

The Lamb shift in the hydrogen atom

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A theoretical expression is derived for the difference $\Delta E_L(1s_{1/2}) - 8\Delta E_L(2s_{1/2})$ in Lamb shifts for the hydrogen atom. The obtained value of the difference is found to be equal to $-187.236(11)$ MHz. The leading logarithmic corrections to the lifetime of the $2p_{1/2}$ level are calculated, and the new value of the lifetime is found to be equal to $1.596\,1887(15) \times 10^{-9}$ s. The existing experimental data on the ratio of the width of this level to the Lamb splitting and the radiative corrections obtained in the investigation are used to obtain new values of the Lamb shift: $E(2s_{1/2}) - E_{2p_{1/2}} = 1057.8576(21)$ MHz, $\Delta E_L(2s_{1/2}) = 1045.0213(26)$ MHz, and $\Delta E_L(1s_{1/2}) = 8172.934(24)$ MHz.

1. INTRODUCTION

Recently reports have appeared of new precise measurements of the frequencies of transitions between the hydrogen-atom levels with different values of the principal quantum number n , aimed at refining the value of the Rydberg constant.^{1–3} The values of the Lamb shift of the $1s_{1/2}$ - and $2s_{1/2}$ -levels play an important role in processing the results of such measurements. The shift of the $2s_{1/2}$ -level is found from the experimental data on the $2s_{1/2} - 2p_{1/2}$ Lamb splitting and the theoretical result for the shift of the $2p_{1/2}$ -level,

$$\Delta E_L(2s_{1/2}) = [E(s_{1/2}) - E(2p_{1/2})]_{\text{expt}} + [\Delta E_L(2p_{1/2}) + \Delta E_{\text{BG}}(2p_{1/2})]_{\text{theor}}, \quad (1)$$

and the value of the Lamb shift of the ground level can be extracted with a high degree of accuracy from the following equation:⁴

$$\Delta E_L(1s_{1/2}) = [\Delta E_L(1s_{1/2}) - 8\Delta E_L(2s_{1/2})]_{\text{theor}} + 8[E(2s_{1/2}) - E(2p_{1/2})]_{\text{expt}} + 8[\Delta E_L(2p_{1/2}) + \Delta E_{\text{BG}}(2p_{1/2})]_{\text{theor}}, \quad (2)$$

where the labels “theor” and “expt” indicate, respectively, theoretical and experimental results. The correction term $\Delta E_{\text{BG}}(2p_{1/2})$, which originates from the effective Dirac equation,⁵ is usually not included in the Lamb shift.

From the theoretical viewpoint the difference in the shifts of the s -level in Eq. (1) is a considerably simpler object than the values in the difference, since either a considerable fraction of the corrections to the size of the Lamb level shifts are strictly proportional to δ_{l0}/n^3 , as is, for instance, the leading logarithmic contribution to the Lamb shift of order $\alpha(Z\alpha)^4 mc^2 \log Z\alpha$, or the part not related to the given factor is moderate, as is the case, for instance, with the Bethe logarithm $\log k_0(nl)$. (For the $1s$ -, $2s$ -, and $1p$ -levels we have 2.984..., 2.811..., and -0.030 ..., respectively.)

This paper examines all the terms in Eqs. (1) and (2). Section 2 discusses the contributions of the one-loop self-

energy of the electron in the Coulomb field of the nucleus to the shift of the p -level and to the difference in the shifts of the s -levels. Section 3 considers the leading two-loop corrections, which in the case of states with $l \neq 0$ are reduced to the known contribution of the electron anomalous magnetic moment, and finding the difference in the shifts of states with zero orbital angular momentum requires calculating the logarithmic corrections of order $\alpha^2(Z\alpha)^6 mc^2 \log^2 Z\alpha$.

The most exact result for the $2s_{1/2} - 2p_{1/2}$ splitting was obtained by Sokolov and Yakovlev,⁶ who directly measured the ratio of the splitting to the width of the $2p_{1/2}$ -level. Later the value of the ratio was revised⁷ because radiative corrections of the relative order of $\alpha(Z\alpha)^2$ to the lifetime of the level were taken into account. However, only some of the corrections in this order were allowed for, and Sec. 4 examines this aspect.

Section 5 discusses the results of the study and compares them to the theoretical and experimental data obtained by other researchers.

