Elastostatic spin waves

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We construct a theory for a new class of magnetic excitations, elastostatic spin waves (ESW), and find that the magnetoelastic interaction plays the same role in their formation as magnetostatic interaction in the formation of magnetostatic spin waves. To classify ESW in finite magnetic materials we use the example of a two-sublattice model for a rhombic antiferromagnet. For a number of cases we obtain the ESW dispersion law.

INTRODUCTION

Two mechanisms of spectrum formation, exchange interaction and magnetodipole interaction,[1-3] occupy the center of the stage in studies of spin waves in various magnetically ordered crystals. In the extreme cases when one of these interactions dominates in a magnetic material, one speaks of either exchange or magnetostatic spin waves. As for another type of interaction, magnetoelastic, its effect on the nature of the spin-wave spectrum is usually considered either in the vicinity of a magnetoacoustic resonance[1] or for small values of the wave vector **k**, when the frequency of the quasiacoustic branch is much lower than the frequency ω of the quasispin branch, $sk \ll \omega$, where s is the velocity of sound. In the latter case the magnetoelastic interaction leads to the appearance of a magnetoelastic gap in the spin-wave spectrum, a gap that manifests itself most vividly in spin-reorientation phase transitions.

Here we will focus on a different region of the magnetoelastic-wave (MEW) spectrum, namely, the region where the wave vectors are fairly large, $k \gg k_r$ (where k_r is the wave vector corresponding to the region of magnetoacoustic resonance), in which $\omega \ll sk$. In many respects this region of the MEW spectrum is similar to magnetostatic waves (MSWs), the only difference being that now the elastic subsystem of the crystal acts as the electromagnetic subsystem. Just as in describing MSWs one uses the equations of magnetostatics instead of the general Maxwell equations, in describing the section of the MEW spectrum of interest to us we use the equations of elastostatics $\partial\sigma_{ik}/\partial x_k = 0$ (where σ_{ik} is the stress tensor) instead of the dynamical equations of elasticity theory. Hence, by analogy with MSW, we call the specific MEW-spectrum branches *elastostatic spin waves*, or ESW.

In contrast to the region of small wave vectors, where the spin subsystem of the crystal is the "rapid" subsystem, for ESWs the elastic subsystem is the "rapid" one. The role of the latter role is reduced to forming an indirect (non-Heisenberg) spin–spin coupling via a quasistatic-phonon field. The long-range nature of this interaction, as the reader will shortly see, leads to a quasinonanalytical[1)] dependence of the ESW frequency on the components of the wave vector just as the long-range nature of magnetostatic interaction leads to a nonanalytical MSW dispersion law in the limit $k \to 0$.

Naturally, both magnetoelastic and magnetodipole interactions are always present in a magnetic material. However, in the antiferromagnets (AFMs) considered here, the magnetoelastic interaction is exchange-enhanced, while the magnetodipole interaction is exchange-inhibited and hence can be ignored.

In addition, ESWs manifest themselves most vividly when the effect of the exchange interaction on the spectrum, notwithstanding the condition $k \gg k_r$, has not yet suppressed the other contributions to the ESW dispersion law. For this the characteristic speeds of the spin waves must be much smaller than s, with the result that T_N is much lower than T_D (where T_N and T_D are the Néel and Debye temperatures, respectively), which occurs in many AFMs. The condition $k \gg k_r$ is fairly stringent, but for ordinary ESWs in finite magnetic materials it can be achieved owing to the component of the wave vector normal to the sample surface. By virtue of size quantization, this component is equal to $\pi n/d$ (where d is the sample thickness and n is an integer) and for fairly thin samples (and $n \neq 0$) can exceed k_r. In this situation, naturally, ESWs can be of either internal or surface (quasisurface) nature (as MSW), and localization of ESWs near the surface can even occur (for appropriate frequencies and wave vectors) when exchange spin waves and MSWs are only bulk waves.

1. GENERAL RELATIONS

For the present study we consider an AFM whose magnetic subsystem can be described within the scope of the two-sublattice model. As shown in Refs. 4 and 5, a suitable approach to describing the dynamics of such an AFM is to employ a Lagrange function written in terms of the unit antiferromagnetism vector $\mathbf{l} = \mathbf{L}/|\mathbf{L}|$, where $\mathbf{L} = \mathbf{M}_1 - \mathbf{M}_2$, with $\mathbf{M}_{1,2}$ the magnetization vectors of the sublattices. With allowance for the elastic subsystem of the crystal the Lagrangian density for an AFM with rhombic magnetic and elastic anisotropy can be written in the form

$$\mathscr{L} = \mathbf{M}_0^2 \left\{ \frac{\alpha}{2c^2} \dot{\mathbf{l}}^2 - \frac{\alpha}{2} (\mathrm{grad}\, \mathbf{l})^2 - \frac{\beta_2}{2} l_z^2 - \frac{\beta_1}{2} l_y^2 - w_{\mathrm{me}} \right\} + \frac{\rho}{2} \dot{\mathbf{u}}^2 - w_{\mathrm{e}}, \qquad (1)$$

where the dot stands for the time derivative, α is the inhomogeneous coupling constant, $c = \frac{1}{2} g M_0 \sqrt{\alpha\delta}$ the characteristic speed (which coincides with the minimal phase

