Self-organization of the critical state in Josephson lattices and granulated superconductors

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A number of models of a Josephson medium and granulated superconductors are studied. It is shown that an important parameter is the quantity $V \sim j_c a^3/\Phi_0$, where j_c is the Josephson-current density, a is the granule size, and Φ_0 is the quantum of flux. In the limit $V \gg 1$ the continuum approximation is inapplicable. In this case the Josephson medium is transformed into a system in which pinning is realized on elementary loops that incorporate Josephson junctions. Here, nonlinear properties of these junctions obtain. The equations obtained for the currents of the Josephson lattice are identical to the standard formulation in the problem of self-organized criticality,[1] while in granulated superconductors a problem of self-organized criticality with a different symmetry arises—a problem not of sites, but of loops. From the point of view of the critical state in granulated superconductors the concept of self-organized criticality radically changes the entire customary picture. The usual equations of the critical state describe only the average values of the magnetic field in the hydrodynamic approximation. However, it follows from the concept of self-organized criticality that the critical state has an extremely complicated structure, much more complicated than that which follows from the equation of the critical state. In particular, the fluctuations of various quantities in the critical state are much stronger than the ordinary statistical fluctuations, since there are large-scale fluctuations of the currents and fields, with a power-law (scaling) behavior that extends up to scales of the order of the size of the system, as in a turbulent medium. If there is current flow (e.g., in the presence of a weak electric field), this flow will be turbulent with large-scale fluctuations of the currents and fields in space and time. To all appearances, the problem of self-organized criticality is just as universal as the problem of fractality. On the other hand, the basic equations in it reflect all the features of pinning—hysteresis and threshold behavior. Therefore, the self-organization of the critical state of a superconductor is a natural realization of this extremely general problem.

1. INTRODUCTION

Recently, great interest has arisen in the problem of self-organized criticality (see, e.g., Refs. 1–5). This concept is applied to a very large class of dissipative dynamical systems. In the course of their evolution these systems arrive at a critical state which, in the subsequent evolution, is self-sustaining and does not require exact tuning of any external parameters for its existence. Therefore, it is called a self-organizing critical state, in contrast to the usual critical state that arises in, e.g., phase transitions. The concept of self-organized criticality is extremely general, and has been applied to such diverse spheres of science as geophysics, economics, astrophysics, and the physics of condensed matter.

However, not unexpectedly in view of such universality, almost all the information on self-organized criticality has been obtained by computer modeling. Experimental data, at least in condensed-matter physics, are available only in measurements on a sandpile.[6] Naturally, we should like to have more-interesting physical systems with self-organized criticality. It would appear that a natural candidate here is charge-density waves, in the study of which the concept of self-organized criticality first arose.[7] However, recent very detailed papers devoted to the study of charge-density waves (see, e.g., Refs. 8 and 9) have shown that they do not exhibit self-organized criticality, but an ordinary phase transition of the pinning-depinning type, associated with a precise tuning of the parameters.

On the other hand, the phenomenon of the critical state of a hard superconductor of the second kind has been known for a long time (see, e.g., Refs. 10 and 11). The critical state itself arises as a result of the dynamics of the fields or currents, and is subsequently also self-sustaining. In this sense it is very similar to systems with self-organized criticality, but in the standard theory the critical state does not possess the spectrum of fluctuations that are inherent to self-organized criticality. This is due to the continuum character of the equations describing the critical state. Recently it has been shown that if one takes into account the discreteness of the pinning centers the standard critical state possesses the properties of self-organized criticality.[12] Thus, it is clear that the critical state in superconductors in the presence of internal discrete structures of some kind is an entirely natural candidate for self-organized criticality. In such systems the actual critical state and spectrum of fluctuations inherent to self-organized criticality should be realized.

In the present paper we shall consider a number of systems of this kind—namely, ordered Josephson lattices and

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ordered granulated superconductors. These two classes of systems are described mathematically by very similar equations, and therefore we shall consider them together. The very important parameter V appears in all these equations. For $V \ll 1$ we can go over to the continuum limit, and the problem reduces to well known equations. We shall consider the opposite case $V \gg 1$. In this case it is found that in the system there are a large number of metastable states, the existence of which leads to the appearance of a critical state in the system, while transitions between these states create a spectrum of fluctuations that are characteristic for self-organized criticality.