2. CONTRIBUTION OF THE ONE-LOOP SELF-ENERGY OF THE ELECTRON IN THE COULOMB FIELD OF THE NUCLEUS

It is customary to represent the contribution of the one-loop self-energy operator of an electron in a Coulomb field as

$$\Delta E_L^{\text{SE}}(nlj) = \frac{\alpha(Z\alpha)^4}{\pi n^3} mc^2 \left(\frac{m_r}{m}\right)^3 F(Z\alpha), \quad (3)$$

where m is the electron mass, m_r , the reduced mass, Z the charge of the nucleus in units of electron charge (it is equal to unity for the hydrogen atom but is customarily retained to allow for classification of contributions), and the function $F(Z\alpha)$ is the following series in $Z\alpha$:

$$F(Z\alpha) = A_{41} \log \frac{1}{(Z\alpha)^2} + A_{40} + (Z\alpha)A_{50} + (Z\alpha)^2 \times \left(A_{62} \log^2 \frac{1}{(Z\alpha)^2} + A_{61} \log \frac{1}{(Z\alpha)^2} + A_{60} \right) + \dots$$

The expansion coefficients in (4) for the hydrogen levels $n=1$ and $n=2$ were examined in detail by Sapirstein and Yennie.⁸ An important property of this series is that a considerable fraction of the coefficients (A_{41} , A_{50} , and A_{62}) are proportional to δ_{l0} and independent of the principal quantum number n . Until recently, all the coefficients written explicitly in (4) for the levels $n=1$ and $n=2$ were known except A_{60} .

In addition to the results for the expansion coefficients, the values of the function $F(Z\alpha)$ for hydrogenlike ion are known through numerical integration, without resorting to an expansion in the parameter $Z\alpha$. The common approach to processing such results is the following. First, all the known quantities are subtracted from $F(Z\alpha)$, and the difference function [customarily denoted by $C(Z\alpha)$,] which includes A_{60} and higher-order terms in $Z\alpha$, is approximated by the following formula (see, e.g., Refs 10 and 11):

$$C_{nlj}(Z\alpha) = A_{60}(nlj) + (Z\alpha) \left(A_{71}(nlj) \log \frac{1}{(Z\alpha)^2} + A_{70}(nlj) \right). \quad (5)$$

The more involved approximations incorporate higher-order terms in $Z\alpha$.

As Ref. 4 notes, the coefficients A_{71} must be proportional to δ_{l0} and independent of the quantum number n . The logarithmic term is expected to be the correction related to the Dirac wave function

$$[\psi_{ns}^{\text{Dir}}(r \sim m^{-1})]^2 \approx [\psi_{ns}^{\text{Schr}}(r \sim m^{-1})]^2 \times \left[1 + (Z\alpha)^2 \log \frac{1}{Z\alpha} \right] \quad (6)$$

that is, the correction to the contribution of order $\alpha(Z\alpha)^5 mc^2$ that contains no logarithms; hence,

$$A_{71} = \frac{1}{2} A_{50} = 2\pi \left(\frac{139}{128} - \frac{1}{2} \log 2 \right) \delta_{l0} \quad (7)$$

(cf. the results for the vacuum-polarization contribution¹¹).

Because there is no logarithmic term in (5) in the case of the hydrogen atom ($Z=1$), it is possible to find the quantities

$$G_{1s_{1/2}}(\alpha) - G_{2s_{1/2}}(\alpha) = 0.865(21), \quad (8)$$

$$G_{2p_{1/2}}(\alpha) = -0.936(14), \quad (9)$$

with greater accuracy than by fitting the function $F(Z\alpha)$ via three parameters. Equation (8) agrees well with Pachucki's results.¹²

3. CONTRIBUTION OF THE TWO-LOOP SELF-ENERGY OF THE ELECTRON IN THE COULOMB FIELD OF THE NUCLEUS

Now we consider second-order corrections in α to the Lamb shift. The general expression for them is¹³

$$\begin{aligned} \delta E_L^{(2)}(nljm) = & \langle nljm | \Sigma_C^{(1)}(E_{nljm}) \bar{G}_C(E_{nljm}) E_C^{(1)} \\ & \times (E_{nljm}) | nljm \rangle + \langle nljm | \Sigma_C^{(2)}(E_{nljm}) | nljm \rangle \\ & + \langle nljm | \Sigma_C^{(1)}(E_{nljm}) | nljm \rangle \\ & \times \left\langle nljm \left| \frac{\partial \Sigma_C^{(1)}(E)}{\partial E} \right| nljm \right\rangle_{E=E_{nljm}}, \end{aligned} \quad (10)$$

where $\Sigma_C^{(r)}(E)$ is the r -loop one-particle-irreducible operator of the electron self-energy in the Coulomb field of the nucleus, $|nljm\rangle$ and E_{nljm} the wave functions and the energies of states in the Dirac hydrogen atom, and $\bar{G}_C(E)$ the reduced Coulomb Green's function.

