Energy of electromagnetic radiation from arbitrary sources in a transparent dispersive medium. Influence of interference on scattering processes

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General expressions are obtained for the spectral and angular distributions of the energy of transverse electromagnetic waves radiated or scattered by arbitrary sources in a transparent dispersive medium in terms of the Fourier transform of a quantity which describes the sources (currents, variable or nonlinear polarization of the medium, etc.). It is shown with the help of the relations so introduced that the contribution to the energy of the waves due to interference of the wave incident on an arbitrary system of currents and the scattered (reradiated) wave adequately describes the attenuation (or amplification) of the wave. In particular, attenuation of radiation in the medium can be explained by interference of the incident wave and waves reradiated by the oscillators of the medium. It is found that for scattering of a beam of electromagnetic waves with arbitrary divergence by a point oscillator, the interference term should alter the angular intensity distribution. The angular intensity distribution of a transmitted wave is calculated for a Gaussian beam with arbitrary aperture, taking interference into account.

1. INTRODUCTION

Processes of radiation, propagation, and scattering of electromagnetic waves are an important object of study in classical electrodynamics. In this regard, in different problems in plasma physics, nonlinear optics, the theory of accelerators and elementary particle detectors, and in other branches of physics, a variety of methods, sometimes differing widely in form, are used to calculate the characteristics of radiated or scattered waves (see, for example, Refs. 1–3).

One of the methods that are suitable in the case of a medium with arbitrary dispersion is the Green's function method based on the solution of the Maxwell equations for the Fourier transforms of the fields (see, for example, Ref. 3, p. 205). The main drawback of such operational methods is the necessity to return to ordinary $(\mathbf{r},t)$-space (perform the inverse Fourier transform) to compare the theoretical results with observed experimental quantities. However, in the case of wave problems the Fourier transforms of the fields have a physical meaning of their own. Thus it is possible to obtain interesting spectral and angular distributions of the radiation intensity in dispersive transparent media in such problems directly from the Fourier transforms of the fields. Such an approach is used, for example, in the theory of transient radiation in a nonstationary, nondispersive medium.[2,3]

In the present paper a general method is proposed which allows one to reduce the problem of calculating the characteristics of electromagnetic waves in a transparent dispersive medium to the calculation of the Fourier transform of a quantity characterizing the source of the waves that amounts to a generalized polarization. The novelty of the expressions obtained below lies in the lack of any restrictions on the type of sources and the character of the dispersion of the medium.

By way of an illustration of the use of the method, we present an analysis of the influence of interference of the incident wave and the scattered wave on scattering by an oscillator. It will be shown that the attenuation of radiation in an absorbing and scattering medium consisting of independent oscillators can be considered a manifestation of this interference. If a beam of electromagnetic radiation with finite aperture is scattered by an isolated oscillator or a localized group ("cluster") of oscillators, then interference of the incident wave and the reradiated wave leads to the formation of a minimum in the angular total energy distribution and to a narrowing of the beam as a result of scattering.

We write the system of Maxwell's equations in a nonmagnetic medium for the total Fourier transforms of the fields, which are given by

$$\mathbf{X}_{\mathbf{k},\omega}=(2\pi)^{-4}\int d\mathbf{r}dt\mathbf{X}(\mathbf{r},t)\exp[i(\omega t-\mathbf{kr})].$$

We denote by $\mathbf{E}$ and $\mathbf{H}$ the electric and magnetic field vectors. We write the constitutive relations in the form $\mathbf{D}_{\mathbf{k},\omega}=\hat{\varepsilon}(\mathbf{k},\omega)\mathbf{E}_{\mathbf{k},\omega}+\mathbf{P'}_{\mathbf{k},\omega}$, where the frequency- and spatially-dispersive "matrix" is described by the tensor $\hat{\varepsilon}(\mathbf{k},\omega)$, and the "additional" polarization $\mathbf{P'}$ can be due, for example, to nonuniformities, nonstationarity, or nonlinearity of the medium. The latter can serve as a source of electromagnetic waves along with charged particles and currents.

Maxwell's equations take the form

$$(\mathbf{k},\hat{\varepsilon}(\mathbf{k},\omega)\mathbf{E}_{\mathbf{k},\omega})=-4\pi(\mathbf{kP'}_{\mathbf{k},\omega})-4\pi i\rho_{\mathbf{k},\omega},\qquad(1)$$

$$\mathbf{kH}_{\mathbf{k},\omega}=0,\qquad(2)$$

$$[\mathbf{kE}_{\mathbf{k},\omega}]=\frac{\omega}{c}\mathbf{H}_{\mathbf{k},\omega},\qquad(3)$$

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$$[\mathbf{kH}_{\mathbf{k},\omega}]=-\frac{\omega}{c}\,\hat{\varepsilon}(\mathbf{k},\omega)\mathbf{E}_{\mathbf{k},\omega}-\frac{\omega}{c}\,4\pi\mathbf{P}'_{\mathbf{k},\omega}-\frac{4\pi i}{c}\,\mathbf{j}_{\mathbf{k},\omega}. \quad (4)$$

Here $\rho_{\mathbf{k},\omega}$ and $\mathbf{j}_{\mathbf{k},\omega}$ are the Fourier transforms of the charge density and the current density of the external charges, respectively.

Taking the cross product of Eq. (3) with $\mathbf{k}$ and taking Eqs. (2) and (4) into account, we obtain

$$[\mathbf{k}[\mathbf{kE}_{\mathbf{k},\omega}]]+\left(\frac{\omega}{c}\right)^2\hat{\varepsilon}(\mathbf{k},\omega)\mathbf{E}_{\mathbf{k},\omega}=-4\pi\left(\frac{\omega}{c}\right)^2\left(\mathbf{P}'_{\mathbf{k},\omega}+\frac{i}{\omega}\,\mathbf{j}_{\mathbf{k},\omega}\right). \quad (5)$$

Below we consider the case in which the medium matrix is isotropic: $\hat{\varepsilon}(\mathbf{k},\omega)=\hat{\varepsilon}(k,\omega)$ (the perturbation of the properties of the medium can, however, be anisotropic). The generalization to the case of an anisotropic medium does not introduce any fundamental difficulties.

