The structure and elastic properties of a vortex lattice in a thin film of an anisotropic superconductor

A. Yu. Martynovich

Donetsk Physicotechnical Institute, Ukrainian Academy of Sciences, 340114 Donetsk, Ukraine
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The vortex structure and its properties in a thin film of a biaxial superconductor are studied. It demonstrates that the distributions of the magnetic field and current of an isolated vortex and the energy of the pair interaction of vortices are anisotropic only at small distances from the vortex center, while at large distances they are always isotropic. The hexagonal shape of a loosely packed lattice is due to the centrally symmetric interaction of the vortices, the elastic properties of the lattice are completely isotropic, and the anisotropy manifests itself only in a given orientation of the lattice with respect to the crystallographic axes. As induction increases, the lattice transforms into an oblique-angled with rhombic symmetry. The response of a closely packed lattice to compressive strain is always isotropic, while for shearing strain it depends on both the direction of shearing and the magnitude of anisotropy. The symmetry axes of the lattice coincide with the directions of easy and difficult shearing. The elastic modulus of rotation of a vortex lattice determines the stability of equilibrium vortex structures.

1. INTRODUCTION

Recently there have been extensive studies of new high-T_c superconductors whose crystals exhibit highly anisotropic electric and magnetic properties. Numerous theoretical papers describe the effect of the crystal's anisotropy on the structure of an isolated vortex[1–3] and a vortex lattice,[4,5] on the elastic properties of a vortex lattice,[5–7] and on the shape of the equilibrium magnetization curve.[8,9] All these papers, without exception, study magnetic vortices in an infinite superconductor. Actually, however, the main object of investigation is thin films of high-T_c superconductors, the manufacturing of which is the best developed. Usually the thickness d of these fields is of order 0.1 μm and is less than the depth of penetration of the magnetic flux into the high-T_c superconductor. This means that the magnetic properties of high-T_c superconducting films can be described using the Pearl approximation.[10] Pearl[10,11] showed that the structure of a magnetic flux-line vortex emerging at the surface of an isotropic superconductor changes dramatically in the surface layer of thickness λ. The variation of the magnetic field and current changes from exponential to a power law, which leads to a long-range interaction of vortices. However, the main characteristic of a vortex field, the presence of axial symmetry, remains unchanged. From this it follows that however thick the superconducting plate is, the vortex interaction is centrally symmetric and that only a square or hexagonal vortex lattice satisfies the condition that all forces are in equilibrium. In both infinite superconductors[12] and thin-film superconductors[13] the only stable structure is the hexagonal. The short-range interaction of vortices in infinite samples and the long-range interaction in thin films ensure marked differences in the elastic properties of vortex lattices of such superconductors. The elastic moduli of the vortex lattice in infinite samples[14,15] can be compared to the restoring forces in the thin-film lattice.[16,17] The latter prove to be anomalously high in compressive strain, that is, the vortex lattice in a thin film is practically incompressible. The lattice's response to shearing strain proves to be practically the same for both forms of superconducting samples, a fact corroborated by experiments.[17]

This paper studies vortex structures in a thin film of a biaxial superconductor whose anisotropy axes are oriented arbitrarily. All results are compared with the known solutions for an infinite anisotropic superconductor and for an isotropic thin film.

2. THE MAGNETIC STRUCTURE OF A PEARL VORTEX

2.1. In the thin-film approximation,[10,13,18] $d\ll\lambda$, the equilibrium equations for the vector potential $\mathbf{A}(\mathbf{x})$ and the gradient of the phase, $\nabla\psi(\mathbf{x})$ have the form

$$\operatorname{curl}\operatorname{curl}\mathbf{A}=\frac{2}{\Lambda}\,\delta(x_3)\hat{\mu}^{-1}\left(\frac{\phi_0}{2\pi}\nabla\psi-\mathbf{A}\right),\qquad(1a)$$

$$\delta(x_3)\nabla\hat{\mu}^{-1}\left(\frac{\phi_0}{2\pi}\nabla\psi-\mathbf{A}\right)=0.\qquad(1b)$$

Here $\Lambda=2\lambda^2/d$ is the effective depth of magnetic-flux penetration into the sample. The anisotropy of the superconductive properties of the material of the film is specified by the effective-electron-mass tensor $m\hat{\mu}$ ($\det\hat{\mu}=1$), and ϕ_0 is the magnetic flux quantum. The plane of the film coincides with the plane $x_3=0$ of a Cartesian system of coordinates.

