frac{1}{t^{1/2}}. \qquad (34)$$

Substituting (33) and (34) into (25), and using (27) and (29), we find A and also our final expression for L:

$$L(t) = \xi \frac{1}{\Gamma_0 \alpha^{1/3}} \left(\ln^2 \frac{\bar{R}}{r_c} \ln \frac{r_N}{r_c} \right)^{-1/3} \frac{1}{t}. \qquad (35)$$

The constant ξ in this expression is made up of the constants ξ_1, ξ_2, and ξ_3. The constant ξ is proportional to the reconnection probability factor in the course of an intersection, raised to the power $-2/3$.

We can draw several important conclusions from (34) and (35). We first note that the number of kinks over the length scale $\bar{R}(t)$ of the vortex structure is totally independent of the time:

$$N(t)\bar{R}(t) = \frac{N(t)}{\sqrt{L(t)}} \sim \frac{1}{\alpha^{1/3}} (t)^0. \qquad (36)$$

This result means that the size scale of the kinks which determine the relaxation of the vortex structure follows $\bar{R}(t)$ in a linearly [the solution in (34) and (35) was determined for times such that the conditions $\bar{R}(t) \gg \bar{R}(0)$ and $r_N(t) \gg r_N(0)$ hold]. The increase in the number of kinks with decreasing α stems from the increase in the ratio (24). From (35) we find the time scale for the onset of long-range order in a system of size D_0:

$$\tau_V(D_0) \sim \frac{D_0^2}{\Gamma_0 \alpha^{1/3}} \frac{1}{\ln(D_0/r_c)}. \qquad (37)$$

This time and hence the rise time of the intrinsic superfluidity thus depend on the size of the system. In general, this time may be many orders of magnitude greater than the formation time of a quasicondensate. Only in small volumes can the time scale for the appearance of true superfluidity become comparable to the quasicondensate formation time. Since we have $\tau_V \sim 1/\alpha^{1/3}$, we would have $\tau_V \sim 1/\alpha$ in the absence of reconnections [see (22)]. This is very important, especially if the temperature falls below $n_0 U$ when equilibrium is reached, where α falls off very sharply with decreasing T [see (18)]. Reconnection may thus turn out to be extremely important for the possibility of observing superfluidity in nonequilibrium systems which have a finite lifetime.

The system of equations found here for $\dot{N}$ and $\dot{L}$ is valid for an arbitrary system in the limit $\alpha \lessdot 1$; in particular, it is valid for describing the kinetics of a vortex structure in helium. (In this case, of course, α would have a different temperature dependence.) Interestingly, the steady-state solution of Eqs. (25) and (30) leads to the result

$$L \sim (v_{ns}/\Gamma_0)^2 \alpha^{4/3}, \qquad (38)$$

which reproduces well the $L(T)/v_{\rm ns}^2$ dependence found at low T [$\alpha(T) \lessdot 1$] in helium in experiments in which the vortex structure is generated in oppositely directed flows of normal and superfluid components.

4. FORMATION OF NONDIAGONAL LONG-RANGE ORDER

In general, the appearance of topological long-range order and thus macroscopic superfluidity does not in itself imply the onset of nondiagonal long-range order with a constant value of the phase along a homogeneous system. If the nonequilibrium phonon distribution in (9) has not relaxed by this time, it will again disrupt phase correlations at large distances. The reason is the anomalous dependence of $n_{\bf k}$ on k, which leads to an infrared divergence in the phase correlation function. The presence of superfluidity or the absence of nondiagonal long-range order makes a nonequilibrium 3D system very similar to an equilibrium 2D system for $0 < T < T_{\rm KT}$, where $T_{\rm KT}$ is the Kosterlitz–Thouless temperature.

The damping of nonequilibrium phonons naturally falls off with decreasing k. This statement means that the time scales for the formation of nondiagonal long-range order are determined by the relaxation of extremely long sound waves. In a homogeneous system, the following conditions holds for them:

$$kl \ll 1, \qquad (39)$$

where l is the mean free path of the particles. Relation (39) corresponds to the hydrodynamic regime of sound dissipation. By the time t, quasicondensate regions with sizes on the order of $\bar{R}(t)$ are free of vortices. The attenuation of sound with $k \sim \bar{R}^{-1}(t)$ is thus approximately the same as that in a genuine superfluid. We can thus use the superfluid hydrodynamic equations (Ref. 23, for example).

In an analysis of sound in this case, the system of linearized hydrodynamic equations can be reduced to the two following equations when dissipation is ignored:

$$\frac{\partial^2 \rho}{\partial t^2} = \Delta p, \quad \frac{\partial^2 s}{\partial t^2} = \frac{\rho_s s^2}{\rho_n} \Delta T. \tag{40}$$

Here s is the entropy per unit mass. Treating ρ and T as independent thermodynamic variables, we find the following results for small deviations of the parameters in the sound wave:

$$p' = \left(\frac{\partial p}{\partial \rho}\right)_T \rho' + \left(\frac{\partial p}{\partial T}\right)_\rho T', \quad s' = \left(\frac{\partial s}{\partial \rho}\right)_T \rho' + \left(\frac{\partial s}{\partial T}\right)_\rho T'. \tag{41}$$

Substituting (41) into (40), we find a system of equations for ρ' and T'. Equations (40) and (41) hold in the general case. Using these equations for a weakly interacting Bose gas, we find results which are quite different from the results for superfluid helium.