2.1. Energy of transverse electromagnetic waves in an isotropic transparent medium.

As is well-known, in an isotropic medium possessing spatial dispersion, the dielectric constant is a tensor:

$$\hat{\varepsilon}(k,\omega)\mathbf{E}_{\mathbf{k},\omega}=\varepsilon_l(k,\omega)\,\frac{\mathbf{k}(\mathbf{kE}_{\mathbf{k},\omega})}{k^2}+\varepsilon_t(k,\omega)\frac{[\mathbf{k}(\mathbf{kE}_{\mathbf{k},\omega})]}{k^2}, \quad (6)$$

where $\varepsilon_l(k,\omega)$ and $\varepsilon_t(k,\omega)$ are, respectively, the longitudinal and transverse permittivities of the medium, which in the case of an isotropic medium do not depend on the direction $\mathbf{k}$. Expanding the field vector into its longitudinal and transverse components,

$$\mathbf{E}_{\mathbf{k},\omega}=\mathbf{E}^l_{\mathbf{k},\omega}+\mathbf{E}^t_{\mathbf{k},\omega}=\frac{\mathbf{k}(\mathbf{kE}_{\mathbf{k},\omega})}{k^2}+\frac{[\mathbf{k}[\mathbf{kE}_{\mathbf{k},\omega}]]}{k^2},$$

we obtain equations for $\mathbf{E}^l_{\mathbf{k},\omega}$ and $\mathbf{E}^t_{\mathbf{k},\omega}$ from Eq. (5) with the help of Eq. (6):

$$\varepsilon_l(k,\omega)E^l_{\mathbf{k},\omega}=-4\pi\left(\mathbf{P}'_{\mathbf{k},\omega}+\frac{i}{\omega}\,\mathbf{j}_{\mathbf{k},\omega}\right)_l, \quad (7a)$$

$$\left(\frac{\omega^2}{c^2}\,\varepsilon_t(k,\omega)-k^2\right)\mathbf{E}^t_{\mathbf{k},\omega}=-4\pi\,\frac{\omega^2}{c^2}\left(\mathbf{P}'_{\mathbf{k},\omega}+\frac{i}{\omega}\,\mathbf{j}_{\mathbf{k},\omega}\right)_t, \quad (7b)$$

where the indices l and t denote, respectively, the longitudinal and transverse components of the vector inside parentheses on the right-hand side.

Taking the polarization $\mathbf{P}'$ and currents $\mathbf{j}$ as given, we write the solution of equations (7) in the form of a sum of the particular solution and the general solution of the homogeneous equation:

$$E^l_{\mathbf{k},\omega}=-\frac{4\pi}{\varepsilon_l(k,\omega)}\left(\mathbf{P}'_{\mathbf{k},\omega}+\frac{i}{\omega}\,\mathbf{j}_{\mathbf{k},\omega}\right)_l+\mathscr{E}_l(\mathbf{k},\omega)\delta[\varepsilon_l(k,\omega)] \quad (8a)$$

$$\mathbf{E}^t_{\mathbf{k},\omega}=-4\pi\frac{\omega^2}{c^2}\,\frac{[\mathbf{P}'_{\mathbf{k},\omega}+(i/\omega)\mathbf{j}_{\mathbf{k},\omega}]_t}{\omega^2\varepsilon_t(k,\omega)/c^2-k^2}+\vec{\mathscr{E}}_t(\mathbf{k},\omega)\delta\left(\frac{\omega^2}{c^2}\,\varepsilon_t(k,\omega)-k^2\right). \quad (8b)$$

In Eqs. (8) the amplitudes of the harmonics $\vec{\mathscr{E}}_l(\mathbf{k},\omega)$ and $\vec{\mathscr{E}}_t(\mathbf{k},\omega)$, which describe the electromagnetic radiation from the sources external to the system under consideration, are determined by the initial and boundary conditions. The corresponding terms differ from zero only if ω and k satisfy the dispersion relations for the longitudinal (8a) or transverse (8b) waves (only in this case do the arguments of the δ-function vanish).

The assumption that the sources on the right-hand side of Eq. (8) are given assumes the lack of any back effect of the radiation field on the sources. The corresponding approximation for the radiation of moving charges is tantamount to neglecting the radiative reaction force; in nonlinear optics it corresponds to the so-called prescribed field approximation.

In what follows we assume that the medium does not absorb, thus:

$$\mathrm{Im}\,\varepsilon(k,\omega)=0.$$

In order to separate out the wave field in Eqs. (8), we perform the inverse Fourier transformation with respect to frequency:

$$\mathbf{E}_{\mathbf{k}}(t)=\int_{-\infty}^{\infty}\mathbf{E}_{\mathbf{k},\omega}e^{-i\omega t}d\omega.$$

We obtain

$$E^l_{\mathbf{k}}(t)=-4\pi\int_{-\infty}^{\infty}d\omega e^{-i\omega t}\,\frac{[\mathbf{P}'_{\mathbf{k},\omega}+(i/\omega)\mathbf{j}_{\mathbf{k},\omega}]_l}{\varepsilon_l(k,\omega)}+\left[\frac{\mathscr{E}_l(\mathbf{k},\omega_l)e^{-i\omega_l t}}{|\partial\varepsilon_l(k,\omega)/\partial\omega|_{\omega=\omega_l}}+(\omega_l\to-\omega_l)\right], \quad (9a)$$

$$\mathbf{E}^+_{\mathbf{k}}(t)=-4\pi\int_{\infty}^{\infty}d\omega e^{-i\omega t}\,\frac{\omega^2}{c^2}\,\frac{[\mathbf{P}'_{\mathbf{k},\omega}+(i/\omega)\mathbf{j}_{\mathbf{k},\omega}]_t}{\omega^2\varepsilon_t(k,\omega)/c^2-k^2}+\left[\frac{\mathscr{E}_t(\mathbf{k},\omega_t)e^{-i\omega_t t}}{|\partial(\omega^2\varepsilon_t(k,\omega)/c^2)/\partial\omega|_{\omega=\omega_t}}+(\omega_t\to-\omega_t)\right]. \quad (9b)$$

