Effect of a magnetic field on the equation of state and mass of a neutron star

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We study the effects of strong magnetic fields, temperature, and high matter density—which are typical of pulsars—on an object consisting only of neutrons. The interaction of the neutrons is described by the Lagrangian of the relativistic mean-field theory (Walecka's model). An equation of state is derived, and it is shown that the object under study is degenerate and its polytropic index is identical to that of a white dwarf. The parameters of a neutron star are estimated as functions of the magnetic field intensity on the basis of the solutions of the Oppenheimer–Volkoff equations.

Modern methods make it possible to take into account the effect of quantum phenomena on the parameters of macroscopic objects, establishing in the process a connection between phenomena occurring at different scales. In particular, the behavior of neutrons in a magnetic field is of interest in application to such remarkable astronomical objects as neutron stars. In addition to colossal density, temperature, and magnetic-field intensities, which are all taken into account in this paper, neutron stars have an admixture of charged particles, which have a considerable effect on the structure and radiation of the star.

Thus, to describe a neutron star we must know the equation of state of its constituent matter. One way to formulate an equation of state at high densities ($\rho \sim 10^{15}$ g/cm^3) is to construct a relativistic Lagrangian that describes the interaction of bare nucleons. Such a model was first proposed by Walecka.[1] In Walecka's model, attraction is mediated by the exchange of scalar mesons and repulsion is mediated by the exchange of massive ω vector mesons. The typical equations of state above the formation point of neutron droplets is discussed, for example, in Refs. 2 and 3. In the present paper we study (on the basis of the model mentioned above) the effect of an external uniform constant magnetic field, finite temperature, and density. The effect of the field will be taken into account by proceeding from the spectrum of a single-particle equation with all its attendant consequences. It is obvious from the foregoing constraints on the problem that a complete solution would require a considerable advance in a number of physical fields, and allowance for many effects. In spite of this, however, we used the relations that we derive below for the pressure and number density as functions of the temperature in the so-called low-temperature limit to describe a neutron star. We assume a spherically symmetric "standard" metric for space–time. In spite of the qualitative nature of the results, even in such a primitive model in which the magnetic field is taken into account, the mass of the "star" is in the $2M_\odot$ range. This is obviously in better agreement with the astronomical observations[2,4] than the corresponding results obtained neglecting the magnetic field. This result may not be model-dependent on the type of nucleon–nucleon interaction, but the complete answer to this question requires a serious check and study of the other models claiming to give a more complete description of the physics of neutron stars.

1. THERMODYNAMIC POTENTIAL OF A NEUTRON GAS IN A MAGNETIC FIELD

As we indicated above, the effective Lagrangian in Walecka's model[1] has the form

$$\mathscr{L} = \bar{n}\left(i\gamma_\mu \partial^\mu - m_0 + g_\sigma \sigma - g_\omega \omega_\mu \gamma_\mu - \frac{i}{2} M_N \sigma_{\mu\nu} F^{\mu\nu} \right) n + \tfrac{1}{2}\left(\partial_\mu \sigma \partial^\mu \sigma - m_\sigma^2 \sigma^2\right) + \tfrac{1}{2} m_\omega^2 \omega^\mu \omega_\mu - \tfrac{1}{4} G^{\mu\nu} G_{\mu\nu}, \tag{1}$$

where $\sigma_{\mu\nu} = (\gamma_\mu \gamma_\nu - \gamma_\nu \gamma_\mu)/2$. In this model the strong interaction is mediated by two types of particles, σ and ω mesons. The scalar σ meson is identified with the $\pi\pi$ resonance and is responsible for the attraction between nucleons. The repulsion between nucleons at short distances can be approximated by introducing a single vector meson ω^μ, corresponding to the term $G_{\mu\nu}G^{\mu\nu}/4$ in the Lagrangian (1), apart from the mass term, where the tensor $G_{\mu\nu} = \partial_\mu \omega_\nu - \partial_\nu \omega_\mu$. It turns out that this model has mean-field solutions, and pions do not contribute in the mean-field approximation.[5] This corresponds[6] to the restoration of chiral symmetry in dense nuclear matter, in which the pion effective mass approaches zero and the pion degrees of freedom become collective, so that even the formation of a pion condensate is possible.