Similar to (3), we can write an expansion in powers of $Z\alpha$ for (10). We can easily verify that the radiative corrections¹⁾ of order $\alpha^2(Z\alpha)^5 mc^2$ and $\alpha^2(Z\alpha)^6 mc^2 \log^3(Z\alpha)$ (see Ref. 15) are proportional to $\delta_{l0} n^{-3}$. Corrections of the leading order $\alpha^2(Z\alpha)^4 mc^2$ contain only one term that is not proportional to $\delta_{l0} n^{-3}$, is related to the anomalous magnetic moment, and is well known (see, e.g., the review paper of Sapirstein and Yennie⁸). Still, for s -levels this contribution is proportional to n^{-3} , while for p -levels it is nonzero and constitutes the leading two-loop contribution. The leading contribution to the difference of s -levels $\Delta E_L(1s_{1/2} - 8\Delta E_L(2s_{1/2}))$ is of order $\alpha^2(Z\alpha)^6 mc^2 \log^2(Z\alpha)$.

We can easily verify¹⁵ that, in the Fried-Yennie gauge¹⁶ for radiation photons, only the first term on the right-hand side of (10) contains triple-logarithmic corrections, and only the terms with one Coulomb exchange in each mass operator $\Sigma_C^{(1)}$ contribute, that is, only Feynman diagrams with two form-factors are essential here:

$$\begin{aligned} \delta E_L^{(2)}(nljm) = & \sum_{n'l'j'm' \neq nljm} \\ & \times \frac{\langle nljm | \Sigma_C | n'l'j'm' \rangle \langle n'l'j'm' | \Sigma_C | nljm \rangle}{E_{nljm} - E_{n'l'j'm'}}, \end{aligned} \quad (11)$$

where the reduced Coulomb Green's function $\bar{G}_C(E)$ is represented in the form of a sum over all states of the discrete and continuous spectrum.

We can now verify that the corrections to the shift difference that contain the square of a logarithm originate from the same diagrams represented by (11). The contribution of the discrete spectrum in the sum over the states in (11) is the simplest to calculate. As noted in Ref. 15, this contribution contains only the square of a logarithm, with both logarithms originating from matrix elements between s -states:

$$\begin{aligned} \langle nljm | \Sigma_C | n'l'j'm' \rangle = & \delta_{l0} \delta_{l'0} \frac{\psi_{n00}(0) \psi_{n'00}(0)}{\psi_{100}(0) \psi_{100}(0)} \\ & \times \frac{8}{3\pi} \alpha(Z\alpha)^4 mc^2 \log \frac{1}{Z\alpha}, \end{aligned} \quad (12)$$

where $\psi_{nlm}(r)$ are the Schrodinger wave functions of the respective states in the coordinate representation.

Next we examine the sums over s -states and drop the Kronecker symbols. Substituting (12) into (11), we arrive at an expression for the discrete-spectrum contribution:

$$\delta E_L^d(ns) = \frac{64}{9\pi^2} \alpha^2 (Z\alpha)^8 m^2 c^2 \log^2 \frac{1}{Z\alpha} \times \sum_{n' \neq n} \frac{1}{n^3 n'^3} \left[\frac{(Z\alpha)^2 m}{2n'^3} - \frac{(Z\alpha)^2 m}{2n^2} \right]^{-1}, \quad (13)$$

or

$$\delta E_L^d(ns) = -\frac{128\alpha^2 (Z\alpha)^6 m c^2}{9\pi^2 n^3} S_n \log^2 \frac{1}{Z\alpha}, \quad (14)$$

where

$$S_n = \sum_{n' \neq n} \frac{n^2}{n' (n'^2 - n^2)}. \quad (15)$$

This sum can easily be calculated:

$$S_n = -\sum_1^{n-1} \frac{1}{q} + \frac{1}{4n}. \quad (16)$$

The expression for the leading part of the contribution of the continuous spectrum can be obtained via analytic continuation of (13). Indeed, the wave functions of the s -states of the discrete and continuous spectra have the form²⁾

$$\psi_{n00}(r) = \frac{1}{\sqrt{4\pi}} R_{n0}(0) \exp\left\{-\frac{\gamma}{n} r\right\} F\left(-n+1, 2, \frac{\gamma}{n} 2r\right) \quad (17)$$

and

$$\psi_{k00}(r) = \frac{1}{\sqrt{4\pi}} C_{k0} \exp\left\{-i \frac{\gamma}{z} r\right\} F\left(iz+1, 2, i \frac{\gamma}{z} 2r\right), \quad (18)$$

where

$$\gamma = Z\alpha m, \quad (19)$$

$$z = \frac{\gamma}{k}, \quad (20)$$

k is the wave vector of states from the continuous spectrum, $F(\alpha, \beta, x)$ the confluent hypergeometric function, which is equal to unity at zero values of x , and the normalization constant C_{k0} is specified by the following relation:

$$(C_{k0})^2 = [R_{10}(0)]^2 \frac{2\pi}{\gamma z [1 - \exp(-2\pi z)]}. \quad (21)$$

Comparison of (17) and (18) readily shows that the above wave functions differ only by the normalization condition and the substitution

$$n \rightarrow -iz. \quad (22)$$

In terms of the dimensionless variable z the formulas for the continuous spectrum become less cumbersome.

The contribution of the continuous spectrum, obtained via an analytic continuation of (13), yields a cubed logarithm.¹⁵ The expression for the cubic contribution has the form

$$\delta E_L^{\text{cube}}(ns) = -\frac{128\alpha^2 (Z\alpha)^6 m c^2}{9\pi^2 n^3} \times \int_0^\infty \frac{dz z \log^2 z}{[1 + (z/n)^2][1 - \exp(-2\pi z)]}. \quad (23)$$

Two logarithms originate from the matrix elements [cf. Eq. (12)], and the third appears as a result of logarithmic integration over momenta. In view of this, the contribution is provided by the intermediate states with nonrelativistic wave numbers k substantially larger than the characteristic atomic momenta:

$$\gamma \ll k \ll m, \quad \text{or} \quad \frac{\gamma}{m} \ll z \ll 1.$$

Hence, the argument of the logarithm in the matrix element is the dimensionless momentum k , rather than γ , as in the case with the discrete spectrum [see Eq. (12)].

The leading contribution in (23) contains the cube of the low-energy logarithm.¹⁵ To obtain it we must expand the integrand in the logarithmic range $\gamma/m \ll z \ll 1$:

$$\frac{z}{[1 + (z/n)^2][1 - \exp(-2\pi z)]} \approx \frac{1}{2\pi} + \frac{1}{2z}. \quad (24)$$

The first term on the right-hand side leads to a nonlogarithmic contribution of order $\alpha^2 (Z\alpha)^5 m c^2$ (see Ref. 14), and the second, after logarithmic integration over momenta, leads to the triple-logarithmic contribution¹⁵

$$\delta E_L^{\text{cube}}(ns) = -\frac{64\alpha^2 (Z\alpha)^6 m c^2}{27\pi^2 n^3} \log^3 \frac{1}{Z\alpha}. \quad (25)$$

Clearly, states with nonzero values of the orbital angular momentum l correspond to nonlogarithmic matrix elements, and in this case there can be neither the cube nor the square of the logarithm.

Obtaining the double-logarithmic contributions to the energy of the s -levels is somewhat difficult and cannot be directly reduced to calculating the analytic continuation discussed above. It has proved convenient to seek directly the corresponding contribution to the difference

$$[\delta E_L(1s_{1/2}) - 8\delta E_L(2s_{1/2})]^{(c)} = -\int_0^\infty \frac{dk}{2\pi} \left[\frac{|\langle 1s | \Sigma_C(E_1) | ks \rangle|^2}{\gamma^2/2m + k^2/2m} - \frac{(\psi_{100}(0))^2 |\langle ns | \Sigma_C(E_n) | ks \rangle|^2}{(\psi_{n00}(0))^2 \gamma^2/2mn^2 + k^2/2m} \right]. \quad (26)$$