2. TWO-DIMENSIONAL LATTICE OF JOSEPHSON JUNCTIONS

We shall consider first a rather simple model—a two-dimensional lattice of Josephson junctions, introduce all the equations for this model and analyze them in detail, and then show how to generalize the results obtained to various models of granulated superconductors.

Suppose that we have a large two-dimensional Josephson junction of size $L \times L$, lying in the xy plane. A Josephson current flows along the z axis. Let the phase in the first superconductor be φ_1 and let the phase in the second be φ_2. Then for the gauge-invariant phase difference $\varphi = \varphi_1 - \varphi_2$ we have in the resistive model the following equation:[13,14]

$$j_0 \sin \varphi + \frac{\Phi_0}{2\pi\rho_0} \frac{\partial \varphi}{\partial t} = \frac{\Phi_0}{16\pi^2 \lambda_L} \Delta \varphi + j_1, \tag{1}$$

where Δ is the two-dimensional Laplacian, j_0 is the Josephson-current density, j_1 is the injection-current density, Φ_0 is the flux quantum, λ_L is the London penetration depth, and ρ_0 is the surface resistance of the junction. Now let $j_0(x)$ be nonzero and equal to j_M for

$$an - \frac{l}{2} < x < an + \frac{l}{2}, \quad am - \frac{l}{2} < y < am + \frac{l}{2}, \tag{2}$$

with $l \ll a$. Thus, we obtain a multi-junction squid, with junctions of size l^2 at the sites of a square lattice with lattice constant a. We shall call this system a two-dimensional Josephson lattice. For $l \ll a$ it is easy to derive from (1) the following equations for $\varphi_{n,m} = \varphi(an,am)$:

$$j_M \sin \varphi_{n,m} + \frac{\Phi_0}{2\pi\rho_0} \frac{d\varphi_{n,m}}{dt} = \frac{j_\varphi}{2\pi} \Delta_{n,m}(\varphi) + j_{1,n,m},$$

$$j_\varphi = \frac{\Phi_0}{8\pi l^2 \lambda_L},$$

$$\Delta_{n,m}(\varphi) = \varphi_{n+1,m} + \varphi_{n-1,m} + \varphi_{n,m+1} + \varphi_{n,m-1} - 4\varphi_{nm}. \tag{3}$$

Equation (3) is the discrete analog of (1), and it is necessary only to convince oneself of the correctness of the coefficient of $\Delta_{n,m}$. This is easily done as follows. We pass to the continuum limit in (3); then, obviously,

$$\Delta_{n,m}(\varphi) \to a^2 \Delta \varphi,$$

$$\frac{j_\varphi}{2\pi} \Delta_{n,m} \to \frac{\Phi_0}{16\pi^2 \Lambda_L} \frac{a^2}{l^2} \Delta \varphi. \tag{4}$$

The factor a^2/l^2 in (4) appears because in the continuum limit in (3) j_0 and ρ_0 in (1) must be replaced by

$$\langle j_0 \rangle = j_M \frac{l^2}{a^2}, \quad \langle \rho_0 \rangle = \rho_0 \frac{a^2}{l^2}. \tag{5}$$

It is convenient to represent (3) in the form

$$V \sin \varphi_{n,m} + \tau \frac{d\varphi_{n,m}}{dt} = \Delta_{n,m}(\varphi) + 2\pi F_{n,m},$$

$$V = 2\pi \frac{j_M}{j_\varphi}, \quad F_{n,m} = \frac{j_{1n,m}}{j_\varphi},$$

$$\tau = V\tau_0, \quad \tau_0 = \frac{\Phi_0}{2\pi\rho_0 j_M}. \tag{6}$$

In (6) we have introduced the very important dimensionless quantity V, and we shall assume that $V \gg 1$. It is this parameter that determines the presence of a large number of metastable states at the sites. In fact, if we separate the term with $\varphi_{n,m}$ on the right-hand side of (6), we obtain

$$V \sin \varphi_{n,m} + \tau \frac{d\varphi_{n,m}}{dt} + 4\varphi_{n,m} = \varphi_0,$$

$$\varphi_0 = 2\pi F_{n,m} + \varphi_{n+1,m} + \varphi_{n-1,m} + \varphi_{n,m+1} + \varphi_{n,m-1}. \tag{7}$$