In Eqs. (9) the frequencies of the longitudinal ω_l and transverse ω_t waves are roots of the corresponding dispersion equations and depend on k. The expression $(\omega_{l,t}\to-\omega_{l,t})$ means that a term differs from the previous one only by a change of sign before $\omega_{l,t}$. Such structure in the expression in Eqs. (9) describing the field incident from outside is due to the evenness of the real part of $\varepsilon_t(k,\omega)$ in ω, and also to the assumption made in Eq. (9a) of the absence of gyration [the presence of only even powers in the expansion of $\omega_l(k)$]. If the dispersion equation has more than one pair of roots $\pm\omega_i(k)$, then instead of one term describing the external field in Eqs. (9), there now enters a sum of terms corresponding to all these frequencies.

In what follows we will restrict the analysis to transverse waves described by Eq. (9b). Using Eq. (9a), it is possible to develop an analogous technique for determining the characteristics of longitudinal waves.

In the integral in Eq. (9b) we replace the variable ω by $q = \sqrt{(\omega/c)\varepsilon_t(k,\omega)}$. The limits of integration do not change since $\varepsilon\to 1$ as $\omega\to\pm\infty$, i.e., $\omega\to qc$, and in the transparency regions $\partial q/\partial\omega>0$ and the absorption regions they are assumed to be wide enough to ensure that $q(\omega)$ is single-valued. The assumption of transparency of the medium does not lead to substantial inaccuracies if the frequencies $\omega_t(k)$ of the waves radiated by the sources do not fall in the absorption regions.

The new integrand has poles corresponding to electromagnetic waves only at $q=\pm k$ since $\partial q/\partial\omega=0$ only near the resonance frequencies—outside the transparency regions, but as $\varepsilon_t\to 0$, obviously $q^2\propto\varepsilon_t\to 0$ and $q^2/\varepsilon_t\to$const. The detour around the poles in the calculation can be chosen, for example, by introducing a small decay:

$$\varepsilon_t\to\begin{cases}\varepsilon_t+i0, & \omega>0,\\ \varepsilon_t-i0, & \omega<0\end{cases}$$

the signs are different for $\omega>0$ and $\omega<0$ since the imaginary part of the dielectric constant is odd in the frequency). This substitution corresponds to shifting the poles $q=\pm k$ into the lower half-plane. Closing the integration contour by an arc of infinite radius in the lower half-plane, we obtain after substituting the results expressed by Eq. (9b)

$$\mathbf{E}_{\mathbf{k}}^t(t)=\mathbf{a}_{\mathbf{k}}^+e^{-i\omega(k)t}+\mathbf{a}_{\mathbf{k}}^-e^{i\omega(k)t}, \tag{10}$$

where

$$k=\frac{\omega}{c}\sqrt{\varepsilon_t(k,\omega)}, \tag{11}$$

$$\mathbf{a}_k^\pm=4\pi^2 i\frac{k\vec{\mathscr{P}}^t_{\mathbf{k},\pm\omega(k)}}{(\partial k/\partial\omega)_{\pm\omega(k)}\varepsilon_t[\pm\omega(k),k]}, \tag{12}$$

$$\vec{\mathscr{P}}_{\mathbf{k},\omega}=\mathbf{P}'_{\mathbf{k},\omega}+\frac{i}{\omega}\mathbf{j}_{\mathbf{k},\omega}+\frac{\varepsilon_t(k,\omega)}{8i\pi^2k^2}\vec{\mathscr{E}}_t(\mathbf{k},\omega). \tag{13}$$

In the calculations it was taken into account that the lack of absorption (ε_t even in ω and k) means that $\omega(k)=-\omega(-k)$, since the frequency $\omega(k)$ satisfies Eq. (11). The quantity $\vec{\mathscr{P}}_{\mathbf{k},\omega}$ in expression (12) is a generalized polarization. It is specifically this quantity, as can be seen from formulas (10) and (12), that determines the total wave field radiated and scattered by an arbitrary source. In expression (13) the variable and nonlinear components of the polarization of the medium, the currents, and the electromagnetic waves incident on the system give additive contributions. Consequently, expressions (10)–(13) subsume the solution of any problem on the radiation or scattering of transverse electromagnetic waves by prescribed sources in a transparent medium.

$\mathbf{E}_{\mathbf{k}}^t(t)\exp(-i\mathbf{kr})$ is the contribution to the plane wave expansion of $\mathbf{E}$ corresponding to the wave vector $\mathbf{k}$ [and frequency $\omega(k)$]. In order to find $\mathbf{E}^t(\mathbf{r},t)$, it is necessary to perform the inverse Fourier transformation in $\mathbf{k}$. However, it turns out to be possible to obtain a general formula for the spectral and angular distributions of the energy of the radiated (scattered) waves directly from Eqs. (10)–(13).

The classical results pertaining to the energy of electromagnetic waves in dispersive media apply quasimonochromatic fields (see, for example, Refs. 3 and 4). In the general case under consideration here, the field is not monochromatic. However, for the expression obtained below for the energy of the waves concentrated in a small region of the spectrum $d\mathbf{k}$, the results of taking account of dispersion turn out to be the same as usual (see in this regard also Ref. 3). In the Appendix it is shown that the energy contained in the interval $d\mathbf{k}$ for the field described by Eq. (10) has the form

$$W_{\mathbf{k}}d\mathbf{k}=2\pi^2\left\{\frac{\partial}{\partial\omega}[\omega\varepsilon_t(k,\omega)]_{\omega=\omega(k)}+\varepsilon_t(k,\omega)\right\}|\mathbf{a}_{\mathbf{k}}^+|^2d\mathbf{k}. \tag{14}$$