Equations (1a) and (1b) must be solved subject to the condition

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$$\text{curl}\,\nabla\psi = 2\pi \sum_\nu \int d^3\mathbf{x}_\nu\, \delta(\mathbf{x}-\mathbf{x}_\nu), \tag{2}$$

which determines $\nabla\psi$ for given positions $\mathbf{x}=\mathbf{x}_\nu$ of the source of the magnetic field. The system of equations (1) and (2) is not complete: the quantities $\mathbf{A}$ and $\nabla\psi$ can be defined only to within the gradient of an arbitrary function. As a supplementary condition we select the equation

$$\text{div}\,\nabla\psi = 0, \tag{3}$$

which fixes the gauge of the gradient of the phase and hence of the vector potential.

Let us examine an isolated magnetic vortex in an infinite film. The solution to Eqs. (2) and (3) is

$$\nabla\psi = 1/[\mathbf{n}\mathbf{x}] \tag{4}$$

if the normal to the film, $\mathbf{n}$, coincides with the vortex axis. Deviation of the vortex axis $\mathbf{x}_\nu$ from the normal vector $\mathbf{n}$ leads to additional terms in (4) that are symmetric under the substitution $\mathbf{x}\leftrightarrow -\mathbf{x}$. Since allowing for these terms changes nothing in the results obtained below, we consider only the solution (4).

2.2. For the solution of the system of equations (1a) and (1b) we take the integral representation of the vortex current in the film:

$$\mathbf{j}(\mathbf{x}) = \frac{1}{4\pi^2}\int d^2q\, \mathbf{j}(\mathbf{q})\exp(i\mathbf{q}\mathbf{x}),$$
$$\mathbf{j}(\mathbf{q}) = \frac{ic\phi_0}{2\pi d}\,\frac{\mathbf{q}\times\mathbf{n}}{q+\Lambda(\mathbf{q}\times\mathbf{n})\cdot\hat{\mu}\cdot(\mathbf{q}\times\mathbf{n})}. \tag{5}$$

The distribution of the magnetic field of a vortex throughout all space has exactly the same dependence on anisotropy. We see that there are two asymptotic regions for a vortex in a superconducting film. The distant region, $x\gg\Lambda$, is characterized by the predominant contribution of small q to (5). In this region the field and current distributions are practically independent of $\hat{\mu}$ and coincide with well-known expressions for an isotropic film. In the near region, $x\ll\Lambda$, vortex currents are essentially anisotropic. Equation (5) depends only on three components of the anisotropy tensor, μ_{11}, μ_{12}, and μ_{22}, which can be represented by a tensor $\mathcal{M}$ of rank 2. Proper selection of the principal axes X_1 and X_2 allows $\mathcal{M}$ to be diagonalized:

$$M_{11} = \tfrac{1}{2}\text{Tr}\,\mathcal{M} - \tfrac{1}{2}\sqrt{(\text{Tr}\,\mathcal{M})^2 - 4\det\mathcal{M}},$$
$$M_{22} = \tfrac{1}{2}\text{Tr}\,\mathcal{M} + \tfrac{1}{2}\sqrt{(\text{Tr}\,\mathcal{M})^2 - 4\det\mathcal{M}}, \tag{6}$$
$$M_{12} = 0.$$

Let us determine the orientation of the principal axes X_1 and X_2. For the initial system of coordinates we take a Cartesian system whose x_1 axis coincides with the projection of the crystallographic axis $\mathbf{a}$ onto the plane of the film. Let us calculate the components of the $\hat{\mu}$ tensor in the initial system. The X_1 principal axis is rotated from the x_1 axis in the direction of the x_2 axis of the initial system of coordinates by the angle

$$\tfrac{1}{2}\tan^{-1}\left(\frac{2\mu_{12}}{\mu_{11}-\mu_{22}}\right).$$

In biaxial superconductors this angle is zero only if one of the crystallographic axes lies in the plane of the film. In a uniaxial superconductor this angle is always zero and the X_2 axis is directed along the projection of the anisotropy axis $\mathbf{c}$ in the plane of the film. In terms of the principal axes of the tensor $\mathcal{M}$ the current distribution near the vortex core, $x\ll\Lambda$, has the form

$$\mathbf{j}(\mathbf{q}) \propto \frac{\mathbf{q}\times\mathbf{n}}{q_1^2M_{22}+q_2^2M_{11}}.$$

