Wave beams in cubically nonlinear nondispersive media

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The process of self-action of beams of highly distorted wide-spectrum waves propagating in nondispersive media with cubic nonlinearity is examined. It is shown that in such media trapezoidal sawtooth-like waves form, whose self-action is accompanied by nonlinear energy dissipation at shock fronts. There is a marked beam narrowing in a self-focusing medium, but in the focal region the amplitude increases unsubstantially. The properties of the nonlinear field equations for the beams are investigated, and both exact and approximate solutions are found.

1. INTRODUCTION

The experimental observation of certain nonlinear phenomena in the spectra of nondispersive waves is beset with considerable difficulties due to the simultaneous operation of a large number of resonant interactions between the spectral components. As a result of the interactions, the spectrum gets enriched with a variety of harmonics and combination frequencies; these take away a considerable part of energy and reduce whatever effect we are interested in.[1] In particular, self-action effects occur in their "pure" form only in highly dispersive media, in which higher harmonics cannot interact resonantly with the fundamental frequency wave. Here, nonlinearity affects only the amplitude and phase characteristics of the signal without actually enriching its spectrum; this allows one to speak of the "self-action" of a quasi- harmonic wave.

Generally, however, the term self-action may be given to a nonlinear change in the behavior of any wave object provided the object is quasistable, that is, the nonlinearity involved does not lead to its destruction or to its effective interactions with other wave entities. In this sense it is possible to speak of the self- action of quasiharmonic signals, which are stable in highly dispersive media, as well as of the self-action of objects with marked nonlinear properties. Examples of these latter include solitons, shock waves, and, typical of acoustics, periodic sawtooth perturbations and solitary shock pulses, whose shape is stable and asymptotically universal for a wide class of initial signals.[1]

Among other fundamental wave phenomena is listed the self-action of spatially modulated signals (wave beams); a good case in point is the self-focusing effect.[2] In dispersive media this phenomenon is described by the Schrödinger type equation[3,4]

$$i\omega \frac{\partial A}{\partial z}=\frac{c}{2}\,\Delta_\perp A+\gamma\omega^2|A|^2A. \tag{1}$$

Here A is the amplitude of the wave of frequency ω traveling with a speed c along the z axis coinciding with the wave beam axis. The Laplacian $\Delta_\perp$ is taken with respect to the transverse coordinates x, y; γ is the cubic nonlinearity coefficient. It is known that Eq. (1) describes the instability of the initial plane wavefront in the cases when its intensity exceeds a critical value.[4] Then the amplitude of any spatial perturbation harmonic (perturbation being an increment to the plane nonlinear wave amplitude) grows exponentially as a function of distance z. Thus, in a self-focusing medium ($\gamma>0$) the plane wave is unstable; it breaks down into separate focusing beams each carrying a power of the order of the critical one.[5]

The propagation of waves in nondispersive media gives rise to a strong distortion of temporal profiles and to the formation of sawtooth waves of complex spectral composition.[1,5] The approach based on Eq. (1) is here inapplicable in principle and one should start with a more general equation,[6]

$$\frac{\partial}{\partial\tau}\left[\frac{\partial u}{\partial z}+\gamma u^2\,\frac{\partial u}{\partial\tau}-\delta\,\frac{\partial^2 u}{\partial\tau^2}\right]=\frac{c}{2}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)u=\frac{c}{2}\,\Delta_\perp u. \tag{2}$$

Equation (2) describes a wave field u. Unlike Eq. (1) for the amplitude A, it contains an extra variable $\tau=t-z/c$, the time in the traveling coordinate system. For a correct description of the shock fronts that form in the medium, a high-frequency dispersion with a coefficient δ is included. Note also that Eq. (2) is similar to the equations governing the nonlinear acoustics of bounded beams,[7] but differs from them in the type of nonlinearity involved.

In acoustical problems Eq. (2) may correspond to a beam of shear waves in a solid[8]; the variable u then has a meaning of the vibrational velocity of material particles. In Ref. 9 it is shown that the occurrence of self-focusing in a cubically nonlinear medium should lead to a new type of surface waves. Finally, the field approach (2) is becoming important in optical problems involving the self-action of femtosecond laser pulses.[10]

A formal change from (2) to (1) is possible if in Eq. (2) one sets $u=A(x,y,z)\exp(i\omega\tau)+$ compl. conj., $\delta\to0$. However, this approach, which is common in nonlinear optics and relies on an *a priori* assumption about the spectral composition of the wave, often fails when applied to nondispersive media.

2. PLANE WAVE IN A MEDIUM WITH CUBIC NONLINEARITY

For a plane wave, Eq. (2) assumes the form of the modified Burgers equation,

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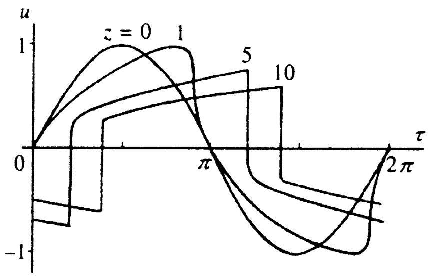

FIG. 1.

$$\frac{\partial u_1}{\partial z_1}+u_1^2\frac{\partial u_1}{\partial \tau_1}-\Gamma\frac{\partial^2 u_1}{\partial \tau_1^2}=0. \quad (3)$$

Here we have used the nondimensional notation

$$u_1=\frac{u}{u_0},\quad \tau_1=\omega\tau,\quad z_1=\gamma\omega u_0^2 z=\frac{z}{z_s},\quad \Gamma=\frac{\delta\omega}{\gamma u_0^2}=\frac{z_s}{z_a}. \quad (4)$$

The constant u_0 (ω) has the meaning of the characteristic amplitude (frequency) of the wave field; z_1 is distance in units of the shock formation length z_s; Γ is the ratio of the nonlinearity (z_s) and absorption (z_a) scales. In what follows the subscript 1 on the variables z_1, τ_1, u_1 will be dropped.

Of particular interest is the case of small Γ, when dissipation effects are unimportant everywhere except for the immediate vicinity of the shock fronts, steep portions of the profile, of duration $\Delta\tau\sim\Gamma$, that form as the wave propagates.[1] For $\Gamma=0$, the solution to Eq. (3) is written in terms of the implicit function

$$u(z,\tau)=\varphi(\tau-u^2 z), \quad (5)$$

where $\varphi(\tau)=u(0,\tau)$ is the initial wave profile.