We take the following values for the masses and interaction constants:

$$m_\sigma = 550 \text{ MeV}, \quad m_\omega = 783 \text{ MeV},$$

$$g_\sigma^2/4\pi = 9.2, \quad g_\omega^2/4\pi = 20.$$

The interaction of a neutron with the electromagnetic field is described in the Lagrangian by a term with an anomalous magnetic moment M_N (AMM).

Using the fact that the number of particles $Q = n^+ n$ is a dynamical invariant (Noether charge[7]) in our system, we introduce a chemical potential in accordance with the scheme described in detail in Ref. 8.

Let the fields σ and ω_μ be represented as coordinate-independent average values $\bar{\sigma}$ and $\overline{\omega_0}$, which form a background for fluctuations $\sigma'(x)$ and $\omega'(x)$:

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$$\sigma=\bar{\sigma}+\sigma'(x),\quad \omega_\mu=\delta_{0\mu}\bar{\omega}_0+\omega'_\mu(x).$$

Then the mean-field approximation $\sigma'(x)=0$ and $\omega'_\mu(x)=0$, with the addition of the term $\mu n^+ n$ into Eq. (1), will make it possible to treat the neutron gas as a free gas with effective parameters

$$m_0\rightarrow m=m_0-g_\sigma\bar{\sigma},\quad \mu\rightarrow\mu_n=\mu-g_\omega\bar{\omega}_0.$$

We assume that the electromagnetic field in Eq. (1) is an external, constant and uniform, purely magnetic field: $\mathbf{H}=(0,0,H)$ and $\mathbf{E}=0$. It is well known[9] that the energy spectrum of a charged particle with an AMM in such a field has the form

$$E^2=p^2+(\sqrt{m^2+2neH}+\xi MH)^2, \tag{2}$$

where m is the particle mass, e is the particle charge, p is the projection of the momentum in the direction of the magnetic field H, n is the number of the Landau level, $\xi=+1$, and M is the AMM of the particle.

The thermodynamic potential is defined in the standard manner[10]

$$\Omega(\mu,\theta)=-\theta\sum_k \ln\left[1+\exp\left(-\frac{E_k-\mu_n}{\theta}\right)\right],$$

where E_k is the energy spectrum of a nucleon whose mass is replaced by the effective mass and μ_n is the effective chemical potential.

The thermodynamic potential engendered by the spectrum (2) has been studied before, for example, in Refs. 11 and 12.

A neutral particle interacts with the field via the AMM. For this reason, the energy spectrum can be represented by an equation derived from the generalized Dirac equation corresponding to the Lagrangian (1):

$$E^2(p)=p_3^2+(\sqrt{m^2+p_\perp^2}+\xi B)^2, \tag{3}$$

where $B=M_N H$ and $M_N=6.25\cdot 10^{-12}$ eV/G. This spectrum can also be obtained formally from Eq. (2) when the electric charge is zero and the transverse momentum is unquantized. The same expression is employed in Ref. 13.

We replace the summation in the definition of the thermodynamic potential Ω with our spectrum (3), which is continuous as a function of p, by integration over momentum, and we take into account the antiparticles in the initial Lagrangian and in the corresponding functional integral. We then represent the total potential Ω in the form

$$\Omega=\Omega_{\text{particles}}+\Omega_{\text{antiparticles}}=\Omega(\mu)+\Omega(-\mu).$$

Expanding $\ln(1+x)$ in a series, we obtain the representation

$$\Omega=-2\theta V\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}\cosh(\mu\beta k)\int d^3p e^{-k\beta E(p)}. \tag{4}$$

Here we use the effective values for m and μ, and we denote the temperature by $\theta=\beta^{-1}$. We use the system of natural units in which $\hbar=c=1$.