We shall assume that φ_0 is some constant. Then (7) is the equation of a one-junction squid, and has been studied by many authors. It has been shown[13,14] that for $V \gg 1$ the energy of the squid has many (of order V) metastable states, and it is this fact which leads to the existence of such important phenomena as flux quantization and hysteresis. It is easy to see that in the complete equation (6) these properties are also preserved, and indeed determine all the interesting phenomena in our system.

We shall consider first the static properties of Eq. (6) for $F_{n,m} = 0$. For simplicity we shall consider quasi-one-dimensional solutions, i.e., we shall assume that $\varphi_{n,m}$ does not depend on m. Then from (6) we obtain

$$V \sin \varphi_n = \varphi_{n+1} + \varphi_{n-1} - 2\varphi_n. \tag{8}$$

This equation was considered in, e.g., Ref. 15. It is easy to see that for $V \gg 1$ the following solution holds

$$\varphi_n = 2\pi K \frac{n(n+1)}{2} + \delta,$$

$$K = \mathrm{Int} \frac{V}{2\pi},$$

$$\Psi = \arcsin \frac{2\pi K}{V} \approx \frac{\pi}{2} - \sqrt{2\left(1 - \frac{2\pi K}{V}\right)},$$

$$f_{n+1/2} = \frac{1}{2\pi} (\varphi_{n+1} - \varphi_n) = (n+1)K. \tag{9}$$

It is easy to show that $f_{n+1/2}$ is the total number of flux quanta through a loop bounded by two junctions and two superconductors. It can be seen from (9) that the magnetic field $h(x_n)=h(an)\sim f_{n+1/2}$ between junctions increases linearly with distance. This implies that we have a critical state with a critical current $j_c\sim V$. It is easy to show that this is a strongly excited, metastable state.

We now return to (6). We introduce the following notation:

$$z_{n,m}(\varphi)=\frac{1}{2\pi}\Delta_{n,m}(\varphi)+F_{n,m},$$

$$z_c=\frac{V}{2\pi}. \tag{10}$$

The solution (9) corresponds to

$$z_{n,m}(\varphi)=K=\mathrm{Int}\, z_c. \tag{11}$$

In this notation Eq. (6) has the form

$$z_{n,m}(\varphi)=z_c\sin\varphi_{n,m}+\frac{\tau}{2\pi}\frac{d\varphi_{n,m}}{dt}. \tag{12}$$

It can be seen from (12) that in the static case we always have

$$z_{n,m}(\varphi)\leqslant z_c. \tag{13}$$

We now consider the dynamics. We assume that $F_{n,m}$ are integers (this is convenient for what follows). It is possible to start the analysis of the dynamics from the initial conditions $\varphi_{n,m}=0$, $F_{n,m}=0$, and then increase $F_{n,m}$. However, we shall proceed differently. We specify the initial conditions as follows:

$$F_{n,m}=K,\quad \varphi_{n,m}=\Psi,\quad \Delta_{n,m}(\varphi)=0, \tag{14}$$

where K and Ψ are defined in (9). It is not difficult to see that (14), like (9), is a solution of our equation, but has an entirely different structure. In (9) we have $F_{n,m}=0$, while in (14) we have $\Delta_{n,m}(\varphi)=0$. Below, it will be seen that in the general case any situation is possible; in particular, we can have $F_{n,m}\gg z_c$ and $|\Delta_{n,m}(\varphi)|\gg z_c$, $\Delta_{n,m}<0$, but then, naturally, the condition (13) is necessarily fulfilled, and then $F_{n,m}$ and $\Delta_{n,m}$ almost cancel each other. Since the term $\Delta_{n,m}(\varphi)$ involves the surface current of the junction, in the case $F_{n,m}\gg z_c$ a strong injection current cannot pass through the junction and flows over the surface of the junction. Here, naturally, the boundary conditions on the edges of the junction are very important. The situation described in (9) corresponds to the case when everything is determined by the external magnetic fields and there is no injection current.