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speed of the spin waves), $M_0=|\mathbf{M}_{1,2}|$, β_1 and β_2 the anisotropy constants, $\mathbf{u}$ the displacement vector, ρ the crystal's density, and

$$w_e=\frac{1}{2}(c_{11}u_{xx}^2+c_{22}u_{yy}^2+c_{33}u_{zz}^2)+(c_{12}u_{xx}u_{yy}+c_{13}u_{xx}u_{zz}) +c_{23}u_{yy}u_{zz}+2(c_{44}u_{yz}^2+c_{55}u_{xz}^2+c_{66}u_{xy}^2), \quad (2)$$

$$w_{me}=(p_{11}l_x^2+p_{12}l_y^2+p_{13}l_z^2)u_{xx}+(p_{21}l_x^2+p_{22}l_y^2+p_{23}l_z^2)u_{yy} +(p_{31}l_x^2+p_{32}l_y^2+p_{33}l_z^2)u_{zz}+2(p_{44}l_yl_zu_{yz}+p_{55}l_xl_zu_{xz} +p_{66}l_xl_yu_{xy}), \quad (3)$$

with $u_{ik}=\frac{1}{2}(\partial u_i/\partial x_k+\partial u_k/\partial x_i)$ the strain tensor, and $c_{\alpha\beta}$ and $p_{\alpha\beta}$ the elastic moduli and the magnetoelastic coupling constants, respectively.

It is well known that due to magnetoelastic interaction some components of the strain tensor are nonzero in the ground state of a magnetic material, which leads to the renormalization of magnetic anisotropy constants.[1,4] For our model (1)–(3) the renormalization has the form

$$\beta_1\to\tilde\beta_1=\beta_1-\left(\frac{p_{66}^2}{c_{66}}+\frac{g_1}{\Delta}\right), \qquad \beta_2\to\tilde\beta_2=\beta_2-\left(\frac{p_{55}^2}{c_{55}}+\frac{g_2}{\Delta}\right), \quad (4)$$

where g_1, g_2, and Δ are rather complicated combinations of the elastic and magnetoelastic constants $c_{\alpha\beta}$ and $\rho_{\alpha\beta}$.

Next we assume that $\tilde\beta_{1,2}>0$. Here, in the ground state of the AFM, the vector $\mathbf{l}$ lies along the x axis, or $l_x^{(0)}=1$ and $l_{y,z}^0=0$.

In analyzing the elementary excitation spectrum it has proved convenient to parametrize the unit vector $\mathbf{l}$ by two angular variables θ and φ:

$$\mathbf{l}=(\sin\theta\cos\varphi,\sin\theta\sin\varphi,\cos\theta). \quad (5)$$

Now, assuming that

$$\theta=\theta^{(0)}+\vartheta,\quad \varphi=\varphi^{(0)}+\psi,\quad u_i=u_i^{(0)}+\tilde u_i, \quad (6)$$

where $\theta^{(0)}$, $\varphi^{(0)}$, and $u_i^{(0)}$ correspond to the ground state of the AFM ($\theta^{(0)}=\pi/2$ and $\varphi^{(0)}=0$), we expand the Lagrangian (1) in a power series in small deviations from the equilibrium state ($\vartheta,\psi,\tilde u\ll 1$) up to second-order terms inclusive: $\mathscr{L}\approx\mathscr{L}_1+\mathscr{L}_{\infty}\mathscr{L}_{\epsilon}$. The term $\mathscr{L}_0$ corresponds to the ground state of the AFM, $\mathscr{L}_1$ vanishes owing to the equations of motion, and the term $\mathscr{L}_2$ describing small oscillations has the form

$$\mathscr{L}=M_0^2\left\{\frac{\alpha}{2c^2}(\dot\theta^2+\dot\varphi^2)-\frac{\alpha}{2}[(\mathrm{div}\,\vartheta)^2+(\mathrm{div}\,\psi)^2] -\frac{\beta_2'}{2}\vartheta^2-\frac{\beta_1'}{2}\psi^2\right\}+\frac{\rho}{2}\dot{\tilde{\mathbf{u}}}^2-\tilde w_e-2M_0^2(p_{66}\tilde u_{xy}\psi -p_{55}\tilde u_{xz}\vartheta), \quad (7)$$

where $\tilde w_e$ differs from w_e of (2) in that u_{ik} is replaced by $\tilde u_{ik}$, and the effective constants β_1' and β_2' are given by the following formulas:

$$\beta_1'=\tilde\beta_1+\frac{p_{66}^2M_0^2}{c_{66}},\quad \beta_2'=\tilde\beta_2+\frac{p_{55}^2M_0^2}{c_{55}}. \quad (8)$$

Note that it is the discrepancy between $\beta_{1,2}'$ and $\tilde\beta_{1,2}$ that causes a magnetoelastic gap to appear in the spin-wave spectrum (at the spin-reorientation phase transition point one of the constants $\tilde\beta_1$ and $\tilde\beta_2$ vanishes, while the spin-wave activation frequencies, which are determined by the effective constants $\beta_{1,2}'$, remain finite).