Suppose that a harmonic signal $\tau=\sin\tau$ is specified at the input to the nonlinear medium. The process by which this profile becomes distorted in the medium is shown in cted in Fig. 1. In contrast to quadratically nonlinear media, here all the parts of the profile shift in one and the same (positive) direction parallel to the τ axis. Thus, a nonlinear addition to the wave velocity arises, capable of producing self-focusing of the beams.

In the evolution process, the wave profile (5) ceases to be unique. Taking into account the finite value of Γ makes it necessary to construct discontinuities in the solution (5). Since Eq. (3) implies the integral $\int_0^{2\pi}u\,d\tau=\text{const}$, it follows that the discontinuity can be achieved, as in the solution of the conventional Burgers equation, by using the "equal area" rule.[1]

However, starting at some point constructing the a certain the profile is no longer such a trivial procedure. This problem has been treated in detail in Ref. 11; here we will only quote the main results necessary for the subsequent discussion.

The solution everywhere satisfies the condition $u(z,\tau+\pi)=-u(z,\tau)$, so the half-periods of the wave are all distorted in the same way. For $z=1$, in each of the periods two shock fronts begin to form, a "compression" front and a "rarefaction" front. Specifically, let us consider the shock near the point $\tau=\pi$ (Fig. 1). Let $A(z)=u(z,\tau_s(z)-0)$ and $B(z)=u(z,\tau_s(z)+0)$ be the values of the field u to the left and to the right of the shock, and τ_s be the position of the shock on the time axis. For $z=1$ we have $\tau_s=3\pi/4+1/2$, $A=B=1/\sqrt{2}$. As the distance z is increased further, the jump amplitude $\Delta u=A-B$ grows; until $z=z_*\approx 9.601$, the required construction may be carried out on the basis of the solution (5), with the aid the "equal areas" rule. For $z=z_*$, it turns out that $A+2B=0$. The use of the solution (5) at distances $z>z_*$ is no longer incorrect. To illustrate this point, consider a step-shaped wave, or "jump" (Fig. 2).

Let the front of the jump be at the point $\tau=\tau_s$, and A and B be the values of u to the left and to the right of the front. The velocity $d\tau_s/dz$ of the jump in the comoving coordinate system is found, for $\Gamma\to 0$, from Eq. (3) by means of the equal-areas rule[1]:

$$d\tau_s/dz=(A^2+AB+B^2)/3. \quad (6)$$

As to the displacement velocity of the smooth portions of the profile, we see from the solution (5) that it equals $d\tau/dz=u^2$, i.e., has the values A^2 to the left and B^2 to the right of the shock. The jump will be stable if the smooth portions of the profile (Riemann waves) come running onto it from either side. In other words, to the left of the front the perturbations must move faster, and to the right, slower than the shock: $B^2\leqslant(A^2+AB+B^2)/3\leqslant A^2$. This condition is equivalent to the set of inequalities $(A-B)(A+2B)\geqslant 0$, $(B-A)(B+2A)\leqslant 0$. For the sake of definiteness, let $A>0$, $A>B$. Then the second inequality holds automatically, whereas the first yields

$$A+2B\geqslant 0. \quad (7)$$

If the condition (7) is not met, the initial jump is not stable; as the wave propagates, the jump rearranges itself in such a

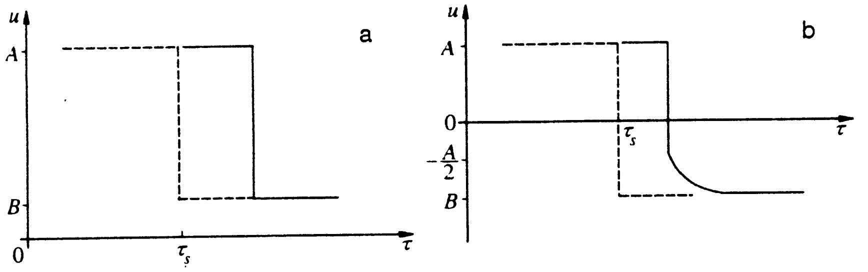

FIG. 2.

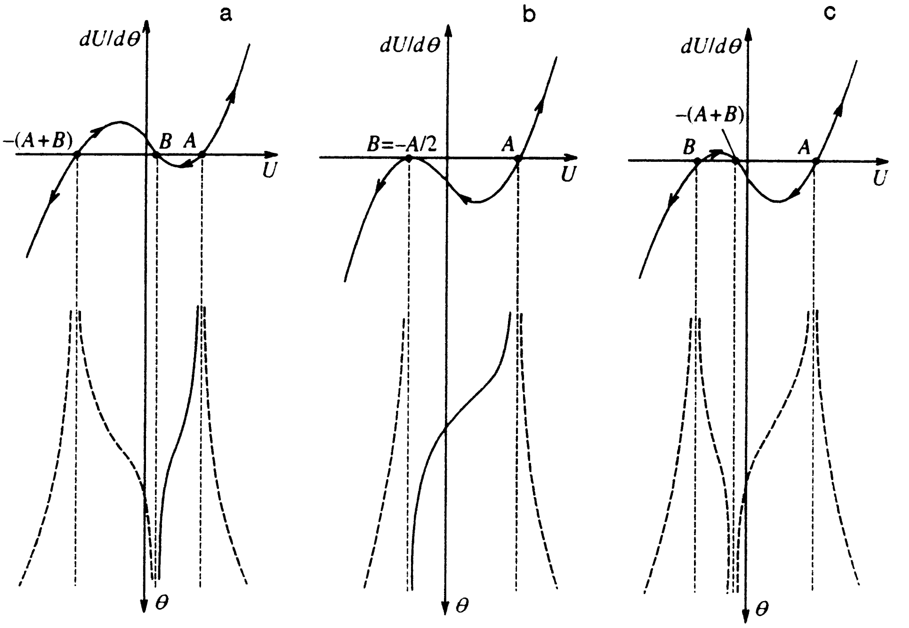

FIG. 3.

way that to the right of the front the wave is distorted by the perturbations "running away" from the shock. A new shock front is established which satisfies the condition $A+2B=0$. Note that $d\tau_s/dz=B^2$, i.e., the perturbation just ahead of the shock moves with the same velocity as the shock itself (Fig. 2b).