We now return to the dynamics. We start from (14), and, at one site n_0,m_0, add one unit to F_{n_0,m_0} (the injection current can be freely varied in accordance with the wish of the experimenters). Then, at this site, we obtain $z_{n_0,m_0} = F_{n_0,m_0} = K + 1 > z_c$, since $\Delta_{n,m}(\varphi)=0$. It can be seen from (6) or (10) and (12) that now $d\varphi_{n_0,m_0}/dt \neq 0$. We must solve the system of equations (6) with $F_{n_0m_0} = K + 1$, $F_{nm}=K$ for $nm\neq n_0m_0$ and with initial conditions $\varphi_{n,m}=\Psi$. It is clear that in the general case the solution of this system is rather complicated. For $V\gg 1$, however, we can find an approximate solution. First of all, we note that when $\varphi_{n,m}$ changes by an amount of order π the term $V\sin\varphi_{n,m}$ changes by V, while the term $\Delta_{n,m}(\varphi)$ changes by an amount of order $\pi\ll V$. Furthermore, it is known[14] that for $V\gg 1$ the equation (12) for a single Josephson junction with a constant $z_{n,m}>z_c$ with the condition $z_{n,m}-z_c\ll z_c$ has a solution that varies slowly over a large time interval $T\gg\tau_0$, and then $\varphi_{n,m}$ changes by 2π over an interval τ_0. This enables us to solve our equations as follows. We shall assume that $\Delta_{n,m}(\varphi)$ and (consequently) $z_{n,m}(\varphi)$ are piecewise-continuous functions that change at the point at which $\varphi_{n,m}$ changes by 2π. Then we shall approximate $\varphi_{n,m}$ in $\Delta_{n,m}(\varphi)$ and in $z_{n,m}(\varphi)$ by the quantity $\pi/2+2\pi p_{n,m}$, with integer $p_{n,m}$; i.e., we set

$$\varphi_{n,m}\approx 2\pi p_{n,m}+\frac{\pi}{2},$$

$$\Delta_{n,m}(\varphi)\approx 2\pi\Delta_{n,m}(p),$$

$$z_{n,m}(\varphi)\approx\Delta_{n,m}(p)+F_{n,m}. \tag{15}$$

Since $\Delta_{n,m}(p)$ is an integer and we assumed that $F_{n,m}$ is also an integer, $z_{n,m}$ is an integer as well. Solving (12) for φ_{n_0,m_0} with $z_{n_0,m_0} = K + 1$ and the condition $\varphi_{n_0,m_0}(0) = \Psi$, we obtain

$$\varphi_{n_0,m_0}(t)=2\arctan\left\{1+\sqrt{2(\alpha_1-1)}\times\tan\frac{\sqrt{\alpha_1-1}(t+t_0)}{\sqrt{2}\tau_0}\right\},$$

$$t_0=\frac{\sqrt{2}\tau_0}{\sqrt{\alpha_1-1}}\arctan\left\{\frac{\tan\left[\frac{\pi}{4}-\frac{\sqrt{1-\alpha_0}}{\sqrt{2}}\right]-1}{\sqrt{2(\alpha_1-1)}}\right\},$$

$$\alpha_1=\frac{2\pi(K+1)}{V}>1,\quad \alpha_0=\frac{2\pi K}{V}<1,\quad t_0<0, \tag{16}$$

where τ_0 is defined in (6). Since, as we have already said, the solution (16) varies slowly at first and then changes rapidly by 2π, it is convenient to take as the matching point the time t_1 at which $\varphi_{n_0,m_0}(t)$ changed by π and became equal to $3\pi/2$. We then obtain

$$t_1=-t_0+\frac{\sqrt{2}\tau_0}{\sqrt{\alpha_1-1}}\arctan\left\{\frac{\tan\frac{3\pi}{4}-1}{\sqrt{2(\alpha_1-1)}}\right\}. \tag{17}$$