Starting from the Lagrangian (7), we can easily obtain the equations of motion for the variables ψ, ϑ, and u_i. If we assume that all these variables are proportional to $\exp[i(\mathbf{k}\cdot\mathbf{r}-\omega t)]$, we can write the corresponding system of equations as follows:

$$(c_{11}k_x^2+c_{66}k_y^2+c_{55}k_z^2-\rho\omega^2)u_x+(c_{12}+c_{66})k_xk_yu_y +(c_{13}+c_{55})k_xk_yu_z-ip_{kk}k_y\psi+ip_{55}k_z\vartheta=0, \quad (9)$$

$$(c_{12}+c_{66})k_xk_yu_x+(c_{22}k_x^2+c_{66}k_x^2+c_{44}k_z^2-\rho\omega^2)u_y +(c_{23}+c_{44})k_yk_zu_z-ip_{66}k_x\psi=0, \quad (10)$$

$$(c_{55}+c_{13})k_xk_zu_x+(c_{44}+c_{23})k_yk_zu_y+(c_{33}k_z^2+c_{55}k_x^2 +c_{44}k_y^2)u_z+ip_{55}k_x\psi=0, \quad (11)$$

$$\left(\frac{\alpha\omega^2}{c^2}-\alpha\mathbf{k}^2-\beta_2'\right)\vartheta+ip_{55}(k_xu_z+k_zu_x)=0, \quad (12)$$

$$\left(\frac{\alpha\omega^2}{c^2}-\alpha\mathbf{k}^2-\beta_1'\right)\psi-ip_{66}(k_xu_y+k_yu_x)=0. \quad (13)$$

The system of equations (9)–(13) describes the full spectrum of MEWs in a rhombic AFM, a spectrum that consists of five branches. The respective dispersion equation can be written in general form but is cumbersome, so that analysis is difficult. Hence, as noted in the Introduction, two regions of the MEW spectrum are usually studied, the region where the wave vector is small and the region of magnetoacoustic resonance.

In the elastostatic approximation of interest to us, where $\omega\ll sk\sim\sqrt{c_{\alpha\beta/\rho}}k$, the term $\rho\omega^2$ in the coefficients of Eqs. (9)–(11) should be discarded. But even with this the system of equations (9)–(13) remains fairly complicated although the respective dispersion equation describes two branches of elementary excitations rather than five. For this reason, below we consider various particular cases of orientation of the wave vector $\mathbf{k}$ with respect to the crystallographic axes.

A. $k_x=0$. The system of equations (9)–(13) splits into two independent subsystems. Equations (10) and (14) describe purely elastic excitations, not related to the angular variables ψ and ϑ and, therefore, of no interest to us. The other three equations, (9), (12), and (13), which describe coupled magnetoelastic oscillations in the elastostatic limit, can easily be analyzed. The corresponding dispersion equation has the form

$$\left(\frac{\alpha\omega^2}{c^2}-\alpha\mathbf{k}^2-\beta_1'\right)\left(\frac{\alpha\omega^2}{c^2}-\alpha\mathbf{k}^2-\beta_2'\right)(c_{66}k_y^2+c_{55}k_z^2)$$

$$+\left(\frac{\alpha\omega^2}{c^2}-\alpha\mathbf{k}^2-\beta_2'\right)p_{66}^2k_y^2+\left(\frac{\alpha\omega^2}{c^2}-\alpha\mathbf{k}^2-\beta_1'\right)p_{55}^2k_z^2=0. \tag{14}$$

If the anisotropy constants β_1' and β_2' differ considerably, for instance, if $\beta_1'\ll\beta_2'$, Eq. (14) (allowing for the smallness of the dimensionless magnetostriction constants $\zeta_1=p_{66}^2M_0^2/c_{66}\ll 1$ and $\zeta_2=p_{55}^2M_0^2/c_{55}\ll 1$) yields the equations for the two ESW branches:

$$\omega_1^2\approx\omega_{10}^2+\frac{c^2\zeta_1}{\alpha}\frac{\gamma k_z^2}{k_y^2+\gamma k_z^2}, \tag{15}$$

$$\omega_2^2\approx\omega_{20}^2+\frac{c^2\zeta_2}{\alpha}\frac{\gamma k_y^2}{k_z^2+\gamma k_y^2}, \tag{16}$$

where $\gamma=c_{55}/c_{66}$, and $\omega_{10}^2=c^2\tilde{\beta}_1/\alpha+c^2\mathbf{k}^2$ and $\omega_{20}^2=c^2\tilde{\beta}_2/\alpha+c^2\mathbf{k}^2$ specify the frequencies of the two branches of the spin-wave spectrum calculated without taking into account the coupling with the elastic subsystem.

B. $k_z=0$. The system of equations (9)–(15) splits into two subsystems; in one the variables ϑ and u_z are coupled, in the other the variables ψ, u_x, and u_y.

The first subsystem, which we call B1, describes the ESW branch whose dispersion law is similar to the one described in case A:

$$\omega^2\approx\omega_{20}^2+\frac{c^2\zeta_2}{\alpha}\frac{k_y^2}{\gamma' k_x^2+k_y^2}, \tag{17}$$

where $\gamma'=c_{55}/c_{44}$.

The second subsystem, B2, describes ESWs with a more complicated dispersion law:

$$\omega^2\approx\omega_{10}^2+\frac{c^2\zeta_1}{\alpha}\frac{k_x^2k_y^2}{mk_x^4+nk_y^2+2lk_x^2k_y^2}, \tag{18}$$

where

$$m=\frac{c_{11}c_{22}}{c_{11}c_{22}-c_{12}^2},\quad n=\frac{c_{22}c_{66}}{c_{11}c_{22}-c_{12}^2},$$

$$2l=1-\frac{2c_{12}c_{66}}{c_{11}c_{22}-c_{12}^2}.$$

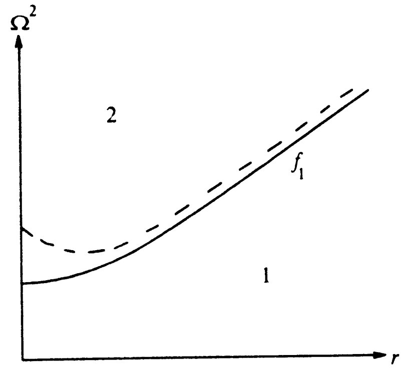

FIG. 1. Partitioning the (r,Ω^2) plane in the A_z case ($\mathbf{n}\parallel OZ$ and $k=k_y$).