Taking into account that

$$\left.\frac{\partial k}{\partial\omega}\right|_{\omega(k)}=\frac{k}{\omega}+\frac{k}{2\varepsilon}\frac{\partial\varepsilon}{\partial\omega}=\frac{k}{2\varepsilon\omega}\left\{\frac{\partial}{\partial\omega}[\omega\varepsilon(\omega)]+\varepsilon(\omega)\right\}$$

substituting (13) into (12) and (12) into (10), we have from Eq. (14)

$$W_{\mathbf{k}}d\mathbf{k}=\frac{(2\pi)^6\omega^2(k)}{c\sqrt{\varepsilon_t(k,\omega)}(\partial k/\partial\omega)_{\omega(k)}}|\vec{\mathscr{P}}^t_{\mathbf{k},\omega(k)}|^2d\mathbf{k}. \tag{15}$$

The corresponding expression in a nondispersive medium has the form

$$W_{\mathbf{k}}d\mathbf{k}=(2\pi)^6\frac{\omega^2(k)}{\varepsilon}|\vec{\mathscr{P}}^t_{\mathbf{k},\omega(k)}|^2d^3\mathbf{k},\quad \omega(k)=kc/\sqrt{\varepsilon}. \tag{16}$$

It may be more convenient to consider the field distribution not in $\mathbf{k}$, but in ω and solid angle. The corresponding formula follows from Eq. (15):

$$W\left(\omega\frac{\mathbf{k}}{k}\right)d\omega do=(2\pi)^6\frac{\omega^4}{c^3}\sqrt{\varepsilon_t(k,\omega)}|\vec{\mathscr{P}}^t_{\mathbf{k}(\omega),\omega}|^2d\omega do, \tag{17}$$

where it is necessary to consider k as a function of ω (11), and do is the element of solid angle. Comparing Eq. (17) with the corresponding formula obtained in the absence of frequency dispersion leads to a result analogous to that obtained in Ref. (3): it is possible to take account of dispersion already in the final expression for $W(\omega,\mathbf{k}/k)$ by making the substitution $\varepsilon\to\varepsilon(k,\omega)$.

To determine the intensity of the waves with polarization characterized by the unit vector $\mathbf{e}$, it is sufficient to substitute $\mathbf{e}\cdot\vec{\mathscr{P}}^t_{\mathbf{k},\omega}$ for $\vec{\mathscr{P}}^t_{\mathbf{k},\omega}$ on the right-hand side of (15) or (17).

Equations (15) and (17) reduce any problem on radiation or scattering by prescribed sources of transverse electromagnetic waves in an isotropic transparent medium to the calculation of the Fourier transforms of the generalized polarization (13).

It is worthy of note that the proposed method allows one without effort to analyze the interference of waves emitted by several sources of various types, simultaneously. Further, in the semiclassical treatment one can include in the generalized polarization $\vec{\mathscr{P}}$ the quantum-mechanically defined polarization. Note also that it is not hard to allow for terms in $\vec{\mathscr{P}}$ corresponding to quadrupole, magnetic-dipole, and other types of radiation.

In a certain sense, expressions (15) and (17) correspond to the formal solution of Maxwell's equations for wave fields if the action of radiation fields is taken into account in $\vec{\mathscr{P}}$ (13). The assumption of the absence of such action ("prescribed sources") allows one to use these formulas for specific calculations. In this sense the assumption amounts to solution of the problem in first-order perturbation theory. The specific physical meaning of the parameter in which the perturbation theory is constructed depends on the situation under consideration. The formal criterion of validity of the approximation of prescribed sources for calculating the energy of the waves with wave vector $\mathbf{k}$ is $|\vec{\mathscr{P}}^{(1)}_{\mathbf{k},\omega(k)}| \ll |\mathscr{P}_{\mathbf{k},\omega(k)}|$ where $\vec{\mathscr{P}}^{(1)}_{\mathbf{k},\omega(k)}$ is an additional term in the generalized polarization due to the radiation field.

Note, finally, that the formulas for the spectral and angular field distributions are usually derived for the field in the far zone. No such assumption was made above. Nevertheless, the expressions obtained with the help of formulas (15) and (17) coincide with the well-known expressions obtained in this way (see Sec. 2.2 below), because going out to the far zone means, in a certain sense, highlighting just one of a set of radiated harmonics $\mathbf{E}_{\mathbf{k},\omega}\exp[i(\mathbf{kr}-\omega t)]$ (or, more accurately, a group with wave vector from $\mathbf{k}$ to $\mathbf{k}+d\mathbf{k}$)). Formulas (15) and (17) describe the energy of just such a set of harmonics. They are valid in the near zone as well as the far zone, which is the reason for the coincidence with the results of Sec. 2.2.

2.2. Special cases. Comparison with known results.

Radiation of moving charges. Let the generalized polarization be due only to the field of a moving point charge of magnitude q:

$$\mathbf{j}(\mathbf{r},t)=q\mathbf{v}(t)\delta[\mathbf{r}-\mathbf{r}_0(t)],$$

where $\mathbf{r}_0(t)$ and $\mathbf{v}(t)=dr_0/dt$ are respectively the trajectory and velocity of the charge. Substituting the corresponding Fourier transform in (13) and employing Eq. (17), we obtain the generalization of the well-known formula for the intensity of the radiation field of a point charge to the case of dispersive media:

$$W\left(\omega,\mathbf{n}\equiv\frac{\mathbf{k}}{k}\right)=\frac{q^2\omega^2\sqrt{\varepsilon}}{4\pi^2c^2}\left|\int_{-\infty}^{\infty}d t\mathbf{v}(t)\times\exp\left[i\omega\left(t-\frac{\mathbf{n}\mathbf{r}_0(t)\sqrt{\varepsilon}}{c}\right)\right]\right|^2, \qquad (18)$$

where $\varepsilon\equiv\varepsilon_t(k,\omega)$. In particular, for $\mathbf{r}_0=\mathbf{v}t$ where $v=\text{const}>c/\sqrt{\varepsilon}$, Eq. (18) reduces to the expression for Čerenkov radiation. In addition, Eq. (18) makes it possible, in principle, to calculate synchrotron radiation and undulator radiation in a dispersive medium.