The reader can see that both the constant-current lines and the constant-field lines are ellipses whose diagonals coincide with the principal axes of $\mathcal{M}$. By a scale transformation of vectors $\mathbf{q}$,

$$q_1' = q_1\left(\frac{M_{22}}{M_{11}}\right)^{1/4}, \quad q_2' = q_2\left(\frac{M_{11}}{M_{22}}\right)^{1/4}, \tag{7}$$

and the coordinate transformation

$$x_1' = x_1\left(\frac{M_{11}}{m_{22}}\right)^{1/4}, \quad x_2' = x_2\left(\frac{M_{22}}{m_{11}}\right)^{1/4} \tag{8}$$

the field and current distributions in the near region of the vortex are reduced to the isotropic form. This statement is universal, that is, is valid for all types of anisotropy and for all orientations of the crystallographic axes in the superconductor.

It is obvious why inversion of the longitudinal magnetic field[2] is absent in thin anisotropic films. In a bulk superconductor the rotation of a magnetic flux line occurs over distances exceeding λ. Pearl films are clearly not thick enough for this, with the result that anisotropy in thin films leads only to a redistribution of magnetic flux lines in the plane of the film.

3. THE VORTEX-LATTICE ELASTIC ENERGY

3.1. The free energy of a superconductor in the London approximation is

$$\mathcal{F} = \frac{\phi_0}{4\pi c}\int d^3x\, \nabla\psi\cdot\mathbf{j}. \tag{9}$$

Here $\mathbf{j}$ is the current density, and $\nabla\psi$ the gradient of the phase of the vortex system in the film. Employing the solutions (4) and (5) for an isolated vortex, we get

$$\mathcal{F} = \frac{\phi_0^2}{16\pi^3}\int \frac{d^2q\,|S(\mathbf{q})|^2}{q+\Lambda(\mathbf{q}\times\mathbf{n})\cdot\hat{\mu}\cdot(\mathbf{q}\times\mathbf{n})}$$
$$\equiv \int d^q\, V(\mathbf{q})\,|S(\mathbf{q})|^2, \tag{10}$$

where $S(\mathbf{q}) = \Sigma_\mathbf{x}\exp(i\mathbf{q}\cdot\mathbf{x})$ is the structure factor and $\mathbf{x}$ are the coordinates of the vortices in the plane of the film.

The free energy (10) consists of the sum of self-energies of the vortices and the energies of pair vortex interactions. The self-energy of a vortex is

$$\mathscr{F}_0=\int d^2q\, V(\mathbf{q})=d\left(\frac{\phi_0}{4\pi\lambda}\right)^2\frac{1}{\sqrt{\mathbf{n}\hat{\mu}^{-1}\mathbf{n}}}\ln\frac{\Lambda}{\xi}, \tag{11}$$

where ξ is the coherence length.

It is interesting to compare this result with the energy of a vortex in a massive anisotropic superconductor,[2] which can be represented in two ways: in terms of the unit vector $\boldsymbol{\nu}$ of the vortex,

$$\mathscr{F}_0=dF_0\sqrt{\boldsymbol{\nu}\hat{\mu}\boldsymbol{\nu}},$$

or in terms of the unit vector of the external magnetic field $\mathbf{h}=\mathbf{H}/H$,

$$\mathscr{F}_0=dF_0\frac{1}{\sqrt{\mathbf{h}\hat{\mu}^{-1}\mathbf{h}}}.$$

Here $F_0=(\phi_0/4\pi\lambda)^2\ln(\lambda/\xi)$ is the free energy, per unit length, of a vortex in an isotropic superconductor. The last expression has the same dependence on the anisotropy parameters as (11): the vortex structure in a thin film is created only by the component of the external magnetic field that is transverse to the surface of the film, or $\mathbf{h}=\mathbf{n}$.

The energy of the pair interaction of two vortices separated by a distance $\mathbf{R}$,

$$U(\mathbf{R})=\int d^2q\, V(\mathbf{q})\exp(i\mathbf{qR}), \tag{12}$$

exhibits the following asymptotic behavior:

$$\begin{aligned}
U(\mathbf{R})&=d\left(\frac{\phi_0}{4\pi\lambda}\right)^2\frac{1}{\sqrt{\mathbf{n}\hat{\mu}^{-1}\mathbf{n}}}\ln\\
&\quad\times\sqrt{\frac{(\mathbf{Rn})\hat{\mu}(\mathbf{Rn})}{2\Lambda^2(\mathbf{n}\hat{\mu}^{-1}\mathbf{n})}}\ \text{if } R\ll\Lambda,\\
U(\mathbf{R})&=d\left(\frac{\phi_0}{4\pi\lambda}\right)^2\left[\frac{\Lambda}{R}+\frac{\Lambda^2}{R^4}(\mathbf{Rn})\hat{\mu}(\mathbf{Rn})\right.\\
&\quad\left.-\frac{\Lambda^2}{R^4}(\mathbf{R}\hat{\mu}\mathbf{R})\right]\quad R\gg\Lambda.
\end{aligned} \tag{13}$$