The integrand can be conveniently transformed into

$$\frac{|\langle 1s | \Sigma_C(E_1) | ks \rangle|^2 - (\psi_{100}(0)/\psi_{n00}(0) \text{bigr})^2 |\langle ns | \Sigma_C(E_n) | ks \rangle|^2}{\gamma^2/2mn^2 + k^2/2m} + |\langle ns | \Sigma_C(E_1) | ks \rangle|^2 \left(\frac{1}{\gamma^2/2m + k^2/2m} - \frac{1}{\gamma^2/2mn^2 + k^2/2m} \right). \quad (27)$$

Obviously, the double logarithmic contribution can originate from two sources. First, the square of the logarithm can be contained in the low-momentum ($k \approx \gamma$) contribution with the logarithmic matrix elements. Second, such corrections appear as a result of logarithmic integration ($\gamma \ll k \ll m$) that allows for the nonlogarithmic part of one of the matrix elements. Partitioning the integrand in the form (27) separates these two contributions. Clearly, the first term in (27) does not lead to contributions of the orders discussed in this paper.

Let us consider the second term. The difference in the energy denominators leads to a situation in which only the nonrelativistic energy range, $k \approx \gamma$, is essential for the integral and, hence, the matrix elements can be found to within logarithmic accuracy [see Eq. (12)]. As a result we get (cf. Ref. 15)

$$[\delta E_L(1s_{1/2}) - 8\delta E_L(2s_{1/2})]^{(c)} = -\frac{128\alpha^2(Z\alpha)^6 mc^2 I}{9\pi^2} \log^2 \frac{1}{Z\alpha}, \quad (28)$$

where

$$I = -\int_0^\infty dz \left(\frac{z}{1+z^2} - \frac{z}{2^2+z^2} \right) \left(1 + \frac{1}{\exp(2\pi z) - 1} \right). \quad (29)$$

This integral can easily be evaluated (see formula 3.415.1 in Ref. 18):

$$I = -\frac{1}{2}(\log 2 + \frac{3}{4}). \quad (30)$$

Thus, the contribution of the double-logarithmic corrections to (1) amounts to

$$\delta E_L(1s_{1/2}) - 8\delta E_L(2s_{1/2}) = -\frac{64\alpha^2(Z\alpha)^6 mc^2}{9\pi^2} \log^2 \frac{1}{Z\alpha} \left(\frac{3}{2} - \log 2 \right), \quad (31)$$

or -14.2 kHz.

4. THE LIFETIME OF THE $2p_{1/2}$ -level and the Lamb splitting for $n=2$ in the hydrogen atom

The most exact value of the Lamb splitting $E(2s_{1/2}) - E(2p_{1/2})$ in the hydrogen atom was obtained by Sokolov and Yakovlev,⁶ who directly measured the ratio of the width of the $2p_{1/2}$ -level to the Lamb splitting. It is known (see, e.g. Refs. 19 and 20) that the level width is determined to the first approximation by the dipole radiation:

$$\Gamma_0 = \frac{4\omega^3}{3} |\mathbf{d}_{12}|^2 = \frac{2^{10}\pi}{3^8} \alpha(Z\alpha)^2 Z^2 \text{Ry} \frac{1}{1+m/M}, \quad (32)$$

where ω is the transition frequency, and \mathbf{d}_{12} the dipole matrix element corresponding to this transition.

The relativistic corrections of the relative order $(Z\alpha)^2$ to (32) were found directly in Ref. 6. Allowing for the high accuracy of the measurements, we must also consider second-order corrections in α . These can be found by calculating the imaginary part of (10). A simpler approach consists in examining the QED corrections to the dipole formula. Pal'chikov, Sokolov, and Yakovlev⁷ found some of these corrections. The result is^{6,7}

$$\Gamma = \Gamma_0 \left\{ 1 + (Z\alpha)^2 \left[\log \frac{9}{8} - \frac{32\alpha}{3\pi} \left(\log \frac{1}{(Z\alpha)^2} - \log k_0(1,0) + \frac{1}{8} \log k_0(2,1) + \frac{1}{64} + \frac{19}{30} \right) \right] \right\}. \quad (33)$$

Obviously, the corrections to (32) are corrections to the transition frequency (allowance for the Lamb shift of the initial and final states) and corrections to the dipole matrix elements. It can easily be verified that in (33) we allowed only for the corrections to the transition energy; we must also examine the matrix elements.