Since the energy (3) depends only on p_3^2 and $p_\perp^2$, the integration over momentum separates into two steps: integration over dp_3 and $dp_\perp$. The integration of dp_3 can be easily performed by making the substitution of variables

$$a=\sqrt{m^2+p_\perp^2}+\xi B,\quad x=\sqrt{a^2+p_3^2}.$$

This substitution and the integration give modified Bessel functions, and the momentum integral in Eq. (4) reduces to the form $\int dp_\perp 2aK_1(k\beta a)$. Using the definite integral of the modified Bessel function,[14] we obtain the initial formula for the thermodynamic potential of the model under consideration:

$$\Omega=-\frac{8\pi m^4}{\omega'^3}V\sum_{\xi=\pm1}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k^3}\cosh(r\omega' k)\times\left[\frac{\omega^2 k}{\omega'}K_2(\omega k)+\frac{\pi\gamma}{2}\{1-\omega k[L_0(\omega k)K_1(\omega k)+L_1(\omega k)K_0(\omega k)]\}\right],$$

where $L_\nu(z)$ is the modified Struve function.

Here we have introduced the dimensionless parameters

$$\omega'=m\beta,\quad \omega=\beta(m+\xi B),\quad \gamma=\xi B/m,\quad r=|\mu|/m. \tag{5}$$

It is well known (see, for example, Refs. 8 and 10) that the complete thermodynamic picture can be obtained if the potential Ω is known. Differentiating the expression obtained for Ω and using the properties of the special functions (see Ref. 14), we can obtain an expression for the number density n and the entropy density ε. As an example, we write out the expressions for n and ε—the proper internal energy:

$$n=\frac{8\pi m^3}{\omega'^3}\sum_{\xi=\pm1}\sum_{k=1}^{\infty}(-1)^{k+1}\frac{\sinh(r\omega' k)}{k^2}\times\left[\frac{\omega^2 k}{\omega'}K_2(\omega k)-(\pi\gamma/2)\{1-\omega k[L_0(\omega k)K_1(\omega k)+L_1(\omega k)K_0(\omega k)]\}\right], \tag{6}$$

$$\varepsilon=2p-8\pi\omega^2\left(\frac{m}{\omega'}\right)^4\sum_{\xi=\pm1}\sum_{k=1}^{\infty}(-1)^{k+1}\cosh(r\omega' k)\times[k^{-2}K_2(\omega k)-(\pi\gamma\omega'/2k)K_1(\omega k)[L_1(\omega k)-L_{-1}(\omega k)]]. \tag{7}$$

We note that in the model of a neutron gas interacting with a magnetic field via the AMM, these expressions are accurate right up to the limits of applicability of the one-loop approximation.

2. EQUATION OF STATE AND ASYMPTOTIC EXPANSION

Equations (6) and (7) are too complicated for direct calculations. It is therefore convenient to make use of the fact that $\omega\gg1$ according to astrophysical estimates for a neutron star (for temperatures in the range 10^6–10^9 K, $\omega \simeq \omega' \sim 10^4$–$10^7$), and to employ asymptotic expansions of the special functions.

Using the asymptotic expansions of the functions $K_\nu(z)$ and $L_\nu(z)$ for large values of the argument z,

$$K_\nu(z) \sim \sqrt{\frac{\pi}{2z}}e^{-z}, \quad L_\nu(z) = \frac{e^z}{\sqrt{2\pi z}}$$

and the fact that $L_\nu(z) = L_{-\nu}(z)$ for large values of the argument in terms proportional to e^z (Ref. 14), it is easy to show that only the term proportional to

$$(\gamma\pi/2)\{1 - \omega k[L_0(\omega k)K_1(\omega k) + L_1(\omega k)K_0(\omega k)]\}$$

is not suppressed by a very small exponential factor. Indeed, even for the case $\omega = 10^4$ and $k = 1$, this factor is

$$e^{-\omega} = 10^{\lg e^{-\omega}} = 10^{-\omega \lg e} = 10^{-0.3 \cdot 10^4} = 10^{-3000}.$$

In addition, it should not be forgotten that γ, defined in Eq. (5), is a small parameter—$\gamma = 10^{-8}$–10^{-2} and that $\omega = \omega'(1+\gamma)$. We therefore employ the crudest approximation. The point is that although summing each successive term can contribute significant changes, the first term is by itself a good approximation. This is one of the properties of an asymptotic series.[15,16]