C. $k_y=0$. This case is identical to case B($k_z=0$). The respective dispersion laws in variants C1 and C2 similar to B1 and B2 are described by Eqs. (17) and (18) in which k_y must be replaced by k_z, c_{22} by c_{33}, etc.

Note that the dispersion laws (15) and (18) are characterized by a quasinonanalytical dependence on the wave-vector components. This is due to the long-range nature of the indirect spin-spin coupling (of a non-Heisenberg type) via a quasistatic-phonon field, just as the MSW spectrum is nonanalytical because of the long-range nature magnetostatic coupling.

2. CLASSIFICATION OF ESWs IN FINITE AFMs

The dispersion laws obtained in Sec. 1 correspond to ESWs in an infinite magnetic material. In reality we always deal with finite crystals, in which the presence of a surface can lead, as is well known, to emergence of surface (or quasisurface) excitations of varying nature, and the amplitude of these excitations falls off as the distance from the surface increases. In such oscillations the wave-vector component perpendicular to the surface is no longer independent and, in view of the boundary conditions, is determined by the frequency ω of the wave (or the component of the wave vector along the surface, $k_\parallel$).

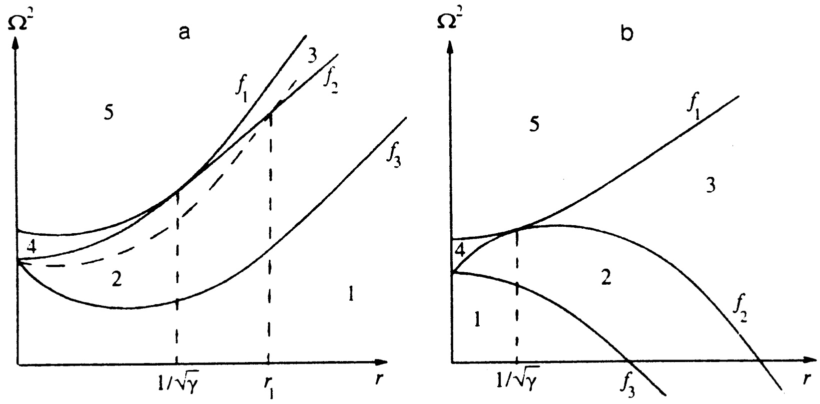

FIG. 2. Partitioning the (r,Ω^2) plane in the A_y case ($\mathbf{n}\parallel OY$ and $k=k_z$). (a) $\gamma<1$, and (b) $\gamma>1$.

Below we classify the various types of ESWs with fixed values of ω and $k_{\parallel}$ for different orientations of the wave vector **k** and the unit vector **n** normal to the surface (a similar MSW classification has been conducted by Bar'yakhtar and Ivanov[6]).

We start with the case A ($k_x=0$) and consider the lower (ω_1) branch of the ESW spectrum. The second branch for this case can be analyzed in a similar manner.

As the dispersion law (16) shows, the components k_y and k_z of the wave vector do not enter into (16) on an equal basis. Hence, for different orientations of **n** the classification of the possible types of ESWs proves to be different, too.

We begin with the case where $\mathbf{n}\parallel OZ$, which we call the A_z case. Assuming $k_y=k$ and $k_z=i\kappa$, we write the dispersion law (16) in dimensionless variables:

$$\Omega^2=\Omega_0^2+r^2(1-q^2)-\frac{\gamma q^2}{1-\gamma q^2}, \qquad (19)$$

where we have introduced the notation

$$\Omega^2=\frac{\omega_1^2}{\omega_{\rm m}^2}, \quad \Omega_0^2=\frac{c^2}{\omega_{\rm m}^2}\frac{\tilde{\beta}_1}{\alpha},$$

$$r=\frac{ck}{\omega_{\rm m}}, \quad q=\frac{\kappa}{k}, \quad \omega_{\rm m}^2=\frac{c^2\zeta_1}{\alpha}.$$

If we plot Ω^2 against q^2, the intersection of the curve by straight lines corresponding to definite values of Ω yields the roots q_i^2 of Eq. (19). The number of these roots and their signs determine the type of ESW: if all q_i^2 are positive, the ESW is of a purely surface nature; if at least one root q_i^2 is negative, this corresponds to an internal wave; and if for a given value of Ω^2 there are no intersections, all the roots are complex-valued and the ESW is a quasisurface one.

In the case A_z considered, Eq. (19) readily implies that for $\Omega^2<\Omega_0^2+r^2$ there are two real roots $q_{1,2}^2$, both positive, which means we have a surface wave. For $\Omega^2>\Omega_0^2+r^2$ one of the roots is negative and hence we have an internal wave. Therefore, the (r,Ω) plane contains two parameter regions, which determine the nature of the possible ESW types [see Fig. 1, where $f_1(r)=\Omega_0^2+r^2$].