The corrections to the energy levels contain the Lamb logarithm $\log Z\alpha$. In what follows we consider only logarithmic contributions. Within the Fried-Yennie gauge¹⁶ for radiation photons such contributions can appear only if we allow for the correction to the wave function of the final state ($1s_{1/2}$), while the corrections to the initial state and radiation operator contain no logarithm.

The correction to the wave function of the $1s_{1/2}$ -state has the obvious form

$$\delta\psi_{1s}(\mathbf{r}) = \sum_{q \neq 1} \psi_{qs}(\mathbf{r}) \frac{\langle qs | \Sigma_C^{(1)} | 1s \rangle}{E_{1s} - E_{qs}}; \quad (34)$$

note that the matrix elements of the one-loop self-energy of the electron in the Coulomb field of the nuclei were discussed earlier [see Eq. (12)]. It is also easy to obtain the correction to the level width:

$$\delta\Gamma_{wf} = \Gamma_0 \frac{4}{3\pi} \alpha(Z\alpha)^4 mc^2 \log \frac{1}{(Z\alpha)^2} \times \sum_{q \neq 1} \left(\frac{\psi_{qs}(0)}{\psi_{1s}(0)} \right)^2 \frac{2\mathcal{D}_q}{E_{1s} - E_{qs}}, \quad (35)$$

where the sum is taken over all the states of the discrete and continuous spectra, and the quantity

$$\mathcal{D}_q = \frac{\mathbf{d}_{q2} \psi_{1s}(0)}{\mathbf{d}_{12} \psi_{qs}(0)} \quad (36)$$

is normalized to unity for the ground state,

$$\mathcal{D}_n = \frac{3^4 n^6}{(n^2 - 4)^3} \left(\frac{n-2}{n+2} \right)^n \quad (37)$$

for $n \neq 2$, and

$$\mathcal{D}_2 = \frac{3^6}{2^6}. \quad (38)$$

The dipole matrix element between the s -state of the continuous spectrum and the $2p_{1/2}$ -level is obtained via analytic continuation.³⁾ Substituting the explicit values of the wave functions (21) and the expressions for the matrix elements readily yields

$$\delta\Gamma_{wf} = -\Gamma_0 \frac{16}{3\pi} \alpha(Z\alpha)^2 \log \frac{1}{(Z\alpha)^2} \times \left[\sum_{n=2}^{\infty} \frac{\mathcal{D}_n}{n(n^2-1)} + \int_0^{\infty} \frac{dz \mathcal{D}_z}{z(z^2+1)[1-\exp(-2\pi z)]} \right], \quad (39)$$

where

$$\mathcal{D}_z = \frac{3^4 z^6}{(z^2+4)^3} \exp \left\{ -2z \tan^{-1} \left(\frac{2}{z} \right) \right\}, \quad (40)$$

and the variable z was introduced earlier in Eq. (20).

Numerical integration and the summation in (11) yield the following result for the width:

$$\Gamma = \Gamma_0 \left[1 + (Z\alpha)^2 \left(\log \frac{9}{8} - \frac{16\alpha}{3\pi} \log \frac{1}{(Z\alpha)^2} (0.49158\dots) \right) \right]. \quad (41)$$

Here we have discarded the nonlogarithmic part of the correction to the transition frequency, since it amounts to only a fraction of the result and retaining it would mean exceeding the accuracy.

5. DISCUSSION

Let us go back to Eqs. (1) and (2). The ‘‘theoretical’’ terms in these equations are found from the following expressions [see Ref. 8 and Eqs. (8), (9), and (31)]:

$$\begin{aligned} \Delta E_L(1s_{1/2}) - 8\Delta E_L(2s_{1/2}) &= \frac{\alpha(Z\alpha)^4}{\pi} mc^2 \left(\frac{m_r}{m} \right)^3 \\ &\times \left\{ -\frac{4}{3} \log \frac{k_0(1s)}{k_0(2s)} \left[1 + 2Z \frac{m}{M} + Z^2 \left(\frac{m}{M} \right)^2 \right] \right\} + (Z\alpha)^2 \\ &\times \left[\left(4 \log 2 - \frac{197}{60} \right) \log \frac{1}{(Z\alpha)^2} \right. \\ &\times \left. \left\{ + \frac{4}{15} \log 2 + \frac{1}{140} + 0.865(21) \right\} \right] \\ &- \frac{7(Z\alpha)^5}{3\pi} mc^2 \left(\frac{m_r}{m} \right)^3 \left(\frac{m}{M} \right) \left(\frac{3}{2} - 2 \log 2 \right) \\ &- \frac{64\alpha^2(Z\alpha)^6 mc^2}{9\pi^2} \log^2 \frac{1}{Z\alpha} \left(\frac{3}{2} - \log 2 \right) \end{aligned} \quad (42)$$

and

$$\begin{aligned} \Delta E_L(2p_{1/2}) &= \frac{\alpha(Z\alpha)^4}{8\pi} mc^2 \left(\frac{m_r}{m} \right)^3 \left\{ -\frac{4}{3} \log k_0(2p) \right. \\ &\times \left[1 + 2Z \frac{m}{M} Z^2 \left(\frac{m}{M} \right)^2 \right] + (Z\alpha)^2 \left(\frac{103}{180} \right. \\ &\times \left. \log \frac{1}{(Z\alpha)^2} - \frac{9}{140} \right) - 0.936(14) \left. \right\} \\ &- \frac{7(Z\alpha)^5}{144\pi} mc^2 \left(\frac{m_r}{m} \right)^3 \left(\frac{m}{M} \right) \\ &- \frac{(Z\alpha)^4}{24} mc^2 \left(\frac{m_r}{m} \right)^2 \left[\frac{\alpha}{2\pi} - 0.3285 \left(\frac{\alpha}{\pi} \right)^2 \right. \\ &\left. + 1.18 \left(\frac{\alpha}{\pi} \right)^3 \right], \end{aligned} \quad (43)$$

where M is the mass of the nucleus. In units of frequency the corresponding results are

$$\Delta E_L(1s_{1/2}) - 8\Delta E_L(2s_{1/2}) = -187.236(11) \text{ MHz} \quad (44)$$

and

$$\Delta E_L(2p_{1/2}) = -12.8335(15) \text{ MHz}. \quad (45)$$

The errors in these expressions are determined by two uncalculated contributions⁴⁾ of order $(Z\alpha)^6(m/M)mc^2$ and $\alpha^2(Z\alpha)^6 mc^2 \log Z\alpha$, respectively. These contributions are estimated at 1 kHz for (45) and 8 kHz for (44).

As noted in Sec. 4, the best result for the Lamb shift for $n=2$ in hydrogen can be found in Ref. 6 if one knows with sufficient accuracy the lifetime of the $2p_{1/2}$ -level, which according to Eq. (41) amounts to

$$\tau_{2p_{1/2}} = 1.5961887(15) \times 10^{-9} \text{ s}; \quad (46)$$

the associated splitting is

$$L(2s_{1/2} - 2p_{1/2}) = 1057.8576(21) \text{ MHz}. \quad (47)$$

The ‘‘theoretical’’ part of the error of the given quantities is completely determined by the uncalculated nonlogarithmic radiative corrections of order $\alpha(Z\alpha)^2$. The logarithmic terms found above contribute -5.1×10^{-15} s to the width and

–2.91 kHz to the splitting. Accordingly, we estimate the constant contributions at 1.5×10^{-15} s and 1 kHz. The experiment described in Ref. 6 introduces an error of 1.9 kHz into (47).

The above result for the Lamb splitting agrees both with the results of other measurements:

$$L(2s_{1/2}-2p_{1/2}) = 1057.862(20) \text{ MHz} \quad (48)$$

according to Ref. 22 and

$$L(2s_{1/2}-2p_{1/2}) = 1057.845(9) \text{ MHz} \quad (49)$$

according to Ref. 23, and with results of theoretical calculations^{12,15,21}

$$L(2s_{1/2}-2p_{1/2}) = 1057.856(5) \text{ MHz}, \quad (50)$$

with the value of the charge proton radius taken from Ref. 24.