Obviously, at great distances the interaction of the vortices is practically isotropic. At small distances, $U(\mathbf{R})$ depends on the direction of $\mathbf{R}$ and the constant-interaction lines are ellipses, as are the constant-current lines.

3.2. Interaction between the vortices lead to their ordering into a regular lattice. To derive the elastic theory of an anisotropic vortex lattice, we consider the displacements $\mathbf{u}(\mathbf{x})$ of the vortices from their equilibrium position and assume that typical displacements of the vortices are small compared to the distances between the vortices. We represent the Fourier trans form of the strains in the form of an integral over the first Brillouin zone:

$$\mathbf{u}(\mathbf{x})=\frac{1}{(2\pi)^2}\int_{BZ}d^2k\,\mathbf{u}(\mathbf{k})\exp(i\mathbf{kx}).$$

We expand the free energy (10) in a power series in the small vortex strains $\mathbf{u}(\mathbf{x})$ about the sites of the equilibrium lattice and restrict our discussion to the harmonic approximation. As a result we get

$$\mathscr{F}=\mathscr{F}_0+du_{ij}\sigma_{ij}+d\int_{BZ}\frac{d^2k}{8\pi^2}\mathbf{u}_i(\mathbf{k})\mathbf{u}_j(-\mathbf{k})\Phi_{ij}(\mathbf{k}). \tag{14}$$

The energy of an unstrained lattice is

$$\mathscr{F}_0=N\int d^2q\, V(\mathbf{q})S(\mathbf{q}),$$

with N the number of vortices in the film.

The strain-linear expansion terms depend only on the components of the distortion tensor

$$u_{ij}=\int_{BZ}d^2k\,\delta(\mathbf{k})ik_ju_i(\mathbf{k})$$

describing homogeneous strains of the vortex lattice. The components of the "stress tensor" σ_{ij} have the form

$$\sigma_{ij}=\frac{1}{d}\int d^2q\, V(\mathbf{q})q_i\frac{\partial S(\mathbf{q})}{\partial q_j}. \tag{15}$$

Note that for real crystals the stress tensor can be defined only because of the short-range of intermolecular forces.[19] In our case the vortex–vortex interaction is essentially long-range. Hence, for the parameters in the expansion of $\mathscr{F}$ we selected $\mathbf{u}(\mathbf{x})$ rather than u_{ij}. The presence in the expansion (14) of the components σ_{ij} identical in form with the true stress tensor is a consequence of the assumptions that the equilibrium vortex lattice must be regular and infinite.

The expansion terms in (14) quadratic in $\mathbf{u}$ are proportional to the vortex-lattice elastic matrix

$$\Phi_{ij}=\frac{1}{d}\int d^2q\, V(\mathbf{q})q_iq_j[S(\mathbf{q}+\mathbf{k})+S(\mathbf{q}-\mathbf{k})-2S(\mathbf{q})]. \tag{16}$$

For real crystals such a matrix is known as the dynamical matrix.

4. EQUILIBRIUM VORTEX STRUCTURES

For infinite superconductors the structure of the vortex lattice can be found by solving the equilibrium equation $\sigma_{ij}=0$. This is not the correct approach if we are dealing with a superconducting film placed in a transverse magnetic field $\mathbf{H}=\mathbf{n}B$. The magnitude of the field uniquely determines the concentration of vortices in the film, $n_L^0=B/\phi_0$. With homogeneous strains in the vortex lattice, the concentration n_L is related to the equilibrium value n_L^0 as

$$n_L=\frac{n_L^0}{(1+u_{11}+u_{22}+u_{11}u_{22}+u_{12}u_{21})^{-1}}.$$

Since in the film $\langle n_L\rangle=n_L^0$, homogeneous strains of the compression–elongation type can occur only if

$$u_{11}+u_{22}=0.$$

Thus, the equilibrium equations of a vortex structure in a thin film are

$$\sigma_{12}=\sigma_{21}=0,\quad \sigma_{11}-\sigma_{22}=0. \tag{17}$$

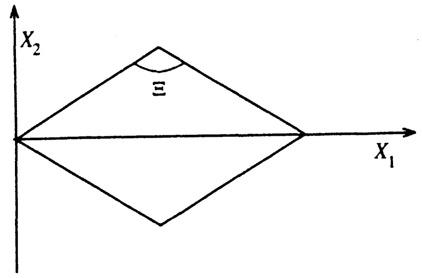

FIG. 1. Orientation of the unit cell of the vortex lattice in an anisotropic superconducting film. X_1 and X_2 are the principal axes of tensor $\hat{M}$.