Classification of the ESW types is radically different when $\mathbf{n}\parallel OY$ (the A_y case). If we put $k_z=k$ and $k_y=i\kappa$ and use the dimensionless variables introduced above, Eq. (16) becomes

$$\Omega^2\approx\Omega_0^2+r^2(1-q^2)+\frac{\gamma}{\gamma-q^2}. \qquad (20)$$

Although Eq. (20) is quadratic in q^2 (as the equation in the A_z case is), partitioning the parameter plane (r,Ω^2) becomes more complicated. It can be shown, however, that the (r,Ω^2) plane contains three characteristic curves,

$$f_1(r)=\Omega_0^2+r^2+1,$$
$$f_{2,3}(r)=\Omega_0^2+r^2(1-\gamma)\pm 2r\sqrt{\gamma}, \qquad (21)$$

which partition the plane into five regions (Fig. 2). In regions *1* and *3*, $q_{1,2}^2$ are positive, which means that these are surface-wave regions. In region *2* all the roots are complex-valued, which corresponds to quasisurface waves. Finally, in region *4* we have $4q_{1,2}^2<0$ and in region *5* we have $q_1^2>0$ and $q_2^2<0$, which means that in both regions the ESWs are internal waves.

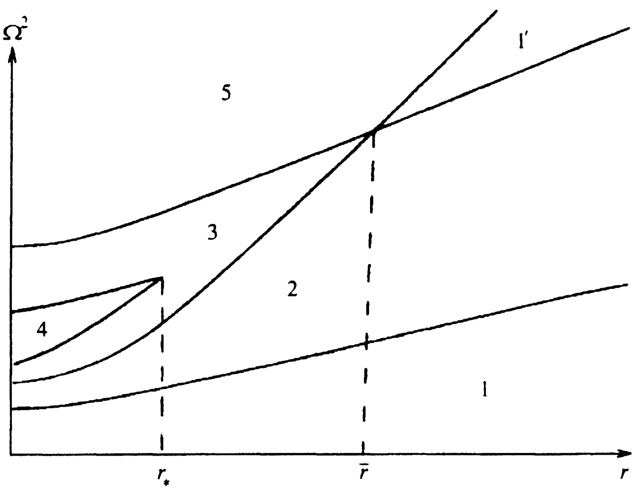

FIG. 3. Partitioning the (r,Ω^2) plane in the B2 case ($\mathbf{n}\parallel OX$ and $k=k_y$).

Now let us analyze case B ($k_z=0$). As noted earlier, the dispersion law (17) corresponding to case B1 is the same as in case A. Therefore, it can easily be shown that the respective partitioning of the (r,Ω^2) plane for $\mathbf{n}\parallel OY$ (the $B1_y$ case) is similar to the A_z case (Fig. 1), and that for $\mathbf{n}\parallel OZ$ (the $B1_z$ case) to the A_y case (Fig. 2).

In the B2 case the ESW dispersion law (18) is symmetric in the k_x and k_y components, with the result that the classification of ESWs for $\mathbf{n}\parallel OX$ and $\mathbf{n}\parallel OY$. Assuming, without loss of generality, $\mathbf{n}\parallel OX$, we write Eq. (18) in the form

$$\Omega^2=\Omega_0^2+r^2(1-q^2)-\frac{q^2}{mq^4-2lq^2+n}. \qquad (22)$$

This equation is cubic with respect to q^2 (rather than quadratic, as in the above cases A and B1), which complicates analysis considerably. Nevertheless, it is still feasible to classify the possible ESW types according to the values of k and ω. The corresponding partitioning of the (r,Ω^2) plane is depicted in Fig. 3.

In the regions *1* and *1'* all three roots of the bicubic equation (22) are real and positive, $q_{1,2,3}^2>0$, that is, these are surface-wave regions. In region *2* there is one positive root, $q_1^2>0$ and two complex-values, which corresponds to quasisurface waves. In region *3* one root is negative, $q_1^2<0$, while the other two, $q_{2,3}^2$, are complex-valued. In region *4* all three roots are negative, $q_{1,2,3}^2<0$. Finally, in region *5* one root is negative, $q_1^2<0$, and the other two are positive, $q_{2,3}^2>0$. Hence, regions *3*, *4*, and *5* correspond to internal ESWs.

The analytical expressions for the curves that partition the (r,Ω^2) plane in the case at hand (B2) are extremely cumbersome, and, hence, are not given here. We note, however, that the characteristic values of parameter r at which the curves intersect, r_* and $\bar{r}$, are given by the following formulas:

$$r_*=\frac{1}{\sqrt{m}}\,(q_+^{4/3}+q_-^{4/3}+q_+^{2/3}q_-^{2/3})^{-3/2}, \tag{23}$$

$$\bar{r}=\frac{2}{\sqrt{m}(q_+^2-q_-^2)}, \tag{24}$$

where $q_\pm^2$ are the roots of the biquadratic equation $mq^4-2lq^2+n=0$. Note that in an elastically isotropic magnetic material $l=m=n=1$ holds, so that we have $q_\pm^2=1$, $r_*^2=3^{-3/2}$ (see Ref. 4), and $\bar{r}\to\infty$.

Since case C ($k_y=0$) is identical to case B ($k_z=0$) as noted in Sec. 1, the three ways of partitioning the (r,Ω^2) plane exhaust all possible ways in which ESWs in a finite AFM can be classified.