It is fairly easy to obtain the values of the Lamb shifts of the s -levels in the hydrogen atom specified by Eqs. (1) and (2):

$$\Delta E_L(2s_{1/2}) = 1045.0213(26) \text{ MHz}, \quad (51)$$

$$\Delta E_L(1s_{1/2}) = 8172.934(24) \text{ MHz}. \quad (52)$$

The result for the ground state has a higher degree of accuracy compared to the theoretical value^{8,12,15}

$$\Delta E_L(1s_{1/2}) = 8172.93(9) \text{ MHz} \quad (53)$$

and to the results of measurements done by Weitz, Schmidt-Kaler, and Hänsch,³

$$\Delta E_L(1s_{1/2}) = 8172.82(11) \text{ MHz}, \quad (54)$$

and those reported in Refs. 1 and 2,

$$\Delta E_L(1s_{1/2}) = 8172.804(83) \text{ MHz}. \quad (55)$$

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¹At present these corrections are known only partially.¹⁴

²Here we employ the notation of Ref. 17 for the wave functions and the relativistic system of units in which $\hbar = c = 1$.

³Cf. the expressions for the photoionization cross section in Ref. 19 and 20.

⁴The first is known²¹ only for the $n=2$ splitting.

¹T. Andreae, W. König *et al.*, Phys. Rev. Lett. **69**, 1923 (1992).

²F. Nez, M. D. Plimmer *et al.*, Phys. Rev. Lett. **69**, 2326 (1992).

³M. Weitz, F. Schmidt-Kaler, and T. W. Hänsch, Phys. Rev. Lett. **68**, 1120 (1992).

⁴S. G. Karshenboim, Yad. Fiz. **58**, in preparation (1995) [Phys. Atom. Nuclei **58** (1995)].

⁵W. A. Barker and F. N. Glover, Phys. Rev. **99**, 317 (1955).

⁶Yu. L. Sokolov and V. P. Yakovlev, Zh. Eksp. Teor. Fiz. **83**, 15 (1982) [Sov. Phys. JETP **56**, 7 (1982)].

⁷V. P. Pal'chikov, Yu. L. Sokolov, and V. P. Yakovlev, Pis'ma Zh. Eksp. Teor. Fiz. **38**, 347 (1983) [JETP Lett. **38**, 418 (1983)].

⁸J. Sapirstein and D. R. Yennie, in *Quantum Electrodynamics*, edited by T. Kinoshita, World Scientific, Singapore, 1990.

⁹P. J. Mohr, Phys. Rev. A **46**, 4421 (1992).

¹⁰P. J. Mohr, Ann. Phys. (N.Y.) **88**, 26, 51 (1974).

¹¹P. J. Mohr, Phys. Rev. Lett. **34**, 1050 (1975).

¹²K. Pachucki, Ann. Phys. (N.Y.) **226**, 1 (1993).

¹³R. Mills and N. Kroll Phys. Rev. **98**, 1489 (1955).

¹⁴M. I. Ėides, S. G. Karshenboim, and V. A. Shelyuto, Yad. Fiz. **57**, No. 4 (1994) [Phys. Atom. Nuclei **57**, No. 4 (1994)].

¹⁵S. G. Karshenboim, Zh. Eksp. Teor. Fiz. **103**, 1105 (1993) [JETP **76**, 541 (1993)].

¹⁶H. M. Fried and D. R. Yennie, Phys. Rev. **112**, 1391 (1958).

¹⁷L. D. Landau and E. M. Lifshitz, *Quantum Mechanics: Non-relativistic Theory*, 3rd ed., Pergamon Press, Oxford, 1977.

¹⁸I. S. Gradshteyn and I. M. Ryzhik, *Tables of Integrals, Series and Products*, 4th ed., Academic Press, New York (1966).

¹⁹H. A. Bethe and E. E. Salpeter, *Quantum Mechanics of One- and Two-Electron Systems*, Springer, Berlin, 1958.

²⁰V. B. Berestetskii, E. M. Lifshitz, and L. P. Pitaevskii, *Quantum Electrodynamics*, 2nd ed., Pergamon Press, Oxford, 1982.

²¹M. Doncheski, H. Grotch, and D. Owen, Phys. Rev. A **43**, 2152 (1991).

²²G. Newton, D. A. Andrews, and P. J. Unworth, Philos. Trans. R. Soc. London **290**, 373 (1979).

²³S. R. Lundeen and F. M. Pipkin, Phys. Rev. Lett. **46**, 232 (1981); Metrologia **22**, 9 (1986).

²⁴L. Hand, D. I. Miller, and R. Willson, Phys. **35**, 335 (1963).

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