In the temperature interval (10), normal excitations have a dispersion relation which is essentially the same as that for free particles. The normal component thus has properties very similar to those of ideal Bose gas. It thus becomes possible to analyze the solution of Eqs. (40) and (41) directly.

We denote by p_T the temperature-dependent part of the motion. In an ideal Bose gas we have, as we know,

$$\left(\frac{\partial p_T}{\partial \rho}\right)_T = 0.$$

We thus have

$$\left(\frac{\partial p}{\partial \rho}\right)_T \approx c^2, \tag{42}$$

where c is the Bogolyubov sound velocity

$$c = \sqrt{n_0 \tilde{U}/m}. \tag{43}$$

On the other hand, we have

$$c_v,\ s,\ \rho_n \sim (T/T_c)^{3/2}, \quad c_p/c_v = 5/3, \tag{44}$$

where c_v and c_p are the specific heats at constant volume and constant pressure, respectively.

Using these relations, we find that the system (40) is amenable to the following self-consistent procedure. In the first equation we can ignore the term with $\Delta T'$, while in the second we can omit the term containing $(\partial^2 T')/(\partial t^2)$. As a result, it follows from the first equation that the sound velocity is given by (43); from the second equation we find

$$\frac{T'}{T} \approx \frac{n_0 \tilde{U}}{T} \frac{\rho'}{\rho}. \tag{45}$$

Actually we have found only one mode: the so-called condensate acoustic mode. However, this is precisely the mode in which we are interested, since the other mode is essentially independent of oscillations of the phase of the condensate wave function.

Since the product ρs is independent of the density in an ideal Bose gas, we can conveniently determine the normal velocity $\mathbf{v}_n$ from the equation

$$\frac{\partial(\rho s)}{\partial t} + \rho s \nabla \mathbf{v}_n = 0.$$

Using (44) we then find

$$v_n \approx 3cT'/2T. \tag{46}$$

Having derived results (45) and (46), we can use the standard procedure for determining the attenuation of sound (Ref. 23, for example). A direct analysis shows that under the conditions assumed here the dissipation due to viscosity is small in comparison with that due to thermal conductivity. As a result, we find the following expression for the time scale for the attenuation of sound:

$$\tau_{\mathrm{ph}}^{-1}(k) \approx \frac{\kappa}{n} \frac{n\tilde{U}}{T} k^2, \tag{47}$$

where κ is the thermal conductivity. A direct calculation of κ puts this result in the form

$$\tau_{\mathrm{ph}}^{-1}(k) \sim \frac{T}{\hbar} \frac{1}{na} k^2 \tag{48}$$

(we are now writing $\hbar$ explicitly again). The time τ_{ph} simultaneously determines the time scale for the damping of phase fluctuations, $\tau_\varphi(k)$. Let us compare the relaxation of a vortex structure with radial scale $\bar{R}$ [see (37)] with $\tau_\varphi(k=\bar{R}^{-1})$:

$$\frac{\tau_V(\bar{R})}{\tau_\varphi(k=\bar{R}^{-1})} \sim \frac{1}{(na^3)^{1/2}} \frac{T}{T_c} \frac{1}{\ln(\bar{R}/r_c)}. \tag{49}$$

At the temperatures under consideration here, this ratio is essentially always much greater than unity. We thus find an important result, which shows that phase fluctuations relax more rapidly than the vortices annihilate and that the onset of topological long-range order is accompanied by the onset of nondiagonal long-range order.

The picture of phonon relaxation changes significantly in a system of bounded size if the mean free path of the particles is set by this size or even becomes larger than D_0 in the case of specular reflection (a "magnetic wall"). In this case, condition (39) is violated, and the hydrodynamic sound-absorption regime is disrupted. A direct scattering of phonons by normal excitations becomes the dominant mechanism. In this case the phonon relaxation time found in Ref. 24 turns out to be

$$\tau_{\mathrm{ph}}^{-1}(k) \approx 30 Tak/\hbar. \tag{50}$$

For $lk \sim 1$, where $l \approx 1/8\pi a^2 n$, expression (50) becomes the same as (48).