It is easy to show that the first two approximations

$$(\pi\gamma/2)\{1 - \omega k[L_0(\omega k)K_1(\omega k) + L_1(\omega k)K_0(\omega k)]\}$$
$$\simeq (\pi\gamma/2)(2 + 105/[512(\omega k)^3])$$

do not depend on either the summation index k or the temperature, and that in carrying out the summation $\Sigma_{\xi=\pm 1}$ in the leading asymptotic approximation, it is primarily the first term, proportional to the cube of the temperature, that survives. After summing over k (see Appendix), the temperature dependence ultimately vanishes completely. Finally, collecting all preliminary calculations, we write

$$p = (7/2048)\pi^2|\gamma|^2 r^6 m^4, \tag{8}$$

$$n = (21/1024)\pi^2|\gamma|^2 r^5 m^3, \tag{9}$$

$$s = (63/2048)\pi^2|\gamma|^2 r^6 m^3 \omega', \tag{10}$$

$$\varepsilon = 2p = (7/1024)\pi^2|\gamma|^2 r^6 m^4. \tag{11}$$

The fact that the temperature dependence has disappeared in the crudest approximation favors the result obtained, since a neutron star is a cold object and the physical situation is such that the neutron gas is a degenerate object. Neglecting this fact, however, we worked under the assumption that the temperature is finite and arrived at the same conclusion that the gas is degenerate. Moreover, if we still wish to determine the temperature dependence, however weak it may be, we can use Joncquière's relation (Eq. (A2) in Appendix). We give below the temperature corrections to the pressure and number density, for example:

$$\delta p = \frac{7}{2048}\frac{B^2}{m^4}\left[5\pi^4\mu^4\theta^2 + 7\pi^6\mu^2\theta^4 - \frac{155}{21}\pi^8\theta^6\right],$$

$$\delta n = -\frac{7}{512}\frac{B^2}{m^4}\left[5\pi^4\mu^3\theta^2 + \frac{7}{2}\pi^6\mu\theta^4\right].$$

We now consider a purely neutron object whose characteristics have already been calculated [Eqs. (8)–(11)]. It is well known[10] that the general conditions of equilibrium and stability have the form

$$\partial\Omega/\partial m = 0, \quad \partial\Omega/\partial\mu = 0.$$

In the mean-field approximation, these conditions are equivalent to the equations of motion for the σ and ω fields, and yield two relations for the three parameters μ_0, $\bar{\omega}_0$, and σ.

Let the free parameter be the effective ratio of the chemical potential to the mass:

$$r = \mu/m = (\mu_0 - g_\omega\bar{\omega}_0)/(m_0 - g_\sigma\bar{\sigma}).$$

This ratio is convenient in that it determines the total proper energy density ρ as a function of pressure:

$$\rho = mn + \varepsilon,$$
$$\rho = (2 + 6/r)p. \tag{12}$$

But we do not know the order of r: after all, we are dealing with a Fermi gas, and in our case there is no restriction $r < 1$, as there is for Bose particles, where the condition $\mu = m$ means that the particles condense. In order to identify the magnetic-field dependence of r, we introduce the parameter α defined by

$$\alpha = m/m_0,$$

which obviously falls in the range $0 < \alpha < 1$ and characterizes the efficiency of nucleon interactions with the scalar field. Then the desired function has the form

$$r = [(1-\alpha)/(a_1\alpha B^2)]^{1/6}, \tag{13}$$

where

$$a_1 = 7\pi^2/512, (g_\sigma/m_\sigma)^2 = 5.16 \cdot 10^{-17}\ \text{eV}^{-2}.$$

We use the expressions (12) and (13) in the next section to derive the mass and radius of a neutron star as a function of the field and the parameter α.

According to Refs. 2, 4, and 17, the surface magnetic fields of pulsars are very high—$H = 10^{12}$–10^{13} G—and in the interior layers of a star the field can reach values of the order of $H = 10^{16}$–10^{18} G. Even neglecting possible twisting of the field, which intensifies the field by another several orders of magnitude, we obtain the following range for investigation:

$$H = 10^{12}\text{–}10^{18}\ \text{G}. \tag{14}$$

Estimates obtained on the basis of the formulas presented above illustrate how the internucleon interaction is realized: for "weak" fields in the range (14) the nucleons interact predominantly via the σ meson, which "stretches" the mass, and at the center of the range the nucleons interact via a change in the chemical potential due to the influence of the ω meson. This result confirms the "naive" consistency of our model (at least to a first approximation), since the ω vector meson approximates repulsion, and it is natural to infer that it plays a larger role in the interior layers.