We write the explicit form of these equations in the system of coordinates associated with the principal axes X_i of the tensor M specified by Eq. (6):

$$(1-\delta_{ij})\int d^2q\, V(\mathbf{q})q_i\frac{\partial S(\mathbf{q})}{\partial q_j}=0, \tag{18a}$$

$$\int d^2q\, V(\mathbf{q})\left(q_1\frac{\partial S\partial}{q_1}+q_2\frac{\partial S}{\partial q_2}\right). \tag{18b}$$

For a lattice that is symmetric under reflection with respect to the X_1 and X_2 the structure factor

$$S(\mathbf{q})=\sum_R \cos(q_1R_1)\cos(q_2R_2)$$

is an even function of q_1 and q_2, and the integrand, on the whole, is an odd function, with the result that the integral in (18a) is zero.

Hence, in high-T_c superconducting films the vortex lattice has the symmetry of a rhombus with diagonal collinear with the X_1 and X_2 axes along which the M tensor assumes its maximum (the X_2 axis) or minimum (the X_1 axis) value.

We determine the shape of the unit cell of the vortex lattice by solving Eq. (18b) for high or low inductions separately.

In the case of a loosely packed vortex lattice ($n_L\Lambda^2\ll 1$) we can ignore in the integral in (18b) the term $\Lambda(q_1^2M_{22}+q_2^2M_{11})$ responsible for the anisotropy. The remaining integral is M-independent and coincides with the similar expression for an isotropic superconductor. The equilibrium vortex structure has the shape of a hexagonal lattice, and the angle Ξ between the translation vectors of the lattice (Fig. 1) is

$$\Xi=\frac{2\pi}{3}, \tag{19a}$$

$$\Xi=\frac{\pi}{3}. \tag{19b}$$

In the case of a closely packed vortex lattice ($n_L\Lambda^2\gg 1$) we should leave in the denominator of (18b) only the terms with q_i^2. The resulting simple expressions are reduced via the scale transformations of coordinates (7) and (8) to the isotropic form. In the "primed" system of coordinates the vortex lattice has a hexagonal structure. An inverse scale transformation yields the following solutions:

$$\Xi=2\tan^{-1}\sqrt{\frac{3M_{22}}{M_{11}}}, \tag{20a}$$

$$\Xi=2\tan^{-1}\sqrt{\frac{M_{22}}{3M_{11}}}. \tag{20b}$$

The monotonic variation of the shape of the vortex lattice and the transition from solution (19) to solution (20) occurs when the field increases from $B\propto\phi_0(\Lambda M_{22})^{-2}$ to $B\propto\phi_0(\Lambda M_{11})^{-2}$.

Let us consider the film of a uniaxial superconductor whose anisotropy axis is inclined by an angle θ from the normal vector $\mathbf{n}$ to the X_2 axis. The principal values of tensor $\hat{\mu}$ are $\mu_a=\mu_b$ and $\mu_c=\mu_a^{-2}$ In this notation the expressions for angle Ξ are

$$\Xi=2\tan^{-1}\sqrt{3(\cos^2\theta+\mu_a^{-3}\sin^2\theta)}, \tag{21a}$$

$$\Xi=2\tan^{-1}\sqrt{\frac{\cos^2\theta+\mu_a^{-3}\sin^2\theta}{3}}. \tag{21b}$$

We compare these expressions with similar solutions for an infinite superconductor:[1]

$$\Xi=2\tan^{-1}\sqrt{\frac{3}{\cos^2\theta+\mu_a^3\sin^2\theta}}, \tag{22a}$$

$$\Xi=2\tan^{-1}\sqrt{\frac{1}{3(\cos^2\theta+\mu_a^3\sin^2\theta)}}. \tag{22b}$$

In discussing the self-energy (11) of a vortex we noted the analogy between the direction of $\mathbf{n}$ in a thin film and that of $\mathbf{H}$ in a massive superconductor. Therefore comparison of (21) and (22) would appear to be justified. The numerical calculation of (21) and (22) for superconductors with $\mu_a<1$ and $\mu_a>1$ is illustrated by Fig. 2. It is clear that at the boundary points $\theta=0$ and $\theta=\pi/2$ the vortex structures in the massive and thin-film samples coincide. For an arbitrary orientation of axis $\mathbf{c}$ the vortex lattices prove to be different.