Radiation in media with variable dielectric constant. For $\vec{\mathscr{P}}(\mathbf{r},t)=\mathbf{P}'(\mathbf{r},t)=\varepsilon_1(\mathbf{r},t)\mathbf{E}_0(\mathbf{r},t)$, not taking spatial dispersion into account, expression (15) transforms into the basic formula of the perturbation theory proposed in Ref. 5 for analyzing radiation arising in a quasistationary field of fixed sources $\mathbf{E}_0(\mathbf{r},t)$ if the variation of the properties of the medium is described by the functional dependence $\varepsilon_1(\mathbf{r},t)$. This method has made it possible to solve a significant number of radiation problems in nonuniform, nonstationary, and moving media.[6]

Nonlinear optics. If $\vec{\mathscr{P}}=\mathbf{P}'$ describes the nonlinear polarization of the medium produced by an electromagnetic wave, then (15) and (17) give the energy of the harmonics in the nonlinear medium in the prescribed field approximation. In this case one can easily convince oneself that the results obtained for plane monochromatic waves coincide with the well-known results of Refs. 7 and 8. It is significant that (15) and (17) are also valid in cases in which the approximation of slowly varying amplitudes, on which the widely accepted theory of nonlinear optical phenomena is based, is not applicable. Therefore, the application of the proposed method to the analysis of the propagation of ultrashort pulses, which cannot be correctly described with the help of simplified equations, is of great interest.

Interference of radiative processes. Expressions (15) and (17) are in general squares of sums of terms proportional to the fields of waves generated by currents, nonstationarity of the medium, etc., and waves incident on the system. The use of these formulas allows one without difficulty to analyze the possible interference of radiative processes corresponding to the cross-products that arise when one takes the square of $\vec{\mathscr{P}}$. Note that the interference of transition radiation and transition scattering with other radiative processes is treated in detail in Ref. 3.

3. INFLUENCE OF INTERFERENCE ON THE SCATTERING OF ELECTROMAGNETIC WAVES BY A CLASSICAL OSCILLATOR

Let us consider the classical problem of scattering of electromagnetic waves by a bound point particle. The usual statement of the problem reduces to a calculation of the reradiated field arising from the oscillations of the charge in the field of the incident wave. Here no account is taken of the possible interference between the scattered (reradiated) and incident wave.

If an electromagnetic wave is incident upon a system of moving charges, then the interference of radiation fields is described in formula (15) by the cross term containing the current **j** and the wave "amplitude" $\vec{\mathscr{E}}_t$:

$$W_{\mathbf{k}}^{\rm int} d\mathbf{k} = -\frac{8\pi^4}{k(\partial k/\partial\omega)_{\omega(k)}} (\mathbf{j}_{\mathbf{k},\omega(k)} \vec{\mathscr{E}}_t^* + \mathbf{j}^*_{\mathbf{k},\omega(k)} \vec{\mathscr{E}}_t) d\mathbf{k}. \tag{19}$$

It is not hard to show that $\int W_{\mathbf{k}}^{\rm int} d\mathbf{k}$ is equal to the total work of the electric field, given by the superposition of harmonics

$$\int d\mathbf{k} \left[\frac{\vec{\mathscr{E}}_t(k,\omega_t)\exp(-i\omega_t t)}{|\partial\, [\omega^2 \varepsilon_t(k,\omega)/c^2]/\partial\omega|_{\omega=\omega_t}} + (\omega_t \to -\omega_t) \right]$$

[cf. Eq. (9b)], on the current **j** taken with the opposite sign. Thus, the interference term (19) describes the attenuation (or amplification if the expression in brackets in Eq. (19) is negative) of the radiation scattered or absorbed (pumped) by the system of currents.

In what follows we limit ourselves to the case $\varepsilon = 1$ and consider only linearly polarized waves.

3.1. Scattering of a plane electromagnetic wave by an oscillator with damping.

In the given case, the field of an incident wave of frequency Ω with wave vector $\boldsymbol{\kappa}$ ($\kappa = \Omega/c$) is described by

$$\mathbf{E}_{\mathbf{k}}(t) = \frac{\mathbf{A}}{2} [\delta(\mathbf{k}-\boldsymbol{\kappa})\exp^{-i\omega t} + \delta(\mathbf{k}+\boldsymbol{\kappa})\exp^{i\omega t}], \tag{20}$$

where **A** is the real wave amplitude (the wave is assumed to be linearly polarized). Here [cf. Eq. (8b)]

$$\vec{\mathscr{E}}_t(\mathbf{k}) = \frac{k}{c} \mathbf{A}\delta(\mathbf{k}-\boldsymbol{\kappa}). \tag{21}$$

The current **j** is created by the steady-state motion of an oscillator with charge q, mass m, natural frequency ω_0, and damping coefficient γ in the field of the incident wave (20):

$$\mathbf{r}_0(t) = \mathbf{a}\cos(\Omega t - \psi), \tag{22}$$

where

$$\mathbf{a} = \frac{q\mathbf{A}/m}{\sqrt{(\omega_0^2-\Omega^2)^2 + 4\Omega^2\gamma^2}}, \quad \tan\psi = \frac{2\Omega\gamma}{\omega_0^2 - \Omega^2}.$$

In Eq. (22) the wave field is assumed to be weak enough that the oscillator can be taken to be nonrelativistic: $a \ll \lambda \ll 2\pi/\kappa$; **a** is a real-valued vector.