The stability condition (7) may be derived in a different way, by considering the internal structure of the jump.[10] We take the corresponding solution to Eq. (3) to be of the form $u(z,\tau)=U(\Theta=\tau-Cz)$, where $C=d\tau_s/dz$. Integrating once and using the relation (6) we find

$$\Gamma \frac{dU}{d\theta}=\frac{1}{3}(U-A)(U-B)(U+A+B). \tag{8}$$

It will be recalled that the boundary conditions are: $U(-\infty)=A$, $U(+\infty)=B$, $A>B$. Equation (8) can be integrated straightforwardly, but the structure of the solution is more conveniently analyzed in the phase plane $(U, dU/d\theta)$ (Fig. 3). The solutions differ qualitatively for three possible cases for which the quantity $A+2B$ is positive, zero, or is negative. These three situations there correspond Figs. 3a, b, and c.

For $A+2B>0$ (Fig.3a) there exist a solution to Eq. (8) which describes the structure of a shock front of width $\Delta\tau\sim\Gamma$, the transition to the boundary values proceeding exponentially. In the case $A+2B=0$ (Fig. 3b), the cubic polynomial on the right-hand side of Eq. (8) has a degenerate root $U=B=-A/2$, with a consequence that the transition to $U=B$ proceeds not exponentially but rather with a slower power law. Finally, for $A+2B<0$ (Fig. 3c) none of the solutions to Eq. (8) satisfies the boundary conditions. Thus, the requirement for a regular shock transition leads to the inequality (7).

Let us now return to the discussion of the sinusoidal initial signal. Its profile is distorted in such a way that as $A+2B\to 0$ as $z\to z_*$. At large distances $z>z_*$, according to Eq. (7) the quantity $A+2B$ cannot decrease further and the condition $A+2B=0$ must be satisfied.ed. The quantity B is maintained at the level $-A/2$ because a new wave starts to "flow out" of the shock front.

Now consider the corresponding characteristics of Eq. (3) (for $\Gamma=0$) and the shock trajectory (Fig. 4). For $z<z_*$ the entire region (τ, z) is covered by characteristics originating in the region of the initial data, the straight line $z=0$. The trajectory of the jump forms as a result of intersections and has a slope intermediate between those of the characteristics that intersect. For $z=z_*$ it is found that the characteristic approaching the jump trajectory from the right is a tangent. For $z>z_*$, the "shadow region" lying between the jump trajectory and this characteristic (and dashed in Fig. 4) is not reached by any of the characteristics stemming from the initial data region. Further, as pointed out, the velocity of perturbations at the point $\tau=\tau_s+0$ equals the shock velocity. Consequently, the characteristics stemming from the shock trajectory are, at each point, tangents, whereas the trajectory itself is the envelope of the family of these characteristics.

The process we have described is conveniently treated as refraction, at the jump trajectory, of the characteristic lines coming from the left. If U_- is the value of u at a given

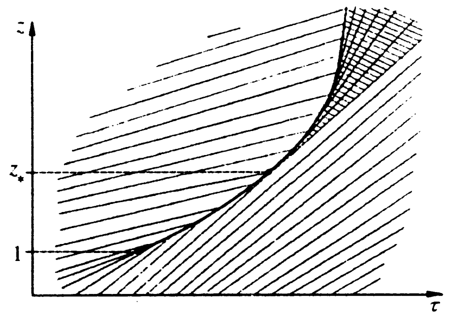

FIG. 4.

characteristic, then after the refraction u changes to $U_+=-U_-/2$. The angle of inclination also changes discontinuously from the value $dz/d\tau=U_-^{-2}$ to $4U_-^{-2}$. Such a treatment implies that, for any z, the region (τ,z) is covered only by those characteristics stemming from the region of the initial data.

In the case of a periodic wave, a multiple refraction of the characteristics takes place. For a sinusoidal initial profile, it proves possible to find the asymptotic waveform for $z\gg 1$ (Ref. 11):

$$u=\sqrt{bU_1(\theta)/z}, \quad -\pi<\theta<0,$$
$$u=-\sqrt{bU_2(\theta)/z}, \quad 0<\theta<\pi, \tag{9}$$

where $\theta=\tau-\tau_s(z)$, $\tau_s=b\ln z+\tau_\infty$, $b=\pi/(3-2\ln 2)$, and τ_∞ is a constant which cannot be predicted from an asymptotic analysis. Numerical calculation yields $\tau_\infty\approx-1.73$. In the expressions (9), the functions $U_1(\theta)$ and $U_2(\theta)$ do not depend on the distance and are given by the relations

$$\frac{\theta}{b}=U_1-\ln\frac{U_1}{4}-4, \quad 1<U_1<4,$$

$$\frac{\theta}{b}=U_2-\ln U_2-1, \quad 1<U_2<4. \tag{10}$$

Thus, as a result of nonlinear evolution, the sinusoidal profile transforms into a sawtooth one (see Fig.1). However, unlike quadratically nonlinear media, here the saw teeth do not have a triangular[1] but rather a trapezoidal shape.

The asymptotic solution (9), (10) is an exact solution of Eq. (3) for $\Gamma=0$. Let us demonstrate this point. Let $\psi=\tau_s(z)$ be the nonlinear phase shift. Changing to the "traveling" time $\theta=\tau-\psi$ we write Eq. (3) in the form

$$\frac{\partial u}{\partial z}+\left(u^2-\frac{d\psi}{dz}\right)\frac{\partial u}{\partial\theta}=0. \tag{11}$$

We seek a solution of the form $u=A(z)U(\theta)$, where $A(z)$ is the wave amplitude and $U(\theta)$ a function describing the temporal profile. We obtain

$$\frac{1}{A^3}\frac{dA}{dz}+\left(U-\frac{1}{U}\frac{1}{A^2}\frac{d\psi}{dz}\right)\frac{dU}{d\theta}=0, \tag{12}$$

from which it inevitably follows that

$$\frac{1}{A^3}\frac{dA}{dz}=-C_1, \quad \frac{1}{A^2}\frac{d\psi}{dz}=C_2, \quad \left(U-\frac{C_2}{U}\right)\frac{dU}{d\theta}=C_1. \tag{13}$$

Here C_1 and C_2 are some constants. Since $dA/dz<0$, $d\psi/dz>0$, these constant are positive. Integrating Eqs. (13) we obtain

$$A(z)=A_0(1+2C_1A_0^2z)^{-1/2}, \tag{14}$$

$$\psi(z)=\psi_0+(C_2/2C_1)\ln(1+2C_1A_0^2z), \tag{15}$$

$$\theta=(U^2-1)/2C_1-(C_2/C_1)\ln U, \tag{16}$$

where $A_0=A(0)$, $\psi_0=\psi(0)$, and it is assumed that $U(0)=1$. The profile (16) is shown in Fig. 5a. The function U is a two-valued function defined for $\theta\geq\theta_0=(C_2-1-C_2\ln C_2)/2C_1$. Note that $U_0^2=C_2$. Such a many-valued solution is physically meaningless, but its rising branch can be used to describe the sawtooth wave profile. To this end, let the saw interval $(-\pi,0)$ be described by the function $U(\theta)$, the interval $(0,\pi)$, by $-U(\theta-\pi)$, and so forth (see Fig. 5b). The smooth portions will be linked by vertical shock lines. It might seem that, in this way, one could construct a whole family of sawtooth solutions differing in the ratio $A:B$. In reality, however, there is only one possibility, the one which is shown in Fig. 1 and corresponds to $A:B=2:-1$.