5. THE ELASTIC PROPERTIES OF VORTEX LATTICES

5.1. We write the elastic matrix $\hat{\Phi}$ (16) in the system of coordinates associated with the wave vector $\mathbf{k}$:

$$\tilde{\Phi}_{11}(\mathbf{k})=\frac{1}{dk^2}\int d^2q\, V(\mathbf{q})(\mathbf{qk})^2[S(\mathbf{q}+\mathbf{k})+S(\mathbf{q}-\mathbf{k})-2S(\mathbf{q})],$$

$$\tilde{\Phi}_{22}(\mathbf{k})=\frac{1}{dk^2}\int d^2q\, V(\mathbf{q})(\mathbf{qk})^2[S(\mathbf{q}+\mathbf{k})+S(\mathbf{q}-\mathbf{k})-2S(\mathbf{q})], \tag{23}$$

$$\tilde{\Phi}_{12}(\mathbf{k})=0.$$

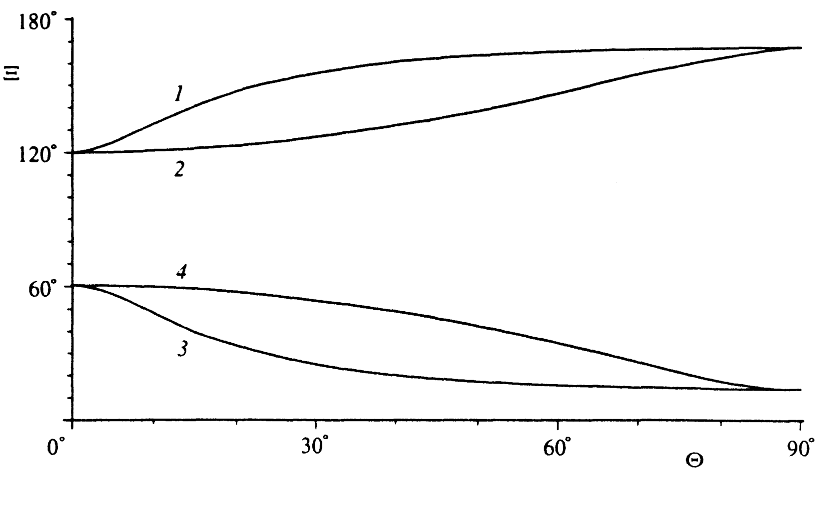

FIG. 2. The vortex lattice angle Ξ as a function of the deviation of the magnetic field from the anisotropy axis of a uniaxial superconductor. Curves *1* and *2* were built for a superconductor with $\mu_c/\mu_a=64$, and curves *3* and *4* for a superconductor with $\mu_a/\mu_c=64$; curves *1* and *3* correspond to thin-film samples, and curves *2* and *4* to massive superconductors.

The component $\tilde{\Phi}_{11}$ describes compression waves ($\mathbf{u}\|\mathbf{k}$) in the vortex lattice, and $\tilde{\Phi}_{22}$ describes shearing strains ($\mathbf{u}\perp\mathbf{k}$).

In the expression for $\tilde{\Phi}_{11}$ the main contribution is provided by vortices separated by a distance greater than Λ. The interaction $\mathfrak{U}$ (13) between such vortices is practically isotropic, and the value of $\tilde{\Phi}_{11}$ in the case of small k is independent of $\hat{\mu}$ and of the direction of $\mathbf{k}$:

$$\tilde{\Phi}_{11}=\frac{B\phi_0 k 2\pi}{d}. \tag{24}$$

A characteristic manifestation of the long-range interaction in the vortex system is the simple dependence $\tilde{\Phi}_{11}$ on the first power of k. For a massive superconductor all the coefficients Φ_{ij} are proportional to k^2 (see Ref. 15), which makes it possible to express the Φ_{ij} in terms of the elastic moduli of the vortex lattice. In the lattice of Pearl vortices, even in the isotropic case, there is no way in which the concept of an elastic modulus can be introduced for compressive strains.

5.2. For closely and loosely packed vortex lattices the quantity $\tilde{\Phi}_{22}$ in (23) depends differently on induction B and the wave vector $k\ll\sqrt{n_L}$.