It can be shown that in (17) it is necessary to choose the following branches of the arctangent (Arctan x is the principal branch of the arctangent):

$$\arctan x=\begin{cases}\mathrm{Arctan}\, x, & 1<x<\infty,\\ \mathrm{Arctan}\, x+\pi, & -\infty<x<1,\end{cases} \tag{18}$$

upon which we obtain

$$t_1=-t_0+\frac{T(\alpha_1)}{2}+\tau_0,$$

$$t_0=-\frac{T(\alpha_1)}{\pi}\arctan\left(\frac{1}{4}\sqrt{\frac{1-\alpha_0}{\alpha_1-1}}\right),$$

$$T(\alpha)=\frac{2\pi\tau_0}{\sqrt{2|\alpha-1|}},$$

$$t_1\approx T(\alpha_1),\quad 1-\alpha_0\gg\alpha_1-1. \tag{19}$$

At time t_1 the phase changed by π, but this change occurred not on the scale of $t_1\sim T(\alpha_1)$, but on the scale of τ_0. In exactly the same way, the further change of phase by π also occurs over a time of order $\tau_0\ll T(\alpha_1)$. Therefore, on a time scale of order T we can assume that the phase changed by 2π at $t=t_1$. This implies that at $t=t_1$ phase slippage occurred, and

$$p_{n_0,m_0}\to p_{n_0,m_0}+1,$$

$$\Delta_{n_0,m_0}(p)\to\Delta_{n_0,m_0}(p)-4,$$

$$\Delta_{n_0\pm1,m_0}(p)\to\Delta_{n_0\pm1,m_0}(p)+1,$$

$$\Delta_{n_0,m_0\pm1}(p)\to\Delta_{n_0,m_0\pm1}(p)+1,$$

$$z_{n_0,m_0}\to z_{n_0,m_0}-4,$$

$$z_{n_0\pm1,m_0}\to z_{n_0\pm1,m_0}+1,$$

$$z_{n_0,m_0\pm1}\to z_{n_0,m_0\pm1}+1. \tag{20}$$

First of all, we note that the rules for change of $z_{n,m}$ in (20) coincide completely with the corresponding rules in papers on self-organized criticality.[1–5] This implies that our model is indeed a system with self-organized criticality. We shall return to this question below.

Next, after the jump of p_{n_0,m_0} occurred, $z_{n,m}$ changed at five sites: n_0,m_0; $n_0\pm1,m_0$; $n_0,m_0\pm1$. At the sites $n_0\pm1,m_0$ and $n_0,m_0\pm1$, $z_{n,m}$ became equal to $K+1$, and the problem reduced to the previous one. At the site n_0,m_0, the quantity z_{n_0,m_0} became equal to $K-3$, and it was necessary for us to solve Eq. (12) with this z and the initial condition $\varphi(t_1)=3\pi/2$. For $t>t_1$ we obtain

$$\varphi_{n_0,m_0}(t)=2\arctan\left\{1-\sqrt{2(1-\alpha_2)}\times\coth\frac{\sqrt{2(1-\alpha_2)}(t-t_1+\tau_0)}{2\tau_0}\right\},$$

$$\alpha_2=\frac{2\pi(K-3)}{V}<1. \tag{21}$$

The expression (16) is true for $0<t<t_1$, while (21) is true for $t_1<t<\infty$. Collecting everything together, and using the expression for $T(\alpha)$ in (19), we obtain

$$\varphi_{n_0,m_0}(t)=2\arctan x(t),$$

$$x(t)=\begin{cases}1+\dfrac{2\pi\tau_0}{T(\alpha_1)}\tan\dfrac{\pi(t+t_0)}{T(\alpha_1)}, & 0<t<t_1,\\ 1-\dfrac{2\pi\tau_0}{T(\alpha_2)}\coth\dfrac{\pi(t-t_1+\tau_0)}{T(\alpha_2)}, & t_1<t<\infty,\end{cases}$$