3. SOLUTION OF THE OPPENHEIMER–VOLKOFF EQUATION

A neutron star, which is prevented from collapsing by the pressure of the neutron gas, is a cold object, for which relativistic effects play an important role. The general-relativistic calculation of the pressure, density, and gravitational fields inside spherically symmetric static stars have been examined in detail in Ref. 18. The conditions of spherical symmetry and hydrostatic equilibrium of an ideal fluid lead to the choice of the "standard" metric, and as a result the Oppenheimer–Volkoff–Tolman equation is obtained from Einstein's equations:[2,4,18]

$$-x^2p'(x)=GM(x)\rho(x)\left[1+\frac{p(x)}{\rho(x)}\right]\left[1+\frac{4\pi x^3\rho(x)}{M(x)}\right]\times\left[1-\frac{2GM(x)}{x}\right]^{-1}. \quad (15)$$

A prime denotes a derivative d/dx, and the following numerical values are used: $G=1.475M_\odot/\text{km}$ and $M_\odot=2\cdot10^{33}$ g. Given $p(x)$ as the function $p[\rho(x)]$, we can study two first-order differential equations: Eq. (15) for $p(x)$ and

$$M'(x)=4\pi x^2\rho(x)dx. \quad (16)$$

Given the equation of state $p(\rho)$ and the initial conditions

$$\rho(0)=\rho_0,\quad M(0)=\rho'(0)=M'(0)=0,$$

the system of differential equations (15) and (16) determines the functions $\rho(x)$, $M(x)$, and $p(x)$. This system must be integrated from the center of the star out to some radius $x=R$ at which $p[\rho(x)]=0$.

To distinguish the ratio of the chemical potential to the mass from the radial coordinate, we relabel the radial coordinate as x. Usually, even in our case the characteristic internal energy density

$$\varepsilon=\rho-mn$$

is proportional to the pressure

$$\varepsilon=(\gamma-1)^{-1}p.$$

The quantity γ is called the polytropic index. In the general case it will not be the ratio of the specific heats unless ε and p are proportional to the temperature, which is not the case in our equations. Nonetheless, the numerical values of γ largely characterize star types (see, for example, Ref. 18):

$\gamma=4/3$—white dwarfs with the largest mass,

$\gamma=5/3$—white dwarfs with the lowest mass,

$\gamma=\infty$—incompressible stars.

In our case $(\gamma-1)^{-1}=2$, i.e., $\gamma=3/2$. Thus a neutron star is similar to a white dwarf with average mass. This qualitative result corresponds to the physical picture according to which the neutron gas is degenerate, the magnetic field being neglected in this analysis.[2,4,18]

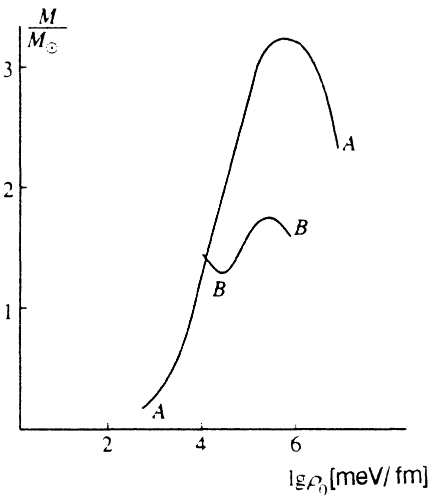

FIG. 1.

We integrated Eqs. (15) and (16) by the Runge–Kutta method[19] with the initial conditions written out above, and obtained the following results.

3.1. Stability

Using the well-known theorems of Refs. 2, 4, and 18 we rewrite the criterion of stability with respect to a particular radial normal mode as

$$\partial M(\rho(0),R)/\partial\rho(0)>0,$$

i.e., above a critical density $\rho_{\text{cr}}(0)$ the star starts to compress spontaneously.