For low inductions ($n_L\Lambda^2\gg 1$),

$$\Phi_{22}(\mathbf{k})=\frac{B^{1/2}\phi_0^{3/2}k^2}{2^5\pi^{3/2}d}C+\frac{B\phi_0\lambda^2}{8\pi^2 d^2}[(\mathbf{kn})\hat{\mu}(\mathbf{kn})-\mathbf{k}\hat{\mu}\mathbf{k}]. \tag{25}$$

The second term, which takes into account the anisotropy of the superconductive properties of the film, is negligible. The first term constitutes the well-known result for an isotropic superconducting film,[17] and the respective lattice sums and the value $C=1.106\,11$ were calculated by Fetter and Hohenberg.[13]

For high inductions ($n_L\Lambda^2\ll 1$), it is convenient, when calculating $\tilde{\Phi}_{22}$, to go over to the primed system of coordinates [Eqs. (7) and (8)], in which $\tilde{\Phi}_{22}$ differs from the isotropic expression $\phi_0^2 k'/64\pi^2\lambda^2$ (see Ref. 15) by an additional factor, $(\mathbf{n}\hat{\mu}^{-1}\mathbf{n})^{-3/2}(k'/k)^2$. Performing the inverse coordinate transformation, we obtain

$$\tilde{\Phi}_{22}(\mathbf{k})=\frac{B\phi_0}{64\pi^2\lambda^2}\frac{[(\mathbf{kn})\hat{\mu}(\mathbf{kn})]^2}{k^2(\mathbf{n}\hat{\mu}^{-1}\mathbf{n})^{3/2}}. \tag{26}$$

The following conclusions can be drawn from (26):

(a) as in the case of a massive isotropic superconductor, $\tilde{\Phi}_{22}\sim k^2$;

(b) for a given orientation of the anisotropy axes in the film, $\tilde{\Phi}_{22}$ assumes its maximum value when the vortex layers are shifted along the X_2 axis with the greatest value of $\mathfrak{M}$, and the minimum value of $\tilde{\Phi}_{22}$ is realized when the shift is in the perpendicular direction;

(c) the greatest and smallest values of $\tilde{\Phi}_{22}$ differ by a factor of $(M_{22}/M_{11})^2$;

(d) for a specific superconducting material $\tilde{\Phi}_{22}$ assumes its extreme values in the case when the axes with the greatest and smallest values of tensor $\hat{\mu}$ lie in the plane of the film;

(e) in the case of a uniaxial superconductor,

$$\tilde{\Phi}_{22}(\mathbf{k})=\frac{\Phi_0^2}{64\pi^2\lambda^2}\frac{[k_2^2+k_1^2(\cos^2\theta+\mu_a^{-3}\sin^2\theta)]^2}{k^2\mu_a(\cos^2\theta+\mu_a^{-3}\sin^2\theta)^{3/2}}. \tag{27}$$

For a fixed value of angle θ, the greatest and smallest values of $\tilde{\Phi}_{22}$ differ by a factor $[1+(\mu_a^{-3}-1)\sin^2\theta]^2$. The projection of the anisotropy axis on the plane of the film determines the direction of difficult shear in superconductors with $\mu_a<1$ and the direction of easy shear in the case of $\mu_a>1$.

Kogan and Campbell[7] calculated the shear modulus for the vortex lattice in a uniaxial superconductor with $\mu_a<1$. Their main finding was that the shear modulus is an anisotropic quantity and reaches its greatest value when the vortices are shifted along the anisotropy. This is also true for a thin film, but the dependence of the shear modulus obtained in Ref. 7 on the parameters μ_a and θ differs from formula (27).

5.3. In the previous section we established that two vortex structures satisfy the equations of equilibrium. Their reactions to compressive and shearing strains are the same and are described by the positive coefficients of the elastic matrix Φ. To establish the stability conditions for these structures we consider the elastic modulus of rotation of the vortex lattice as a whole:

$$C_{\rm rot}=\frac{n_{\rm L}}{d}\int d^2q\, V(\mathbf{q})\left\{-\left(\mathbf{q}\frac{\partial}{\partial \mathbf{q}}\right)-\left[\mathbf{n}\left(\mathbf{q}\frac{\partial}{\partial \mathbf{q}}\right)\right]^2\right\}S(\mathbf{q}). \tag{28}$$

Introducing the elastic modulus $C_{\rm rot}$ as a means of describing rotational strain is justified since in the given case the interaction of vortices is "short-range": at great distances $U(\mathbf{R})$ is independent of direction of $\mathbf{R}$.