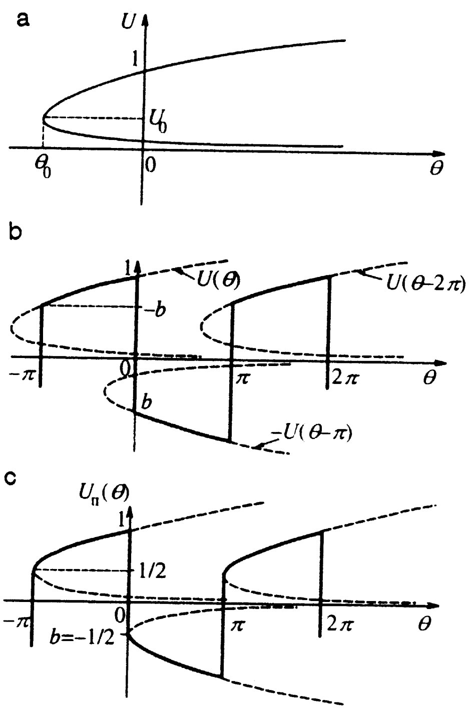

FIG. 5.

Let us prove this. Let $U_n(\theta)$ be a sawtooth solution and $b=B/A=U_n(0+0)$. The requirement for a unique solution yields $|b|\geq U_0$, that is, $b^2\geq C_2$. In other words,

$$B^2\geq C_2A^2. \tag{17}$$

On the other hand, the jump can be stable only if $d\psi/dz\geq B^2$ or, from Eq. (13), if

$$B^2\leq C_2A^2. \tag{18}$$

From Eqs. (17) and (18) it follows that $B^2=C_2A^2$ or $b^2=U_0^2$. It is precisely this case which is illustrated in Fig. 5c. Since $B^2=d\psi/dz$, it follows, as discussed above, that $B/A=-1{:}2$, or $b=-1/2$. This also implies that $U_0=1/2$, i.e.,

$$C_2=1/4. \tag{19}$$

The constant C_1 is found using Eq. (16) and the condition $U(-\pi)=1/2$ to give

$$C_1=(3-2\ln 2)/8\pi. \tag{20}$$

It is readily seen that the solution obtained, Eqs. (14)–(16), (19), (20), is identical to the asymptotic form given by Eqs. (9) and (10).

Now write the first two of Eqs. (13) in a different form,

$$\frac{dA}{dz}=-\frac{\beta}{2}A^3, \quad \frac{d\psi}{dz}=\frac{A^2}{4}, \tag{21}$$

where $\beta=2C_1\approx 0.184$. This formulas show that the sawtooth wave experiences self-action. In fact, in the case of linear absorption the amplitude varies in accordance with the equation $dA/dz=-\alpha A$. Comparing this form with the first of Eqs. (21), for the absorption coefficient we have, formally, that $\alpha(A)=\beta A^2/2$. Further, if one considers the wave velocity c, then from the meaning of the quantity ψ we have $c^{-1}-c_0^{-1}=d\psi/dz$, where c_0 is the velocity of the linear wave (for $A\to 0$). The second of Eqs. (21) implies

$$C^{-1}=C_0^{-1}+A^2/4. \tag{22}$$

Thus, in a cubically nonlinear medium with no dispersion there exist waves which do not change their profile as they travel. These are "trapezoidal sawteeth." They suffer self-action associated with the nonlinear absorption $\alpha=\alpha(A)$ and with the nonlinear dispersion of the propagation velocity, $c=c(A)$. A point of fundamental importance is that both of these phenomena always occur simultaneously because both are due to one and the same nonlinearity.

In dispersive media, in which the model (1) describes the process adequately, the only manifestation of self-action is the amplitude dependence of the wave velocity. So here self-action proceeds in its "pure" form. In nondispersive media, in parallel to self-focusing, the competing nonlinear absorption mechanism always operates. In what follows, the implication of this for the process of self-focusing of intensive wave beams is considered.

3. SELF-ACTION OF BEAMS

Let us consider the nonlinear equation (2). We rewrite it in the dimensionless notation of Eq. (4):

$$\frac{\partial}{\partial\tau}\left[\frac{\partial u}{\partial z}+u^2\frac{\partial u}{\partial\tau}-\Gamma\frac{\partial^2 u}{\partial\tau^2}\right]=\frac{N}{4}\Delta_\perp u. \tag{23}$$

The number

$$N=\frac{z_s}{z_d}=\frac{2c_0}{\gamma\omega^2a^2u_0^0} \tag{24}$$

is equal to the ratio of the nonlinear length z_s, Eq. (4), to the diffraction length $z_d=\omega a^2/2c_0$. It allows one to compare the diffraction and nonlinearity contributions to the distortion of the wave field. Here a is the characteristic initial beam size. The transverse coordinates on the right-hand side of Eq. (23) are normalized to the scale a.

In discussing plane waves we have shown that nonlinearity leads to the absorption and acceleration of a wave. Since the wave amplitude varies across the beam, the latter effect acts to distort the wave front. This self-refraction may show up in the form of self-focusing.

Apart from nonlinear processes, Eq. (23) also describes diffraction, which is important in the focal region. In other regions of space it can often be neglected, which enables one to simplify the analysis by employing approximate methods.