We can see (Fig. 1) the difference from the results obtained by J. Kapusta[20] (line A), who did not study the effect of a magnetic field on the system and immediately took the Fermi momentum p_F as the parameter and assumed temperature independence. As a result he obtained the following range of energy densities for a stable configuration

$$10^2\,\frac{\text{MeV}}{\text{fm}^3}<\rho_0<10^6\,\frac{\text{MeV}}{\text{fm}^3},$$

which in our notation is

$$8.3\cdot10^{-4}\,\frac{M_\odot}{km^3}<\rho_0<8.3\,\frac{M_\odot}{km^3}.$$

However, in our case, i.e., taking into account the magnetic field, the range is narrower and shifted in the direction of low central densities

$$0.02\cdot10^{-4}\,\frac{M_\odot}{km^3}<\rho_0<0.4\,\frac{M_\odot}{km^3}$$

and increasing the intensity of the external field does not significantly decrease this interval. The radius R assumes satisfactory values from 5 to 10 km and the critical mass is not so high.

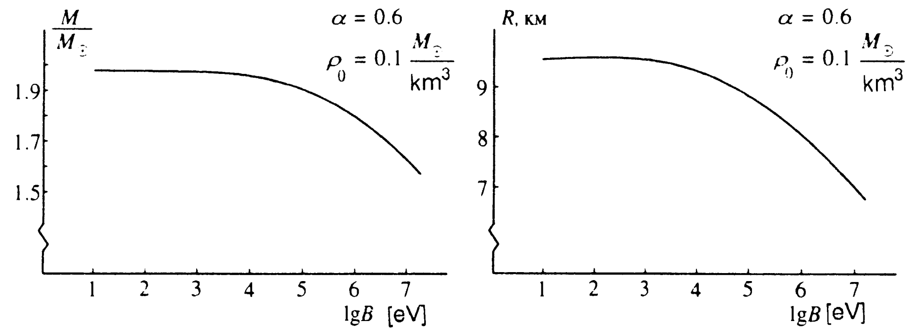

FIG. 2.

3.2. Effect of a magnetic field

We studied first a constant uniform magnetic field and found that starting at 10^{16} G the field significantly compresses the star. The star becomes more compact: The mass and radius decrease from 1.95 to 1.6 solar masses and from 9.5 to 7 km, but the average density

$$\bar{\rho}=3M/(4\pi R^3)$$

increases. This result is realistic. Let us assume that the thermodynamic formulas employed above do not change much if the magnetic field deviates from a constant value. We assume that the third axis remains parallel to the field H, but the field decreases in modulus from the center of the periphery according to the law

$$H(x)=H_0 \exp(-0.1x \ln H_0/H_1),$$

where H_0 and H_1 are the values of the field at the center of the star and at a distance of 10 km from it, respectively. Then the result of differentiating the quantity p/ρ with respect to x is added to the right-hand side of Eq. (15). We found that, first, this change is continuous and, second, the main compression occurs due to a strong field at the periphery and not the center of the star.

3.3. In practice the α dependence is insignificant in the interval $0.01 \leqslant \alpha \leqslant 0.99$

In Sec. 3 we studied the behavior of a neutron medium (in which the internucleon interaction is described by Walecka's model Lagrangian[1]), with a spherically symmetric metric, for densities at the center of the star. The fact that the masses and radii were found to be of the order of the values calculated previously in Refs. 2, 4, 18, and 20 is mainly due to our choice of the initial condition on the central density. Indeed, all thermodynamic quantities are calculated by differentiating the term proportional to $K_\nu(z) L_\nu(z)$ and the magnetic field, i.e., they are of a completely different nature from Refs. 18 and 20. For this reason, we do not discuss the limit of zero magnetic field.

It should be kept in mind, however, that neutrons alone are not enough to construct a complete physical picture of a star whose specific features include processes in which neutrinos, charged fermions, a pion condensate, and possibly also a number of exotic particles, such as the axion, photino, and others, participate. For example, it is noted in Ref. 5 that the pion degrees of freedom significantly influence the structure of the star and the state of its matter.