$$t_1=-t_0+\frac{T(\alpha_1)}{2}+\tau_0,$$

$$t_0=-\frac{T(\alpha_1)}{\pi}\arctan\left[\frac{T(\alpha_1)}{T(\alpha_0)}\right],$$

$$\varphi_{n_0,m_0}(0)=\frac{\pi}{2}-\sqrt{2(1-\alpha_0)},$$

$$\varphi_{n_0,m_0}(\infty)=\frac{5\pi}{2}-\sqrt{2(1-\alpha_2)}. \tag{22}$$

Thus, Eq. (22) describes the transition from the equilibrium state $\varphi(0)$, by addition of unity to F_{n_0,m_0}, to a new equilibrium position $\varphi(\infty)$. Here, φ changes by almost 2π, while p changes by unity.

However, it can be seen from (20) that in this case the values of z (and, consequently, φ) for the nearest neighbors of the site n_0, m_0 also change. In our example, $z_{n,m}$ for the nearest neighbors became equal to $K+1$, and the problem reduced to the previous one. However, this does not always happen. In fact, below, either we shall add unity by means of F_{n_0,m_0} or it will be taken from neighbors and added to the site n_0,m_0, when z_{n_0,m_0} goes over from $K-3$ to $K-2$, and then to $K-1$ and K. After this the cycle will be closed. Thus, we have four equilibrium states $z_{n,m}^{(2,3,4,0)}$ and one nonequilibrium state $z_{n,m}^{(1)}$:

$$z_{n,m}^{(2,3,4,0)}=K-3,K-2,K-1,K,$$

$$z_{n,m}^{(1)}=K+1. \tag{23}$$

The following equilibrium phases correspond to the equilibrium states:

$$\varphi_{n,m}^{(i)}=2\pi p_{n,m}+\frac{\pi}{2}-\sqrt{2(1-\alpha_i)},$$

$$\alpha_i=\frac{2\pi z^{(i)}}{V},\quad i=0,2,3,4. \tag{24}$$

It is obvious that α_1 corresponds to the nonequilibrium situation.

We shall consider, e.g., the transition from the state with $z^{(2)}$ to the state with $z^{(3)}$. We need to solve Eq. (12) with $z_{n,m}=z^{(3)}$ and the initial value $\varphi_{n,m}^{(0)}$ that corresponds to $i=2$ in (24). This situation is obtained when z at the site n, m is increased by unity either as a result of addition to $F_{n,m}$ or as a result of a change of phase at a neighboring site by 2π. The solution has the form

$$\varphi_{n,m}(t)=2\ \arctan\left\{1-\frac{2\pi\tau_0}{T(\alpha_3)}\coth\frac{\pi(t+t_2)}{T(\alpha_3)}\right\},$$

$$t_2=\frac{T(\alpha_3)}{\pi}\arctan\left\{\frac{T(\alpha_3)}{T(\alpha_2)}\right\} \tag{25}$$

with corresponding expressions in the transition from $z^{(3)}$ to $z^{(4)}$ and from $z^{(4)}$ to $z^{(0)}$. These transitions differ sharply from the case of addition of unity to $z^{(0)}$, since the state with $z^{(1)}$ that arises in the latter case is unstable and goes over to that with $z^{(2)}$. We have already considered this case. In transitions of the form (25), $\varphi_{n,m}$ changes from $\varphi^{(2)}$ in (24) to $\varphi^{(3)}$, and so on, without change of p, and in these transitions $\varphi_{n,m}$ changes by an amount of order $\sqrt{1-\alpha_i}\ll 1$, in contrast to the transition from $z^{(0)}$ to $z^{(1)}$ and then to $z^{(2)}$, when $\varphi_{n,m}$ changes by 2π.

Thus, we have described a scheme of transitions through the states $2\to3\to4\to0\to1\to2$, which leads finally to a complete change of phase by 2π at one site.