The Fourier transform of the current corresponding to (22) has the form

$$\mathbf{j}_{\mathbf{k},\omega} = -\frac{q\Omega\mathbf{a}}{(2\pi)^4} \int_{-\infty}^{\infty} dt \sin(\omega t - \psi)\exp\{i[\omega t - \mathbf{k}\mathbf{a}\cos(\Omega t - \psi)]\}. \tag{23}$$

Substituting (23) in Eqs. (21) and (19) and noting that $\boldsymbol{\kappa} \perp \mathbf{a}$, we obtain $W_{\mathbf{k}} \propto \delta(\mathbf{k}-\boldsymbol{\kappa})\delta(\omega-\Omega)$. Integrating over **k**, we arrive at an expression describing the interference contribution to the energy radiated per unit time:

$$\frac{W^{\rm int}}{T} = \int \frac{W_{\mathbf{k}}^{\rm int}}{T} d\mathbf{k} = -\frac{1}{2} q\Omega(\mathbf{a},\mathbf{A})\sin\psi \equiv -\frac{q^2}{8\gamma m} |A|^2 \sin^2\psi, \tag{24}$$

where, in accordance with (22),

$$\sin\psi = 2\Omega\gamma / \sqrt{(\omega_0^2-\Omega^2)^2 + 4\Omega^2\gamma^2} > 0.$$

The interference contribution (24) to the radiated energy is negative. In the absence of damping, the phase shift between the incident wave and the motion of the oscillator $\psi = 0$ and the interference term disappears (the phase shift between the incident and reradiated waves is equal to $\psi + \pi/2 = \pi/2$).

In the plane-wave case under consideration, the energy $W_{\mathbf{k}} \propto \delta(\mathbf{k}-\boldsymbol{\kappa})$ and interference influences only the forward scattering. This is explained by the fact that only in that direction do the spatial periods of the incident and reradiated waves coincide.

According to the above, $W^{\rm int}/T$ in (24) is precisely equal to the work of the incident wave on the oscillator per unit time. Since the energy of motion of the oscillator does not vary in the case of a monochromatic wave, the interference term (24) determines the energy losses of the original wave to scattering and other processes.

In the classical theory of dispersion, the medium in which the wave propagates is considered to be a set of oscillators (22) distributed with some volume density N. If the wave propagates along the z axis and is characterized by an energy flux density $I(z) = W^{(0)}(z)/TS$, where $S \to \infty$ is the area of the wavefront, then (24) can be written in the form

$$W^{\rm int}(z) = -\alpha I(z), \quad \alpha = \frac{\pi q^2}{\gamma mc}\sin^2\psi. \tag{25}$$

Note that the interference contributions from the different oscillators are additive, since the phase of the reradiated wave is in synchrony (shifted by $\psi + \pi/2$) with the phase of the incident radiation. Therefore, for the dependence of the intensity of the main wave on the distance it has advanced in the medium, we obtain, with the help of Eq. (25), the Bouguer–Lambert–Beer law:

$$W^{(0)}(z) = W^{(0)}(0)e^{-\alpha N z}.$$

Here the attenuation coefficient αN, as can easily be shown with the help of relations (25) and (22), coincides with the value calculated in the classical theory of dispersion (see, e.g., Ref. 9).

Thus, any attenuation of radiation in a medium is due to interference between the incident wave and waves that have been reradiated by the oscillators of the medium. This conclusion in a certain sense complements the Ewald–Oseen absorption theorem (see Ref. 10, §2.4), which explains in an analogous way the variation of the velocity of an electromagnetic wave in a medium (the phenomenon of refraction).

3.2. Scattering of a beam with finite aperture.

Let us now consider the situation in which a group of N_0 independent oscillators is located along the axis of an axially symmetric beam of electromagnetic waves. We assume that the oscillators are distributed so compactly (as a "cluster") that the characteristic dimension of the region that they occupy is $d \ll \lambda$, R, where $\lambda \equiv 2\pi/\kappa$ is the wavelength of the radiation and R is the beam aperture. Here (22), as before, describes the motion of each oscillator, and we take $\mathbf{A}$ to be the amplitude of the field oscillations in the main wave at the location of the oscillator. Therefore, the reradiated wave remains invariant in shape—it is determined by the current (23) multiplied by N_0.

The electric field of the wave of an axially symmetric beam propagating along the z axis which has the intensity distribution

$$\mathbf{E}_{z=0}(\rho,t)=\mathbf{F}(\rho;R)e^{i\Omega t}$$

is

$$\mathbf{E}_{\mathbf{k}}(t)=\mathbf{F}(\mathbf{k}_\rho;R)\delta(k_z-\sqrt{\kappa^2-k_\rho^2})e^{i\kappa ct}, \tag{26}$$

where

$$\mathbf{F}(\mathbf{k}_\rho;R)=(2\pi)^{-2}\int dxdy\exp(-ik_xx-ik_yy)\mathbf{F}(\rho;R),$$

$$\kappa\equiv\Omega/c,\quad k_\rho^2\equiv k_x^2+k_y^2.$$

In the limit of a plane wave ($R\to\infty$) we have

$$\mathbf{F}(\mathbf{k}_\rho;R)\to\frac{\mathbf{A}}{2}\delta(k_x)\delta(k_y).$$

Substituting (26) and (23) in Eq. (19) and multiplying by N_0, we obtain, by analogy with Eq. (24),

$$\frac{W^{\text{int}}(\vartheta)}{T}do=-\frac{N_0q\Omega^3}{2\pi c^2}[\mathbf{a},\mathbf{F}(\mathbf{k}_\rho;R)]\sin\psi\cos\vartheta\, do, \tag{27}$$

where $\vartheta=\widehat{(\mathbf{n},\mathbf{z})}$, and $\mathbf{a}$ is defined by Eq. (22). The derivation of Eq. (27) assumes that the oscillators are nonrelativistic: $ka\approx 0$ and the scattering takes place without change of frequency. The contributions of all oscillators are additive.

It follows from Eq. (27) that the width of the angular distribution $W^{\text{int}}(\vartheta)\propto F(\mathbf{k}_\rho;R)$ is greater than that of the angular distribution for the initial beam $W^{(0)}(\vartheta)\propto F^2(k_\rho;R)$. This means that in addition to a decrease in the amplitude of the initial wave, interference leads to a change in the angular distribution of the total radiation. Let us illustrate this effect for a Gaussian beam.