In conclusion of this section an important remark should be made concerning the formulation of problems of ESWs in a semi-infinite magnetic material (the associated problems are studied in detail in Ref. 10 using the example of electroacoustic waves in piezocrystals). There exists an important difference between formulating the problem for the case where the characteristic equation in the system of the form (9)–(13) that determines the value of the wave-vector component normal to the surface has no real root (in our notation this corresponds to all roots q_i^2 being either positive or complex-valued), and for the case where there is at least one, that is, one of the q_i^2 is negative. In the first instance the number of boundary conditions is equal to the number of independent amplitudes, and we are dealing with the problem of formation of a surface (or quasisurface) spin wave. But if at least one q_i^2 is negative, then there are more independent amplitudes than boundary conditions and we are dealing with the problem of reflection of an internal spin wave from the surface of the magnetic material. Here, if the remaining roots of the characteristic equation are imaginary (complex-valued), they describe the so-called satellite surface (or quasisurface) spin waves,[10] which in the present case are induced by indirect spin–spin exchange via the field of quasistatic virtual phonons. Such a situation occurs, for instance, in the A_z case for $\Omega^2>\Omega_0^2+r^2$, in the A_y case (region *5*), and in the B2 case (regions *3* and *5*). But if there are two or more negative roots q_i^2 (the A_y case, region *4*, and the B2 case, region *4*), we are dealing with the problem of multiple-wave reflection of the spin wave without changing the wave polarization. In the analysis of ESW in plates, where the boundary conditions are formulated for two surfaces, the above difference vanishes, and we can always formulate the problem of eigenexcitations in the magnetic material. Here the excitations are the result of interference of bulk and surface oscillations (see below).

3. ESWs IN FINITE AFMs: EXAMPLES

The above classification of the possible types of ESWs realizable in a crystal for a given $\mathbf{k}$ and ω cannot answer the question of what type of ESW exists in a finite AFM for a fixed $\mathbf{k}$ (or ω), since the law governing the dispersion of the wave, $\omega=\omega(\mathbf{k})$, is determined by both the geometry of the problem and the specific form of the boundary conditions.

As the first example of calculating the ESW dispersion law for the surface-wave problem we take a semi-infinite AFM occupying the half-space $y>0$. Suppose that the wave propagates in the plane that is at right angles to the equilibrium antiferromagnetism vector $\mathbf{l}_0$, that is, along the z axis, the A_y case in the notation of Sec. 2.

The boundary conditions at the surface $y=0$ of the magnetic material is selected in the simplest possible form, that is, corresponding to a crystal free from external stresses,

$$\sigma_{ik}n_k\Big|_{y=0}=\sigma_{iy}\Big|_{y=0}=0, \tag{25}$$

and spins whose directions are not fixed at the surface,

$$\frac{\partial\mathbf{l}}{\partial y}\Big|_{y=0}=0,\quad \frac{\partial\psi}{\partial y}\Big|_{y=0}=\frac{\partial\vartheta}{\partial y}\Big|_{y=0}=0. \tag{26}$$

With such a geometry of the problem, the link between the frequency ω, the wave vector along the crystal's surface $k=k_z$, and the quantity $\kappa=-ik_y$, which determines the depth of penetration of an excitation into the magnetic material, is determined by Eq. (20), which is a quartic equation in the variable $q=\kappa/k$. Since we are interested in solutions of the equations of motion (9)–(13) that go to zero as $y\to+\infty$, solutions corresponding to the lower branch of the ESW spectrum, we seek these in the form of two-part waves:

$$\begin{aligned} u_x(y,z,t)&=(u_1e^{-\kappa_1 y}+u_2e^{-\kappa_2 y})e^{ikz-i\omega t},\\ \psi(y,z,t)&=(\psi_1e^{-\kappa_1 y}+\psi_2e^{-\kappa_2 y})e^{ikz-i\omega t}, \end{aligned} \tag{27}$$

(as in Sec. 1, we assume $\tilde{\beta}_1\gg\tilde{\beta}_2$ and, hence, as can easily be shown, $|\vartheta|\ll|\psi|$).

Substituting (27) into the boundary conditions (25) and (26) and using the equations of motion, we arrive at the following relation:

$$q_1^2+q_1q_2+q_2^2=\gamma, \tag{28}$$

where q_1 and q_2 are the two roots of the biquadratic equation

$$r^2q^4-q^2[(\gamma+1)r^2-\Omega^2+\Omega_0^2]-\gamma(\Omega^2-\Omega_0^2-r^2-1)=0, \tag{29}$$

satisfying the condition $\mathrm{Re}\,q_{1,2}>0$.

Solving Eqs. (28) and (29) simultaneously, we find the ESW dispersion law in explicit form:

$$\omega^2=\frac{\tilde{\beta}_1c^2}{\alpha}+c^2k^2+\frac{\gamma}{2}\left\{-c^2k^2+c|k|\sqrt{\frac{4\omega_{\mathrm{m}}^2}{\gamma}+c^2k^2}\right\}. \tag{30}$$

Note that for small values of the wave vector ($ck\ll\omega_{\mathrm{m}}$) the dispersion law (30) is linear,

$$\omega^2\approx\omega_a+\sqrt{\gamma}\omega_{\mathrm{m}}c|k|,\quad \omega_a^2=\frac{\beta_1c^2}{\alpha}, \tag{31}$$

and the quantities κ_1 and κ^2, as Eq. (29) implies, are complex-valued:

$$\kappa_{1,2}\approx\frac{\gamma^{1/4}}{2}\sqrt{\frac{\omega_{\rm m}|k|}{c}}(1\pm i\sqrt{3}). \qquad (32)$$

Hence, for $ck\ll\omega_{\rm m}$ the ESW is a quasisurface one, and the depth of the wave's localization region, $\lambda=(\mathrm{Re}\,\kappa)^{-1}$, is of the order of the spatial period of the amplitude's oscillations, $\Lambda=|\mathrm{Im}\,\kappa|^{-1}$.