3.1. NONLINEAR GEOMETRICAL BEAM ACOUSTICS

In order to go over to the description of beams in terms of the nonlinear geometrical acoustics (NGA) approximation we introduce, as we did for the plane wave case, a new variable $\theta=\tau-\psi(r,z)$. In the following we will consider circular, axially symmetric beams, in which r is the distance from the axis to the observation point measured in units of the beam radius a. The phase shift ψ, unlike the plane waves, now also depends on r. Equation (23) transforms into

$$\frac{\partial}{\partial\theta}\left\{\frac{\partial u}{\partial z}+\left[u^2-\frac{\partial\psi}{\partial z}-\frac{N}{4}\left(\frac{\partial\psi}{\partial r}\right)^2\right]\frac{\partial u}{\partial\theta}+\frac{N}{2}\frac{\partial\psi}{\partial r}\frac{\partial u}{\partial r}+\frac{N}{4}\Delta_\perp\psi u\right\}=\frac{N}{4}\Delta_\perp u.$$

Note that the right-hand side of this relation describes only diffraction, whereas the analogous term in Eq. (23) is responsible for both processes, the diffraction and self-refraction of the wave. Neglecting the right-hand side enables one to integrate this relation with respect to θ to give

$$\frac{\partial u}{\partial z}+\left[u^2\frac{\partial\psi}{\partial z}-\frac{N}{4}\left(\frac{\partial\psi}{\partial r}\right)^2\right]\frac{\partial u}{\partial\theta}+\frac{N}{2}\frac{\partial\psi}{\partial r}\frac{\partial u}{\partial r}+\frac{N}{4}\Delta_\perp\psi u=0. \tag{25}$$

Equation (25) is the generalization of Eq. (11) to the beam case. As before, we will be interested in the sawtooth waves

$$U(z,r,\theta)=A(z,r)U(\theta), \tag{26}$$

where A is the, as yet, unknown wave amplitude and $U(\theta)$ is the previously found trapezoidal-saw profile. We use the last of Eqs. (13) to obtain

$$\left(U-\frac{1}{4U}\right)\frac{dU}{d\theta}=\frac{\beta}{2}, \tag{27}$$

using Eqs. (19) and (20). Substituting (26) into (25) we find

$$\frac{\dfrac{\partial A}{\partial z}+\dfrac{N}{2}\dfrac{\partial\psi}{dr}\dfrac{\partial A}{\partial r}+\dfrac{N}{4}\Delta_\perp\psi A}{A^3}+U\frac{dU}{d\theta}-\frac{1}{U}\frac{dU}{d\theta}\,\frac{\dfrac{\partial\psi}{\partial z}+\dfrac{N}{4}\left(\dfrac{\partial\psi}{\partial r}\right)^2}{A^2}=0.$$

Using (27), this gives

$$\frac{\partial A}{\partial z}+\frac{N}{2}\frac{\partial\psi}{dr}\frac{\partial A}{\partial r}+\frac{N}{4}\Delta_\perp\psi A=-\frac{\beta}{2}A^3, \tag{28}$$

$$\frac{\partial\psi}{\partial z}+\frac{N}{4}\left(\frac{\partial\psi}{dr}\right)^2=\frac{1}{4}A^2. \tag{29}$$

Thus, a pair of equations is obtained which describe the behavior of the wave amplitude [transport equation (28)] and of the wave phase [eikonal equation (29)]. The meaning of this system becomes clear if one introduces the angle of incidence of rays, $V=(N/2)\partial\psi/\partial r$, and the cross-sectional area of the ray tube, S,

$$\left(\frac{\partial}{\partial z}+V\frac{\partial}{\partial r}\right)S=S\,\frac{1}{r}\frac{\partial}{\partial r}(rV). \tag{30}$$

The operator $\partial/\partial z+V\partial/\partial r$ corresponds to the differentiation $\partial/\partial l$ along the acoustic ray. With this in mind, and using the relation (30), we write Eqs. (28) and (29) in the form

$$\frac{\partial A}{\partial l}+\frac{A}{2S}\frac{\partial S}{\partial l}=-\frac{\beta}{2}A^3, \tag{31}$$

$$\frac{\partial V}{\partial l}=\frac{N}{8}\frac{\partial}{\partial r}A^2. \tag{32}$$

Equation (31)implies that the wave amplitude along the ray occurs for two reasons: because the ray-tube cross sectional area changes, and because of nonlinear absorption. Equation (32) indicates that the nonlinear refraction takes place: the angle of incidence of the ray changes in the presence of a transverse gradient of the wave amplitude.

3.2. ABERRATION FREE SELF-FOCUSING

We consider next the paraxial region with a wave front taken to be parabolic

$$\psi=\psi_0(z)+\frac{z^2}{N}\frac{1}{f}\frac{df}{dz}. \tag{33}$$

Here $f(z)$ is a function describing, to linear approximation, the change in the beam width and in the amplitude of the wave at the beam axis.[5] From Eq. (29)

$$\frac{d^2f}{dz^2}=\frac{N}{8}f\left.\frac{\partial^2}{\partial r^2}A^2\right|_{r=0}. \tag{34}$$

Equation (28) for the parabolic front (33) has the exact solution

$$A=\frac{1}{f}\Phi\left(\frac{r}{f}\right)\left[1+\beta\Phi^2\left(\frac{r}{f}\right)\int_0^z\frac{dz'}{f^2(z')}\right]^{-1}, \tag{35}$$

in which the function $\Phi(r)=A(z=0,r)$ describes the initial transverse beam profile. Once we have the solution (35) it is not difficult to specify Eq. (34) from which the unknown function f has to be found. In particular, for the Gaussian beam $\Phi=\exp(-r^2)$, Eq. (34) takes the form

$$f^3\frac{d^2f}{dz^2}=-\frac{N}{2}\left[1+\beta\int_0^z\frac{dz'}{f^2(z')}\right]^{-2}. \tag{36}$$

The boundary conditions

$$f(z=0)=1,\quad \frac{df}{dz}(z=0)=\frac{1}{R} \tag{37}$$

correspond to an initial beam having (in the dimensionless notation) unit width and a radius of curvature R. Equation (36), we note, is a complex nonlinear integrodifferential equation with a second derivative. Therefore it appears surprising that the Cauchy problem (36), (37) has an exact solution

$$f(z)=\left(1+\frac{z}{R}\right)\left(1+\frac{\delta_1 z}{1+z/R}\right)^{\delta_2/(\delta_1+\delta_2)}\times\left(1-\frac{\delta_2 z}{1+z/R}\right)^{\delta_1/(\delta_1+\delta_2)}, \tag{38}$$

where $\delta_{1,2}=(\sqrt{\beta^2+2N}\pm\beta)/2$.