For a loosely packed lattice ($n_{\rm L}\Lambda^2 \ll 1$) the integral in (28) can be calculated analytically. For the vortex structure (19a) we have

$$C_{\rm rot}\simeq 11.6(M_{22}-M_{11})^3\frac{2^9B^3\lambda^{12}}{\phi_0 d^{10}}, \tag{29a}$$

while for the vortex lattice described by (19b) we have

$$C_{\rm rot}\simeq -11.6(M_{22}-M_{11})^3\frac{2^9B^3\lambda^{12}}{\phi_0 d^{10}}. \tag{29b}$$

These expressions show that the sign of the modulus $C_{\rm rot}$ determines the stability of a vortex structure. A lattice is stable if one of its elementary translation vectors is directed along the axis with the greatest value of tensor $\mathcal{M}$. This statement is also true for massive superconductors.[5]

For a closely packed lattice ($n_{\rm L}\Lambda^2 \gg 1$) a numerical calculation of (28) leads to the following expressions for the elastic modulus $C_{\rm rot}$:

$$C_{\rm rot}=\beta\frac{(M_{22}-M_{11})^3}{(M_{11}M_{22})^{5/2}}\frac{B^{1/2}\phi_0^{3/2}d}{4^4\pi^3\lambda^4} \tag{30a}$$

for solution (20a), and

$$C_{\rm rot}=-\beta\frac{(M_{22}-M_{11})^3}{(M_{11}M_{22})^{5/2}}\frac{B^{1/2}\phi_0^{3/2}d}{4^4\pi^3\lambda^4} \tag{30b}$$

for solution (20b). In these expressions the numerical parameter β is approximately equal to unity and depends very weakly on the anisotropy: for any variations in M_{11} and M_{22} by a factor of 100 (from 0.3 to 30) the value of β varies by a factor of less than two.

In the particular case of a uniaxial superconductor the elastic rotation modulus of a vortex lattice is

$$C_{\rm rot}\simeq \pm(1-\mu_a^3)^3\sin^6\theta.$$

6. CONCLUSION

1. We have studied the structure of an isolated vortex and an equilibrium vortex lattice in a thin film of an anisotropic superconductor. We have calculated the components of the dynamical matrix and the elastic rotation modulus of the lattice. The results are valid for a broad range of magnetic fields from zero to several tesla. For high values of induction the distance between the vortices becomes smaller than d and the Pearl approximation breaks down.

2. The anisotropy of the magnetic properties of thin film is described by the tensor $\mathcal{M}$ of rank 2 given by Eq. (6).

3. At great distances ($\lambda \gg \Lambda$) the structure of a vortex is determined by the fields generated by the vortex in free space. Hence the field, the current (5), and the pair vortex interaction potential $\mathfrak{U}(\mathbf{x})$ (13) are centrally symmetric and do not depend on the anisotropy parameters. In the near region ($x \ll \Lambda$) the vortex field and the vortex interaction with neighbors are anisotropic. The constant-current and constant-field lines are ellipses. The principal axes of tensor $\mathcal{M}$ are the symmetry axes of the vortex structure and fix the direction of difficult and easy shear of vortex chains in the lattice.

4. For low inductions ($B<\phi_0/\Lambda^2$) the vortex lattice always has the hexagonal structure (19), which transforms into an oblique-angled structure (20) as the field strength grows. The lattice orients itself in such a way that one of the elementary translation vectors is directed along the X_2 axis with the greatest value of the tensor $\mathcal{M}$ in the plane of the film.

5. The rigidity of the lattice under compressive strains (24) grows $\propto B^2k$ and is independent of the direction of the compression waves.

6. The reaction of the lattice to shearing strains is different for low (Eq. (25)) and high (Eq. (26)) inductions. For $B<\phi_0/\Lambda^2$ the coefficient Φ_{22} varies like $B^{3/2}k^2$. As the field grows, the dependence on B becomes linear, the dependence on the wavelength disappears, and a dependence on the direction of shearing appears. The lattice manifests the greatest rigidity under a shift of the vortex layers along the X_2 axis, the direction with the greatest value of $\mathcal{M}$, and the least rigidity under a shift in the perpendicular direction.

7. For rotation of the lattice as a whole the potential of vortex interaction is short-range. Rotation strain is described by the elastic rotation modulus (28) and determines the stability of equilibrium vortex structures.

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