For large wave vectors ($ck\gg\omega_{\rm m}$) the ESW dispersion law corresponds to the law of dispersion for an internal spin wave:

$$\omega^2\approx\omega_a^2+\omega_{\rm m}^2+c^2k^2. \qquad (33)$$

In this case κ_1 and κ_2 are real:

$$\kappa_1\approx\sqrt{\gamma}k,\quad \kappa_2\approx\frac{1}{\sqrt{\gamma}k}\left(\frac{\omega_{\rm m}}{c}\right)^2\ll\kappa_1. \qquad (34)$$

Hence, the wave here is of a purely surface nature, and the localization-region depth $\lambda=\kappa_2^{-1}$ grows with k.

In Fig. 2(a) the dispersion law (30) is depicted by a dashed curve. The above analysis shows that for k small the curve lies in region *2*, which corresponds to quasisurface waves. When the magnitude of the wave vector becomes

$$k_1=\frac{3}{2\sqrt{\gamma}}\frac{\omega_{\rm m}}{c}, \qquad (35)$$

the curve finds itself in *3*, a region of purely surface waves.

Since the equations of elastostatics used here are valid for $k\ll k_r\sim\omega_{\rm m}/s$, we conclude that Eqs. (31) and (32) provide a meaningful picture of ESWs in the $k_r\ll k\ll k_1$ interval, which exists when $c\ll s$. Note that the type of surface ESW exists only if one allows simultaneously for both indirect spin–spin exchange via the long-range field of quasistatic elastic strains and inhomogeneous exchange interaction. If we ignore the latter ($c\to0$), there can be no ESW with the dispersion law (30) and the "free" boundary conditions (25).

Clearly, there can be similar surface ESWs in other geometries of the problem: in the $\mathrm{B1}_x$ case ($k_z=0$, $k=k_y$, and $\mathbf{n}\parallel OX$) and in the $\mathrm{C1}_x$ case ($k_y=0$, $k=k_z$, and $\mathbf{n}\parallel OX$). The ESW dispersion laws in these cases coincide with (30) to within constants.

Now we take the same A_y case and examine the dispersion properties of a propagating ESW at wave-vector and frequency values corresponding to region *4* in Fig. 2. As noted earlier, the region contains two negative roots of Eq. (20), $q_{1,2}^2<0$, that is, four purely imaginary values of q. Such a situation in a semi-infinite magnetic material enables formulating the problem of multiple-wave reflection of a spin wave from the surface of the magnetic material without altering the wave's polarization. But we will we consider the propagation ESW in a plate of finite thickness, which means we are dealing with an eigenvalue problem.

Let us assume that the magnetic material occupies the region $|y|<d$. If on both surfaces of the plate the boundary conditions of the (25)–(26) types are still valid, the solution of the equations of motion (9)–(13) corresponding to the internal branch of the ESW spectrum must be sought in the form of a four-partial wave. It can easily be shown that in the given situation there are two independent solutions, the solution symmetric with respect to the $y=0$ plane,

$$u_x(y,z,t)=(u_1\cos p_1y+u_2\cos p_2y)e^{i(kz-\omega t)},$$
$$\psi(y,z,t)=(\psi_1\cos p_1y+\psi_2\cos p_2y)e^{i(kz-\omega t)}, \qquad (36)$$

and the asymmetric solution

$$u_x(y,z,t)=(u_1\sin p_1y+u_2\sin p_2y)e^{i(kz-\omega t)},$$
$$\psi(y,z,t)=(\psi_1\sin p_1y+\psi_2\sin p_2y)e^{i(kz-\omega t)}, \qquad (37)$$

where $p_{1,2}^2=-\kappa_{1,2}^2>0$, and $\kappa_{1,2}^2$ are the roots of Eq. (20).

The corresponding dispersion equation has the form

$$(p_1^2+k^2)\tan 2p_1d=(p_2^2+k^2)\tan 2p_2d \qquad (38)$$

for the symmetric mode (36) and

$$(p_1^2+k^2)\cot 2p_1d=(p_2^2+k^2)\cot 2p_2d \qquad (39)$$

for the asymmetric mode (37).

Analysis of Eqs. (38) and (39) shows that, within the range of wave vectors k and frequencies ω considered here, allowing for the inhomogeneous exchange interaction and the indirect spin–spin coupling via the virtual-phonon field simultaneously leads to the formation of an additional (with respect to the exchange internal wave) internal ($p_{1,2}^2>0$) spin wave of the elastostatic type with the same polarization.

It is impossible to obtain an explicit analytical expression for the dispersion law determined by Eqs. (38) and (39), and one must resort to numerical methods. However, the basic features of this law can easily be analyzed in the limit of fairly small wave vectors ($\omega_{\rm m}/c\gg|\mathbf{k}|$).

In this case Eqs. (38) and (39) describe two drastically different types of bulk spin waves with the same polarization. In the limit considered, the dispersion law of one type is formed primarily by indirect exchange via the field elastostatic phonons; the wave is direct, or $k(\partial\omega/\partial k)>0$, and the corresponding dispersion law is

$$\omega^2\approx\omega_a^2+\omega_{\rm m}^2\frac{k^2}{k^2+p_1^2},\quad p_1\approx\frac{\pi n}{2d}, \qquad (40)$$

where $n=\pm1, \pm2,\dots$.