The result (38) has been obtained owing to the discovery of a change of variables of the form

$$\tilde{z}=z/(1+\lambda z),\quad \tilde{f}=f/(1+\lambda z), \tag{39}$$

under which Eq. (36) is invariant. According to the formulas (39), $\tilde{z}(0)=0$, $\tilde{f}(0)=f(0)$, and

$$\frac{d\tilde{f}}{d\tilde{z}}(0)=\frac{df}{dz}(0)-\lambda f(0)=\frac{1}{R}-\lambda. \tag{40}$$

In Eq. (40), the boundary conditions (37) are taken onto account. Thus, different values of the parameter λ correspond, by Eq. (40), to different curvatures of the initial wave front. Therefore it suffices to find only one particular solution to Eq. (36) and then to apply the transformation (39) to write an arbitrary solution satisfying, for example, the conditions (37) at the boundary of the nonlinear medium.

Such a particular solution does exist:

$$\tilde{f}(\tilde{z})=[1+(\delta_1+\delta_2)\tilde{z}]^{\delta_2/(\delta_1\delta_2)}. \tag{41}$$

This is easily seen by substituting (41) into the integrodifferential Eq. (36). Knowing the solution (41), from Eq. (40) we find the required parameter $\lambda=R^{-1}-\delta_2$. After this, transforming the solution (41) according to the expressions (39), we obtain the general result (38).

We now turn to the analysis of the exact solution (38). It is seen that under the condition $R^{-1}<\delta_2$ the beam must collapse at the distance $z_{SF}=R/(\delta_2 R-1)$. For small radii of curvature of the initial front of the diverging wave, when $R^{-1}>\delta_2$, the beam width can only increase. If at the input into the nonlinear medium the wave front is plane ($R\to\infty$), the dimensionless focusing length is

$$z_{SF}=\frac{1}{\delta_2}=\frac{2}{\sqrt{\beta^2+2N}-\beta}. \tag{42}$$

Equation (42) allows to estimate at what values of the number N, Eq. (24), appreciable self-focusing is possible. Note that the competing processes of nonlinear absorption and diffraction have spatial scales of order 1 and N^{-1}, respectively. The condition that z_{SF}, Eq. (42), be small compared to these scales leads to two inequalities

$$N>2(1+\beta)\approx 2.36,\quad N<0.5-\beta\approx 0.32, \tag{43}$$

which do not hold simultaneously. Hence the process of self-focusing in cubically nonlinear media with no dispersion cannot be important in principle.

3.3. SELF-FOCUSING WITH DIFFRACTION

It is clear that the shape of a wide-band signal (in particular, of a trapezoidal sawtooth) should become distorted in the process of propagation since the components of the wave spectrum diffract differently. However, for N small the dif-

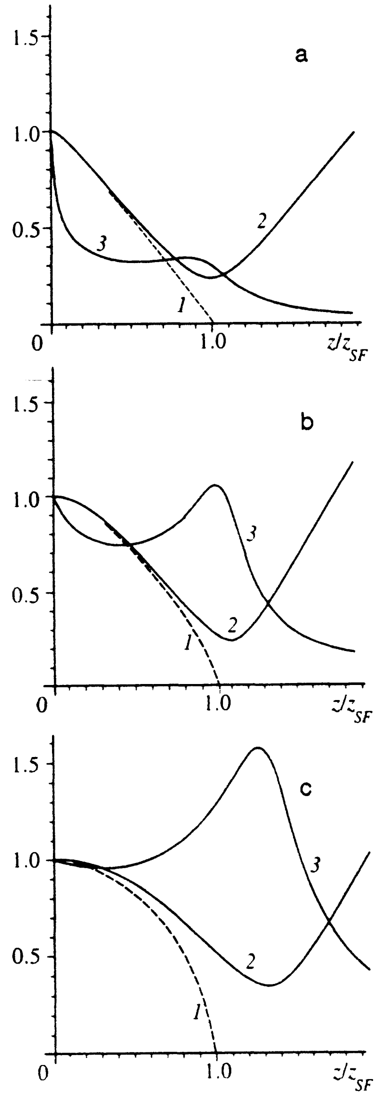

FIG. 6.

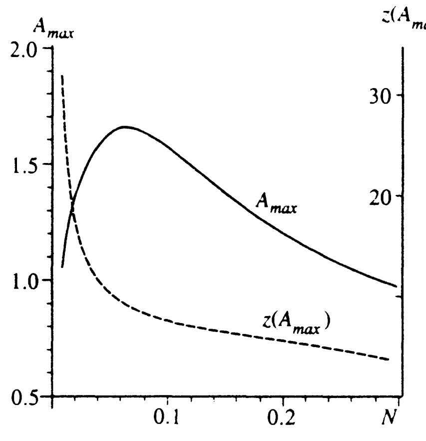

FIG. 7.

fraction effects are not large, and in the most interesting cases this difference may be neglected. If it is assumed that the spatial behavior of the higher harmonics matches the field at the fundamental frequency, then Eq. (36) for the function f should be modified by adding the term N^2 to its right-hand side. The modified equation (36) is also invariant under the transformation (39). However, it has not yet been solved analytically because no particular solutions of the type (41) has been found.

Figures 6 and 7 display the results of numerical integration. The dashed lines 1 in Figs. 6a,b,c show the behavior of the beamwidth-determining function $f(z)$ in the absence of diffraction [that is, the behavior described by Eq. (38)]. The solid lines 2 (for f) and 3 (for the saw amplitude) are constructed with account for the diffraction corrections. Figs. 6 a,b,c correspond to increasing values $N=10^{-3}$(a), 10^{-2} (b), and $10-1$ (c); increasing N enhances the role of diffraction. The nondimensional self-focusing scale [recall that the dimensional scale was normalized to the shock formation length z_s, Eq. (4)] approximately equals 264 (Fig. 6a), 32 (Fig. 6b), and 6 (Fig. 6c). Thus, the characteristic self-focusing length is many times nonlinear length z_s; hence the self-focusing definitely occurs at the stage of strongly pronounced nonlinear wave-profile distortions.

The curves labeled *1* in Fig. 6 are constructed neglecting diffraction. They describe the collapse, the process in which the width of the beam drops to zero. The curves labeled *2* contain the diffraction "bottleneck": the beam narrows to its minimum, which is just beyond the nonlinear focus, and then broadens.