We shall show now how to describe this motion over the whole lattice. We shall apply the idea of piecewise continuity of $z_{n,m}$ for the whole system. As already stated, $\varphi_{n,m}$ changes phase by an amount of order 2π over a time interval $\tau_0\ll T(\alpha_i)$, i.e., on our time scale, instantaneously. In order to reflect this fact in the mathematics, we define $p_{n,m}$ as follows:

$$p_{n,m}=\text{nint}\left\{\frac{\varphi_{n,m}}{2\pi}-\frac{1}{4}\right\}, \tag{26}$$

where nint x is the integer nearest to x. Then, taking into account all that has been said, together with (15), we can rewrite Eqs. (10) and (12) as follows:

$$\frac{\tau}{2\pi}\frac{d\varphi_{n,m}}{dt}+z_c\sin\varphi_{n,m}=z_{n,m},$$

$$z_{n,m}=\Delta_{n,m}(p)+F_{n,m}. \tag{27}$$

The system of equations (26) and (27) is, of course, much simpler than the initial system of equations (6) or (10), (12). The principal simplification in (26), (27) is that the "center of gravity" of the theory has been carried over from the phases $\varphi_{n,m}$ to the $p_{n,m}$ and $z_{n,m}$, which are related to each other. The phases here change only at their own sites and do not interact with each other.

However, these equations are still rather complicated for analysis. This is because in the theory we have five large times $T(\alpha_i)$, and, therefore, on each site the phases move on their own time scales, and, in fact, their motion does not reduce to the solutions (22) and (25), since in the injection time the $z_{n,m}$ can change discontinuously.

Therefore, we shall make one further simplification, which, without changing the physical picture, makes it possible to eliminate the phase from the analysis completely. Let

$$\alpha_1-1\ll 1-\alpha_i,$$

$$T(\alpha_1)\equiv T\gg T(\alpha_i),\quad i=0,2,3,4,$$

$$t_1=T. \tag{28}$$

In this case, as can be seen from (22) and (25), the process involving the transition of z from $z^{(1)}$ to $z^{(2)}$ takes a very long time, viz., $t_1=T$, while the other processes are fast and we shall assume that they occur instantaneously on the scale of the time T. We can now synchronize the times at all sites by setting

$$t_k=kT. \tag{28a}$$

After this we can forget about the motion of the phases on the sites and describe the motions in the language of $p_{n,m}$ and $z_{n,m}$.

Suppose that at time k on the site n, m a transition of $z_{n,m}$ from $z^{(0)}$ to $z^{(1)}$ has occurred; then at time $k+1$, $p_{n,m}$ increases by unity. If we recall that we have $z^{(1)}>z_c$, while $z^{(i)}<z_c$ holds for $i=0,2,3,4$, we have the following very simple closed system of equations:

$$p_{n,m}(k+1)=p_{n,m}(x)+\vartheta\{z_{n,m}(k)-z_c\},$$

$$z_{n,m}(k)=\Delta_{n,m}\{p(k)\}+F_{n,m}(k). \tag{29}$$

In (29) it is easy to eliminate $p_{n,m}$, and we then obtain a nonlinear, discrete diffusion equation with an external source:

$$\begin{aligned}z_{n,m}(k+1)=z_{n,m}(k)&+\vartheta\{z_{n+1,m}(k)-z_c\}\\&+\vartheta\{z_{n-1,m}(k)-z_c\}+\vartheta\{z_{n,m+1}(k)-z_c\}\\&+\vartheta\{z_{n,m-1}(k)-z_c\}-4\vartheta\{z_{n,m}(k)-z_c\}\\&+\xi_{n,m}(k),\end{aligned}$$

$$\xi_{n,m}(k)=F_{n,m}(k+1)-F_{n,m}(k). \tag{30}$$

It is easily verified that Eqs. (29) and (30) correspond completely to the rules (20), and therefore correspond to a system with self-organized criticality.

The entire behavior of our system is determined by two factors—the equations themselves and the boundary conditions. The standard situation considered in problems with self-organized criticality reduces to the fact that $\xi_{n,m}(k)$ at random sites and at random times is equal to unity, while for the rest of the time $\xi_{n,m}(k)=0$. Here, on the edges of the system, we impose the conditions $z_{n,m}=0$ if n and m are outside the system. The relationship of the various conditions to the physics of a discrete Josephson junction will be considered in Sec. 6.