4. CONCLUSIONS

The effects of an admixture of charged particles are far from trivial for the following reasons.

First, the general model Lagrangian of the system and the equations for the averages will change. Correspondingly, the solutions for the averages will change, as will a number of other parameters of the system.

Second, the separation of the main contributions in the expressions for the thermodynamic potential will turn out to be a very complicated problem with different parameters of the system. Thus, for example, the term θs makes the main contribution to the characteristic internal energy density of the charged gas and the corresponding equation of state will be more complicated than in the case of neutral particles.

Third, the system of seven equations with eight unknowns is itself a complicated structure for further investigation. A preliminary study of this system, using the conjectured numerical values of the fields and temperatures at the center of the star, will make it possible to determine whether or not the choice of model and its predictions are qualitatively correct.

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APPENDIX

In order to calculate the leading asymptotic term in $r\omega$ in a sum of the form

$$\Sigma_s(r\omega)=\sum_{k=1}^{\infty}(-1)^{k+1}e^{r\omega k}/k^s$$

outside the circle of convergence $|r\omega|<1$ it is necessary to use Joncquière's relations[16] for the analytic continuation of the function $F(z,s)$ into the range $|z|>1$. Since by definition

$$F(z,s)=\sum_{k=1}^{\infty} z^k/k^s,$$

in our case $z=e^{-r\omega}$ and we are interested in $F(1/z,s)$:

$$-F\left(\frac{1}{z},s\right)=\Sigma_s(z).$$

Joncquière's relation

$$F(z,s)+e^{i\pi s}F\left(\frac{1}{z},s\right)=\frac{2\pi}{\Gamma(s)}\,e^{i\pi/2}\zeta\left(1-s,\frac{\ln z}{2\pi i}\right) \qquad \text{(A1)}$$

is the analytic continuation of $F(z,s)$ into the region outside the circle of convergence. Using the property of the Riemann zeta function for integer values of m

$$\zeta(-m,v)=-B_{m+1}(v)/(m+1),$$

where $m=1,\ 2,\ldots$, and B_m are Bernoulli polynomials, we obtain easily

$$F(z,m)+(-1)^m F\left(\frac{1}{z},m\right)=-\frac{(2\pi i)^m}{m!}\,B_m\left(\frac{\ln z}{2\pi i}\right). \qquad \text{(A2)}$$

The appearance of $i=\sqrt{(-1)}$ in Eq. (A2) even though the initial series are obviously real should not confuse us, since the imaginary part cancels exactly, if the Bernoulli polynomial is written out explicitly as

$$B_n(x)=\sum_{k=1}^{n} C_n^k B_k x^{n-k},$$

where C_n^k are the binomial coefficients and B_k are the Bernoulli numbers, and the fact that $z=-e^{-r\omega}$ is also taken into account, i.e.,

$$(\ln z)/2\pi i=-(r\omega)/(2\pi i)+1/2.$$

For example, for $s=5$

$$B_5\left(-\frac{r\omega}{2\pi i}+\frac{1}{2}\right)=\left(-\frac{r\omega}{2\pi i}\right)^5-\frac{5}{6}\left(-\frac{r\omega}{2\pi i}\right)^3+\frac{7}{12}\left(-\frac{r\omega}{2\pi i}\right).$$

Writing out the right-hand side of Eq. (A2) we obtain

$$F(-e^{-r\omega},s)+(-1)^s F(-e^{r\omega},s)=\sum_{n=1}^{\infty}\frac{(-1)^n}{n^s}\,[e^{-r\omega n}+(-1)^s e^{r\omega n}]=(-1)^s F(e^{r\omega},s).$$

It is obvious that the leading term from the Bernoulli polynomial will be

$$B_s(x)\cong x^s/s!\,.$$

Then

$$(-1)^m F\left(\frac{1}{z},m\right)\simeq-\frac{(2\pi i)^m}{m!}\left[-\frac{r\omega}{2\pi i}\right]^m,$$

and finally we obtain the leading term in $r\omega$ in the desired sum

$$\Sigma_s(r\omega)=\frac{1}{s!}\,(r\omega)^s. \qquad \text{(A3)}$$

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