Of primary interest is the behavior of the amplitude. At short distances it decreases due to the nonlinear dissipation of energy at the fronts of the sawtooth wave. After this the nonlinear self-focusing slows the process down and can even amplify the wave a little in the focal region. Beyond the focus the beam becomes divergent; due to the divergence and nonlinear dissipation, the amplitude of the wave is reduced.

As can be seen from the analysis of the curves in Fig. 6, in the absence of dispersion the cubic nonlinearity does not lead to any essential growth of the amplitude. Even though the beam is markedly narrowed and has a nonlinear constriction in it, the amplification factor is not large because of the fundamentally unavoidable absorption occurring at the shock front of the saw.

Figure 7 is represented the dependence on the number N of the maximum amplitude (solid line) and of the distance at which the maximum is reached (dashed line). The maximum focus value, $A_{\max}\approx 1.65$, occurs at $N\approx 0.06$

3.4. INVARIANCE OF EQ. (23) UNDER A TRANSFORMATION THAT CHANGES THE CURVATURE ON THE FRONT

The fact that Eq. (36) for the beam width f under the change of variables (39) follows the more general symmetry properties of the initial equation (23). In fact, when we apply the transformation

$$\tilde{\tau}=\tau-\frac{\lambda}{1+\lambda z}\frac{r^2}{N}, \quad \tilde{z}=\frac{z}{1+\lambda z},$$

$$\tilde{r}=\frac{r}{1+\lambda z}, \quad \tilde{u}=u(1+\lambda z). \tag{44}$$

Equation (23) becomes

$$\frac{\partial}{\partial\tilde{\tau}}\left[\frac{\partial\tilde{u}}{\partial\tilde{z}}+\tilde{u}^2\frac{\partial\tilde{u}}{\partial\tilde{\tau}}-\tilde{\Gamma}\frac{\partial^2\tilde{u}}{\partial\tilde{\tau}^2}\right]=\frac{N}{4}\tilde{\Delta}_\perp\tilde{u}. \tag{45}$$

Here

$$\tilde{\Delta}=\frac{1}{\tilde{r}}\frac{\partial}{\partial\tilde{r}}\left(\tilde{r}\frac{\partial}{\partial\tilde{r}}\right),\quad \tilde{\Gamma}=\Gamma\frac{1}{(1-\lambda\tilde{z})^2}.$$

It is seen that in the absence of dissipation, i.e., for $\Gamma=\tilde{\Gamma}=0$, Eq. (23) is not changed.

Thus, given but one of the solutions $\tilde{u}$ to Eq. (23), [or (45)] for beams in cubically nonlinear nondispersive media, it is possible to construct a whole family of new solutions

$$u(z,r,\tau)=\frac{1}{1+\lambda z}\tilde{u}\left(\frac{z}{1+\lambda z},\frac{r}{1+\lambda z},\tau-\frac{\lambda}{1+\lambda z}\frac{r^2}{N}\right), \tag{46}$$

differing in the value of the parameter λ, that is, in the initial wave front curvature.

3.5. EQUATIONS FOR THE AVERAGE INTENSITY

In describing the beam in Secs. 3.1 through 3.3, we used the fact that the time profile is stationary and represents a trapezoidal sawtooth (Fig. 1) with an "amplitude" depends on position.

An alternative description is possible, based on the equations for the intensity which are obtained by averaging over the wave period. This approach possesses greater generality and is not limited to quasistationary profiles alone.

In the NGA approximation (see Sec. 3.1), Eq. (23) yields a pair of equations,

$$\frac{\partial\psi}{\partial z}=\frac{N}{4}\left(\frac{\partial\psi}{\partial r}\right)^2=\overline{u^2}, \tag{47}$$

$$\frac{\partial u}{\partial z}+(u^2-\overline{u^2})\frac{\partial u}{\partial\theta}-\Gamma\frac{\partial^2 u}{\partial\theta^2}+\frac{N}{2}\frac{\partial\psi}{\partial r}\frac{\partial u}{\partial r}+\frac{N}{4}\Delta_\perp\psi u=0. \tag{48}$$

Here the overbar denotes a period average. Note that Eq. (47) represents the eikonal equation, and Eq. (48), of the transport equation. However, Eq. (48) contains the timelike variable θ; it differs from ordinary transport equations and may be called the modified Burgers equation.[1]

Multiply Eq. (48) by u and average the resulting relation over the period. For the average intensity $J=\bar{u}^2$ we have

$$\frac{\partial J}{\partial z}+\frac{N}{2}\left(\frac{\partial\psi}{\partial r}\frac{\partial J}{\partial r}+\Delta_\perp\psi J\right)=-\Gamma\overline{\left(\frac{\partial u}{\partial\theta}\right)^2}. \tag{49}$$

The right-hand side of Eq. (49) describes wave absorption due to dissipative and nonlinear effects. In order to calculate it one needs to know the wave profile shape. For ideal media ($\Gamma\to 0$), in the region prior to the shock formation the right-hand side of Eq. (49) equals zero (both for a harmonic signal near the boundary $z=0$ and for a strongly distorted profile for distances up to $z=1$).

In the case of discontinuous waves ($z>1$), nonlinear absorption must be taken into account even in ideal media since at steep fronts $\partial/\partial\theta\sim\Gamma^{-1}$. In order to calculate the right-hand side of Eq. (49), one must employ the solutions to Eq. (8), which describe the structure of the shock fronts. The result of the averaging turns out to be independent of the dissipation Γ:

$$\Gamma\overline{(\partial u/\partial\theta)^2}=\nu J^2, \tag{50}$$

where ν is a coefficient dependent on the shape of the profile.

Thus, the system of equations for the average intensity takes the form

$$\frac{\partial J}{\partial z}+\frac{\partial}{\partial r}(JV)+\frac{1}{r}JV=-\nu J^2, \tag{51}$$

$$\frac{\partial V}{\partial z}+V\frac{\partial V}{\partial r}=\frac{N}{2}\frac{\partial J}{\partial r}. \tag{52}$$

The variable $V=(N/2)\partial\psi/\partial r$ (the angle of incidence of the rays) has already been used in Eqs. (30) and (31).

Where there are no discontinuities the right-hand side of Eq. (50) is equal to zero. In the general case, when the linear and nonlinear absorption mechanisms are both involved, the right-hand side of Eq. (51) will be a more complex function of J and z (which, however, is amenable to calculation for many situations of interest).