For a Gaussian beam with the amplitude distribution

$$\mathbf{E}_{z=0}(\rho,t)=\left(\frac{\mathbf{A}}{2}\right)e^{-\rho^2/R^2}e^{i\Omega t},$$

in the entrance plane, we have

$$\mathbf{F}(\mathbf{k}_\rho;R)=\frac{\mathbf{A}}{8\pi}R^2\exp\left(-\frac{k_\rho^2R^2}{4}\right). \tag{28}$$

Adding the expressions for the intensity of the initial beam $W^{(0)}$, the intensity of Thomson scattering (which in the given case is coherent ($d\ll\lambda$)), and $W^{\text{int}}(\vartheta)$, obtained by substituting (28) into Eq. (27), we obtain the total intensity $W^{\text{tot}}(\vartheta)$ of the electromagnetic radiation in the direction $\mathbf{n}$. Taking the angles $\vartheta\approx 0$ and $\vartheta'=\widehat{(\mathbf{A},\mathbf{n})}\approx\pi/2$, and retaining the dependence on ϑ only in the arguments of the exponents, we have

$$\frac{W^{\text{tot}}(\vartheta)}{T}do=\frac{3cR^2A^2\sin^2\psi}{64\pi^2\nu}\left[\frac{2\pi\nu k^2R^2}{3\sin^2\psi}\exp\left(-\frac{k^2R^2\vartheta^2}{2}\right)+\frac{N_03\pi}{2k^2R^2\nu}-N_0\exp\left(-\frac{k^2R^2\vartheta^2}{4}\right)\right]do, \tag{29}$$

where $\nu\equiv\gamma(3mc^3/2q^2\Omega^2)\equiv\gamma/\gamma^{\text{rad}}\gg 1$.

Taking account of interference leads to nonmonotonicity of the dependence of (29) on angle. The minimum of this dependence is determined by the condition

$$\exp\left(-\frac{k^2R^2\vartheta_{\min}^2}{2}\right)=\left(\frac{3N\sin^2\psi}{4\pi\nu k^2R^2}\right)^2\equiv\left(\frac{W^{\text{int}}(\vartheta=0)}{2W^{(0)}(\vartheta=0)}\right)^2. \tag{30}$$

For weak scattering, $W^{\text{int}}/W^{(0)}\ll 1$, the exponential in Eq. (30) determines the relative intensity level at which the minimum of $W^{\text{tot}}(\vartheta)$ is located. For $\vartheta=\vartheta_{\min}$, the interference term in $W^{\text{tot}}(\vartheta)$ is twice the initial intensity $W^{(0)}(\vartheta_{\min})$, but the main contribution comes from the Thomson scattering W^T, which in our model is coherent, and for the small angles ϑ considered here does not depend on angle. From Eq. (30) we have

$$W^{\text{tot}}(\vartheta_{\min})=W^T\left(\vartheta'=\frac{\pi}{2}\right)\left(1-\frac{\sin^2\psi}{4\pi^2}\right). \tag{31}$$

From Eq. (31) it can be seen that in the given case, the depth of the minimum does not exceed $W^T/4\pi^2$. A plot of the behavior of (29) is shown in Fig. 1.

4. CONCLUSION

The proposed technique for calculating the spectral and angular energy distributions of electromagnetic waves allows one to describe all possible processes of radiation and scattering in a transparent dispersive medium on the basis of the general formulas (15) and (17).

The use in these formulas of the Fourier transforms of quantities characterizing sources that cannot be described in ordinary space extends the class of objects that can be considered. Thus, for example, the beam of electromagnetic waves considered in Sec. 3 can be described analytically in $(\mathbf{r},t)$-space only in the paraxial approximation (in general there is no simple analytic expression for the original function of which expression (26) is the Fourier transform). Using Fourier transforms allows for arbitrary temporal and spatial dispersion of the medium. The use of the method in nonlinear optics makes it possible to go beyond the framework of the approximation of slowly varying amplitudes and weakly divergent beams.

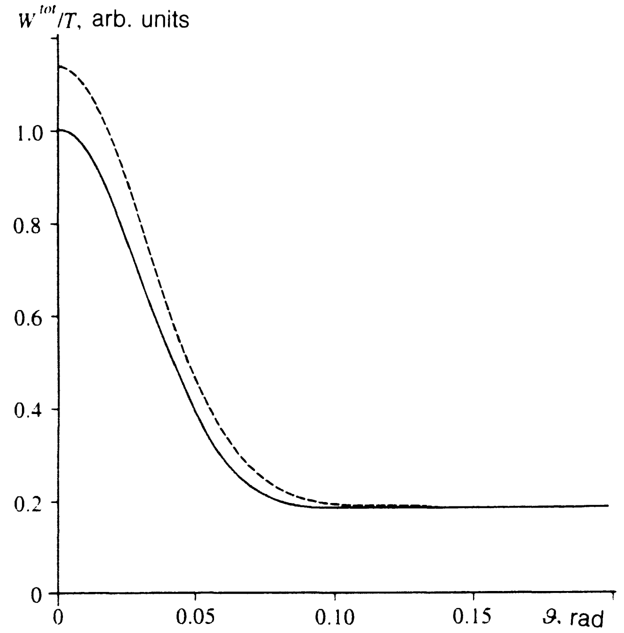

FIG. 1. Angular intensity distribution of a beam of electromagnetic waves scattered by a cluster of oscillators, with interference taken into account (solid line) and without (dashed line), $N=300$, $R=5\lambda$.

Note that within the framework of the proposed method it is easy to take account of the back influence of the radiation field on the sources in second- and higher-order perturbation theory.

The efficacy of this technique in analyzing interference processes attendant upon radiation and scattering has been demonstrated. With its help it has been shown that the attenuation of electromagnetic radiation in a medium can be considered to result from interference between secondary waves reradiated by the molecules of the medium and the incident radiation. The change in the angular distribution of a wave beam scattered by a classical oscillator due to interference between the incident and reradiated waves has been analyzed.