Comparing this result with the limiting value $k \sim D_0^{-1}$ with $\tau_V(D_0)$, we find

$$\frac{\tau_V(D_0)}{\tau_\varphi(D_0)} \sim 10 \frac{T}{T_c} \frac{D_0}{r_c}. \qquad (51)$$

Since the core radius is of order r_c, this relation is meaningful only under the condition $D_0 \gg r_c$. In many cases, the ratio (51) is much greater than unity. The qualitative assertions made above thus remain in force. When the gas parameter is sufficiently small, and T is relatively close to the low-energy boundary of the interval (10), however, the ratio (51) may become less than unity. This means qualitatively that the nondiagonal long-range order is reached after the topological long-range order, and there exists a time interval in which there is true superfluidity but no nondiagonal long-range order. In this case the relaxation time of nonequilibrium phase fluctuations is determined by

$$\tau_\varphi \sim 10^{-2} \frac{\hbar}{T} \frac{D_0}{a}. \qquad (52)$$

5. CONCLUDING COMMENTS

It follows from this analysis that the kinetics of the Bose condensation in an interacting gas is characterized by a hierarchy of relaxation times. The shortest time scale, τ_c given by (2), determines the time scale for the formation of a quasicondensate after the excess of particles which form the condensate at equilibrium reaches the energy interval $\varepsilon < n_0 \widetilde{U}$. This stage is preceded by a stage of evolution in the kinetic region $\varepsilon > n_0 \widetilde{U}$, which is characterized by a collision time $\tau_{\rm kin}$ [see (4)]. Since the condition $\tau_{\rm kin} \gg \tau_c$ holds in a low-density gas, the formation of a quasicondensate as a whole is determined by the time scale $\tau_{\rm kin}$. Beginning immediately thereafter is the stage in which long-range order develops, which is characterized primarily by the annihilation time of the vortex structure, $\tau_V(D_0)$ [see (37)]. This time, which is responsible for the appearance of topological long-range order and thus true superfluidity, turns out to depend on the size of the system: $\tau_V \sim D_0^2/\ln(D_0/r_c)$. This time may be very long. Vortices may annihilate at a surface, which we have not discussed in this paper, but this will not change this estimate in any fundamental way. Finally, there is yet another time, τ_φ, the time scale for the decay of anomalous fluctuations in the smooth part of the phase. This is the rise time of nondiagonal long-range order with a macroscopically determined phase which is constant along the system and, correspondingly, with a δ-function peak in the particle distribution at $\varepsilon = 0$. The time τ_φ also turns out to depend on the size of the system. If the particle mean free path satisfies $l \ll D_0$, then we have $\tau_\varphi \sim D_0^2$. It has been found that the condition $\tau_\varphi < \tau_V$ holds in essentially all cases; this condition means that the onset of topological long-range order is accompanied by the onset of total long-range order. If the condition $l < D_0$ is violated, then both the condition $\tau_\varphi < \tau_V$ and the opposite condition $\tau_\varphi > \tau_V$ can hold, depending on the parameter values. In the latter case, the nondiagonal long-range order sets in last. In this case we have $\tau_\varphi \sim D_0$.

We have a comment here. When α is very small, the so-called Kolmogorov regime may change the essentially dissipative relaxation of the vortex structure discussed in Sec. 3 of this paper. This regime would correspond to dissipative annihilation at arbitrarily small scales, which would be preceded by a nondissipative energy flux (approximately equal to the length of the filaments of the vortex structure) in length-scale space, analogous to the Kolmogorov regime of turbulence. This flux may arise, for example, because of the appearance of smaller scales due to reconnection, and it should not depend on α. This problem deserves a special analysis. Here we will simply point out that experimental results on helium indicate that this regime has not yet arisen at values $\alpha = 10^{-2}$ (Ref. 16).

In this paper we have analyzed primarily the kinetics in a homogeneous system. The inhomogeneous case is distinctive because a Bose condensate forms in a small volume, leading to a sharp local increase in density. Accordingly, all the density-dependent processes should be enhanced.[13] A sufficient condition here would be the formation of a quasicondensate and thus relaxation times on the order of $\tau_{\rm kin}$ (and τ_c, if the time at which the coherent region is reached is reduced sharply because of an interaction with an external medium). At the same time, a sharp decrease in the times τ_V and τ_φ should correspond to small sizes.

An interesting fact has emerged from the attempt to experimentally observe Bose-Einstein condensation in an exciton gas created by a laser pulse in a semiconductor:[4,5] During cooling, the trajectory in the (n,T) plane reached the phase-transition line comparatively rapidly, but did not intersect this line. Consequently, Bose condensation did not occur. Analysis of the kinetics of Bose condensation shows that this could not be caused by the long time scale for the formation of a quasicondensate. We believe that the reason is a "burnup" of a (quasi-) condensate due to a rapid growth of processes which consume excitons (possibly surface recombination). The motion of a trajectory in the course of evolution along a phase-transition line, on the other hand, can be understood if we note that at $c_p/c_v = 5/3$ adiabatic expansion of an exciton gas should lead to specifically this evolution picture. A recent study[25] of the example of spin-polarized atomic hydrogen in a magnetic confinement system clearly demonstrated that a density-dependent decay channel precedes any significant penetration into the Bose–Einstein condensate region.

The question of the possible superfluidity of a nonequilibrium exciton gas, proposed in Refs. 4–6, requires a separate analysis. That problem is complicated because the exciton gas created in a narrow layer of order 1 μm in size expands at a velocity on the order of the sound velocity. This expansion may promote a growth of the vortex structure.

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