When there are no shocks and the medium has no dissipative properties, the system (51), (52) is identical to the well-studied equations of aberrational optical self-focusing.[3,5] As is known, these admit an exact solution which predicts an intersection of the rays at distances $z>N^{-1/2}$. The conditions that this length be small compared to the shock formation and diffraction scales (1 and N^{-1} in the dimensionless notation) are contradictory, thus proving self- focusing to be impossible for smooth profiles ($z<1$).

With the substitution $J=1/B$, Eq. (51) reduces to a linear equation for the new variable,

$$\frac{\partial B}{\partial z}+V\frac{\partial B}{\partial r}-B\left(\frac{\partial V}{\partial r}+\frac{V}{r}\right)=\nu, \tag{53}$$

which can be solved exactly. In the paraxial approximation, the solution for the average intensity J has the form

$$J=\frac{1}{f^2}J_0\left(\frac{r}{f}\right)\left[1+\nu J_0\left(\frac{r}{f}\right)\int_0^z\frac{dz'}{f^2(z')}\right]^{-1}, \tag{54}$$

and Eq. (36), which we have discussed in Sec. 3.2., follows for the function f.

3.6. INVARIANTS OF MOTION AND THE SELF-FOCUSING PROCESS

It is useful to remember that in the absence of dissipation ($\Gamma=0$) the initial equation (23) has the following integrals:

$$I_1=\int u^2 d\tau dr_\perp,\quad I_2=\int\left[(\nabla_\perp w)^2-\frac{2}{3N}u^4\right]d\tau dr_\perp, \tag{55}$$

$\partial I_{1,2}/\partial z=0$. Here $dr_\perp=dxdy$, $\partial w/\partial\tau=u$.

The process of change of the beam width can be described using the (u^2-weighted) "average," over the cross section and time, of the square of its radius,

$$\langle r_{\perp}^2\rangle = I_1^{-1}\int r_{\perp}^2 u^2 d\tau dr_{\perp}. \quad (56)$$

Differentiating Eq. (56), from Eqs. (55) and (23) we obtain

$$\frac{d^2}{dz^2}\langle r_{\perp}^2\rangle = \frac{N^2}{2}\frac{I_2}{I_1} = \text{const}. \quad (57)$$

This equation is readily integrated to give

$$\langle r_{\perp}^2\rangle = \langle r_{\perp}^2\rangle|_{z=0} + z\,\frac{d}{dz}\langle r_{\perp}^2\rangle|_{z=0} + \frac{1}{4}N_{z^2}^2\frac{I_2}{I_1}. \quad (58)$$

The result (58) allows one to determine the position of the focus, at which $\langle r_{\perp}^2\rangle \to 0$.

In particular, for a Gaussian beam of harmonic waves, $u = \exp(-r^2)\sin\tau$, we have $I_1 = \pi^2/2$, $I_2 = \pi^2(1-1/8N)$, and

$$\langle r_{\perp}^2\rangle = \frac{1}{2}\left[1 + N_{z^2}^2\left(1 - \frac{1}{8N}\right)\right]. \quad (59)$$

From this, for the dimensionless self-focusing length we have the expression

$$z_{\mathrm{SF}} = 2\sqrt{2}[N(1-8N)]^{-1/2}. \quad (60)$$

It is seen that self-focusing occurs only for $N < 1/8$, when the shock formation scale is at least a factor 8 less than the diffraction length. The optimal regime corresponds to $N = 1/16$, when $z_{\mathrm{SF}} = 16$. The latest data are consistent with the results of numerical integration given in Sec. 3.3.

Thus, self-focusing occurs at best over a distance of many times the length z_s. This means that the beam cannot collapse before shock fronts have formed. We had already reached a similar conclusion back in Sec. 3.5.

4. CONCLUSION

This paper is devoted to one of the fundamental problems in the theory of nonlinear waves. A possible application is the self- focusing of sound, whose feasibility was first indicated in Ref. 12. Later, the effect of thermal self-action of beams was observed in Refs. 13 and 14 in highly viscous liquids, where it bears a close resemblance to the light self-focusing effect.[2,3] However, in the most important and most typical case of a medium with weak sound absorption, shock waves form and self-action processes operate differently. Investigations of the inertial (thermal) self-action of sawtooth waves (Ref.15, theory; Ref. 16, experiment) have confirmed this observation. As to the inertialess self-action of discontinuous waves, which in nondispersive media is only possible in the presence of cubic nonlinearity (see Ref. 1), ours is the first investigation into the subject.

We have limited consideration to self-focusing media, to which correspond to positive values of the constant Eqs.(1) and (2). The case $\gamma < 0$ (defocusing media), while less interesting physically is of greater practical value. Thus, the thermal defocusing of sawtooth waves often is a primary cause for the reduction in the maximum value of the field in underwater sound focusing,[15] whereas the self-refraction of shock pulses causes the peak pressures in the focus.[17] These phenomena have been observed repeatedly in the operation of high- power acoustic systems in technological and medical applications.[18,19] Therefore the self-defocusing and self-refracting of intensive waves in nondispersive media deserve a separate analysis that would take us beyond the scope of the present paper.

The approaches we have developed may prove useful in describing self-action in media with weak dispersion (for example, for ultrashot light pulses). Dispersion should generally lead to an additional broadening of the shock front thus reducing nonlinear losses. For example, in Eq. (36) the coefficient β should decrease, whereas N will retain its value. As a result, conditions for self-focusing to occur will be more favorable. This, however, is a general statement; to obtain specific estimates, new independent studies are needed.

In summary, then, a study of self-focusing nondispersive beams in nonlinear media has been made which reveals fundamental departure from the familiar results for waves in dispersive media. The resonant interaction between numerous harmonics, and shock front formation in the wave profile, call for a field description on the basis of the modified Khokhlov–Zabolotskaya equations. As the analysis of the equations has shown, the self-focusing process develops over lengths well in excess of the shock formation scale; the wave profile having a trapezoidal sawtooth shape, for which the inclusion of nonlinear damping is crucial even for an ideal nondissipative medium. There is competition between two processes, self-focusing and damping, both generated by the same nonlinearity. As a result, the width of the beam may be reduced substantially while the amplitude of the sawtooth increases little.

Also, a number of remarkable properties of the field equations that govern the beam uations have been found; special methods of treating them for highly distorted, essentially nonharmonic waves have been developed, and exact and asymptotic solutions obtained. It is believed that the universality of the models considered will make it possible to employ the above results in solving a wide variety of diverse problems in wave physics.

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