APPENDIX

To obtain an expression for the energy of the transverse waves $W_{\mathbf{k}}d\mathbf{k}$ imparted to the waves with wave vectors in the interval from $\mathbf{k}$ to $\mathbf{k}+d\mathbf{k}$, note first that

$$W_{\mathbf{k}}=W_{\mathbf{k}}^{e}+W_{\mathbf{k}}^{m}, \tag{A1}$$

where $W_{\mathbf{k}}^{e}$ is the electric field energy in a medium with dispersion, and the magnetic field energy $W_{\mathbf{k}}^{m}$ is equal to the electric field energy calculated without account of dispersion. This can be verified directly by calculation, but it is clear from the fact that $W_{\mathbf{k}}$ describes the energy of a quasi-plane wave in an isotropic medium, where div $\mathbf{E}^{t}=0$.

To derive a formula describing the energy of the waves, it is necessary to start with the expression for its rate of change:

$$\frac{\partial}{\partial t}\int W_{\mathbf{k}}^{e}d\mathbf{k}=\int dV\frac{1}{4\pi}\left(\mathbf{E}^{t}\frac{\partial\mathbf{D}^{t}}{\partial t}\right)$$
$$=\frac{1}{4\pi}\int dV\int d\mathbf{k}d\omega e^{i(\mathbf{kr}-\omega t)}\left(\mathbf{E}^{t}\frac{\partial\mathbf{D}^{t}}{\partial t}\right)_{\mathbf{k},\omega}. \tag{A2}$$

Invoking relation (11) and the relation $\mathbf{D}^{t}=\varepsilon_{t}(\mathbf{k},\omega)\mathbf{E}^{t}$, we obtain from Eq. (A2)

$$\mathbf{E}^{t}\frac{\partial\mathbf{D}^{t}}{\partial t}=\int d\mathbf{k}'\mathbf{a}^{+}(\mathbf{k}')\mathbf{a}^{-}(\mathbf{k}-\mathbf{k}')\delta(\omega-\omega_{\mathbf{k}'}+\omega_{\mathbf{k}-\mathbf{k}'})$$
$$\times[\omega_{\mathbf{k}-\mathbf{k}'}\varepsilon_{t}(|\mathbf{k}-\mathbf{k}'|,\omega_{\mathbf{k}-\mathbf{k}'})-\omega_{\mathbf{k}'}\varepsilon_{t}$$
$$\times(k',\omega_{\mathbf{k}'})]. \tag{A3}$$

If in Eq. (A2) we take the integration limits over V to be infinite, then substituting (A3) in Eq. (A2) gives zero in a nonabsorbing medium. This is to be expected since the energy of the superposition of strictly monochromatic waves is constant in time. If we take the region occupied by the field to be large but finite (which indirectly assumes a field that is not monochromatic), then after integrating over V there remain only harmonics $(\mathbf{E}^{t},\partial\mathbf{D}^{t}/\partial t)_{\mathbf{k},\omega}$ with $k\approx 0$. Here

$$\mathbf{a}^{-}(\mathbf{k}-\mathbf{k}')\approx\mathbf{a}^{-}(-\mathbf{k}'),$$

$$\omega_{\mathbf{k}-\mathbf{k}'}\varepsilon_{t}(|\mathbf{k}-\mathbf{k}'|\omega_{\mathbf{k}-\mathbf{k}'})\approx\omega_{-\mathbf{k}'}\varepsilon_{t}(k',\omega_{-\mathbf{k}'})$$
$$+\frac{\partial}{\partial\omega}[\omega\varepsilon_{t}(k,\omega)]|_{\omega=\omega(-k')}d\omega,$$

where $d\omega=\omega_{\mathbf{k}-\mathbf{k}'}-\omega_{-\mathbf{k}'}$. Assuming, as before, that the medium is nongyrotropic and transparent, and noting that $\mathbf{a}^{+}(\mathbf{k})=[\mathbf{a}^{-}(-\mathbf{k})]^{*}$, after substituting Eq. (A3) into Eq. (A2) we obtain

$$\frac{\partial}{\partial t}\int W_{\mathbf{k}}^{e}d\mathbf{k}=2\pi^{2}\frac{\partial}{\partial t}\left[e^{-itd\omega}\right.$$
$$\left.\times\int d\mathbf{k}\left\{\frac{\partial}{\partial\omega}[\omega\varepsilon_{t}(k,\omega)]\right\}_{\omega=\omega(k)}\left|\mathbf{a}^{+}(\mathbf{k})\right|^{2}\right].$$

Integrating over time and taking the limits $V\to\infty$, $d\omega\to 0$, we have

$$W_{\mathbf{k}}^{e}d\mathbf{k}=2\pi^{2}\left(\frac{\partial}{\partial\omega}[\omega\varepsilon_{t}(\mathbf{k},\omega)]\right)_{\omega=\omega(\mathbf{k})}|\mathbf{a}^{+}(\mathbf{k})|^{2}d\mathbf{k}. \tag{A4}$$

In the absence of dispersion, (A4) becomes

$$W_{\mathbf{k}}^{e}d\mathbf{k}=2\pi^{2}\varepsilon|\mathbf{a}^{+}(\mathbf{k})|^{2}d\mathbf{k}=W_{\mathbf{k}}^{m}d\mathbf{k}, \tag{A5}$$

which agrees, for example, with known results used in the theory of transition radiation in a nonstationary medium.[3] From Eqs. (A4) and (A5), taking account of Eq. (A1), we obtain the total energy of the radiation field [cf. Eq. (14)]:

$$W_{\mathbf{k}}d\mathbf{k}=2\pi^{2}\left\{\frac{\partial}{\partial\omega}[\omega\varepsilon_{t}(k,\omega)]_{\omega=\omega(k)}+\varepsilon_{t}(k,\omega)\right\}|\mathbf{a}_{\mathbf{k}}^{+}|^{2}d\mathbf{k}.$$

Note that taking frequency dispersion into account leads to the appearance of the usual factor in such cases[3,4] in the square brackets.

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