Under the same conditions the second solution represents an ordinary bulk spin wave with a dispersion law

$$\omega^2\approx\omega_a^2+c^2(k^2+p_2^2),\quad p_2\approx\frac{\pi n}{2d}, \qquad (41)$$

and formed primarily by inhomogeneous exchange interaction.

Comparing Eqs. (40) and (41), we conclude that in the vicinity of wave-vector values determined by the condition

$$\omega_{\rm m}^2k^2\left[k^2+\left(\frac{\pi n}{2d}\right)^2\right]^{-1}=c^2\left[k^2+\left(\frac{\pi n}{2d}\right)^2\right] \qquad (42)$$

there is resonant interaction between the above two types of spin excitations, that is, inhomogeneous spin–spin resonance sets in. Here the structure of the spin-wave spectrum is determined by the general formulas (38) and (39), and an important aspect of their formation is to allow for both mechanisms of spin–spin exchange simultaneously. As usual, the interacting modes repel each other, which results in the emergence of windows of nontransparency in frequency for the propagating spin-wave oscillations with a given **k**.

Note that the mechanism of inhomogeneous spin–spin resonance described here is the magnetoelastic analog of the well-known dipole-exchange resonance (see, e.g., Ref. 11), with the nontransparency windows being similar to so-called dipole gaps.

Next we examine ESWs in a plate similar to the one studied above but with a different geometry, namely, the A_x case ($k_x=0$, $k=k_y$, and $\mathbf{n}\parallel OZ$). From the analysis performed in Sec. 2 we conclude that with such geometry in a semi-infinite magnetic substance and with wave-vector and frequency values corresponding to region *2* in Fig. 1 it is possible to formulate the problem of reflection of an internal wave from the surface accompanied by the formation of a satellite surface wave.

Assuming as before that the magnetic material is limited by the planes $z=-d$ and $z=d$, we write the appropriate boundary conditions similar to (25) and (26) as

$$\left.\frac{\partial \mathbf{l}}{\partial z}\right|_{\pm d}=0, \quad \sigma_{iz}\Big|_{\pm d}=0. \tag{43}$$

In a finite plate the solutions of the equations of motion in the geometry considered here have four parts [see Eq. (19)]. Hence, we seek them in the form

$$u_x(y,z,t)=(u_1 \sin p_1 z+\bar{u}_1 \cos p_1 z+u_2 \sin p_2 z + \bar{u}_2 \cos p_2 z)e^{i(ky-\omega t)}, \tag{44}$$

$$\psi(y,z,t)=(\psi_1 \sin p_1 z+\bar{\psi}_1 \cos p_1 z+\psi_2 \sin p_2 z + \bar{\psi}_2 \cos p_2 z)e^{i(ky-\omega t)}.$$

Substituting (44) into the boundary conditions (43), we find that

$$p_1=\frac{\pi n}{2d}, \quad p_2=\pm i\sqrt{k^2+p_1^2+\frac{\eta}{k^2+p_1^2}} \tag{45}$$

where $n=\pm 1, \pm 2, \ldots, \eta$ is a positive constant, and the ESW dispersion law has the form

$$\omega^2=\omega_a^2+\frac{\omega_m^2 p_1^2}{k^2+p_1^2}+c^2(k^2+p_1^2), \tag{46}$$

In Fig. 1, which corresponds to this case, the dispersion law (46) of a propagating internal ESW for $n=1$ is represented by the dashed curve.

An important feature of (46) is the fact that it is nonmonotonic: for $k<k_*$ (where $k_*^2=\omega_m p_1-p_1^2$) the function $\omega=\omega(k)$ decreases, that is, the propagating ESW is a backward wave, or $k(\partial\omega/\partial k)<0$, while for $k>k_*$ it increases, that is, the respective ESW is a direct wave, or $k(\partial\omega/\partial k)>0$. Note that the minimum point of the function $\omega=\omega(k)$ specified by (46), which the function reaches at $k=k_*$, exists only if $d>\pi nc/2\omega_m$. But if this inequality is not met, there is no minimum, and the ESW is a direct wave for all values of k.

Hence, the interference of the exchange internal ($p_1^2>0$) partial wave and the elastostatic satellite surface ($p_2^2<0$) partial wave drastically changes the nature of the dispersion curve of the propagating internal spin wave because, if we do not allow for indirect exchange via the field of virtual elastostatic phonons, the exchange spin wave in the plate is always direct; the respective dispersion law has the form $\omega^2=\omega_a^2+\omega_m^2+c^2(k^2+p_1^2)$.

In conclusion we note that if in the last two examples we put $c=0$, that is, ignore the inhomogeneous exchange interaction, which corresponds to the exchangeless approximation widely used in MSW theory, the waves considered above still have meaning, and their dispersion properties are determined entirely by indirect exchange via the virtual-phonon field. Hence, at spin-oscillation frequencies conforming to the elastostatic criterion $\omega(k)\ll sk$ (see Ref. 5) we can speak of formation in a finite magnetically ordered crystal of a new class of exchangeless excitations—elastostatic spin waves. A meaningful analysis of the spin-wave dynamics of finite magnetic materials (especially magnetic films) is impossible without systematically allowing for the effects of the indirect spin–spin interaction via the virtual-phonon field.

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[1)]We use the term "quasinonanalytical"because the wave vector **k** can be taken equal to zero only formally, since ESW exist only when $|\mathbf{k}|$ is fairly large, or $|\mathbf{k}|\gg k_r$.

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