It is important to note here that we have made two strongly simplifying assumptions, that $F_{n,m}$ and $\xi_{n,m}$ are integers and that $T(\alpha_1)$ is large compared with other times. Physically, it is entirely clear that these assumptions are of absolutely no importance. Nevertheless, it would be very interesting to perform computer modelling without making these assumptions and convince oneself that in this case the system belongs to the same universality class as the system described by Eqs. (29) and (30).

3. RELATIONSHIP BETWEEN THE NONLINEAR DIFFUSION EQUATION AND THE ELECTRODYNAMICS OF A JOSEPHSON LATTICE

In this section we shall examine our equations from another point of view. This will enable us subsequently to write all the necessary equations without derivation.

First of all, we write down the relationship of the phase to the voltage $U_{n,m}$ across the junction:

$$U_{n,m}(t)=-\frac{\Phi_0}{2\pi}\frac{\partial\varphi_{n,m}}{\partial t}. \tag{31}$$

If in a time T there is a phase advance equal to 2π, the average voltage across the junction during this time is equal to

$$\overline{U_{n,m}(t)}=\frac{1}{T}\int_0^T U_{n,m}(t)dt=\frac{\Phi_0}{T}. \tag{32}$$

We have already stated that if at time Tk we have $z_{n,m}<z_c$, then at time $T(k+1)$ the phase will change by 2π, whereas if $z_{n,m}<z_c$ the phase changes little and we can neglect this change. We denote

$$U_{n,m}\left(k+\frac{1}{2}\right)=\frac{1}{T}\int_{kT+T'}^{(k+1)T+T'}U_{n,m}(t)dt, \tag{33}$$

where $T'\ll T$, and T' has been introduced so that we can be sure that a phase jump of 2π has occurred. Then, obviously,

$$U_{n,m}\left(k+\frac{1}{2}\right)=-\frac{\Phi_0}{T}\,\vartheta\{z_{n,m}(k)-z_c\}. \tag{34}$$

Thus, if at $t=Tk$ we have $z_{n,m}>z_c$, then over the time interval $[Tk+T';T(k+1)+T']$ there will have been a voltage pulse with an average value given by (32). From (31)–(34) it is obvious that we also have

$$U_{n,m}\left(k+\frac{1}{2}\right)=-\frac{\Phi_0}{2\pi T}\{\varphi_{n,m}(k+1)-\varphi_{n,m}(k)\}$$

$$=-\frac{\Phi_0}{T}\{p_{n,m}(k+1)-p_{n,m}(k)\}. \tag{35}$$

Comparing (34) and (35) with (29), we see that the first equation (29) and Eq. (34) have a very simple physical meaning. They are simply the I-V characteristics of one junction, since $z_{n,m}$ is the current across the junction in units of j_φ. Next, if we substitute the expression for $U_{n,m}$ in terms of p_{nm} in (35) into the second equation (29), we obtain

$$z_{n,m}(k+1)-z_{n,m}(k)=-\frac{T}{\Phi_0}\Delta_{n,m}\left\{U\left(k+\frac{1}{2}\right)\right\}+\xi_{n,m}(k). \tag{36}$$

It is not difficult to see that Eqs. (36) are Maxwell equations, written on a discrete lattice and for a discrete time. We shall show this. First of all, we introduce two quantities $f^{(x)}_{n,m+1/2}(k)$ and $f^{(y)}_{n+1/2,m}(k)$, which have the meaning of the numbers of flux quanta across the gap between two junctions and two superconductors in the x or y directions. The meaning of this notation can be seen from Fig. 1. Then (36) can be represented in the form

$$f^{(y)}_{n+1/2,m}(k)-f^{(y)}_{n-1/2,m}(k)-f^{(x)}_{n,m+1/2}(k)+f^{(x)}_{n,m-1/2}(k)$$

$$=z_{n,m}(k)-F_{n,m}(k),$$

$$f^{(x)}_{n,m+1/2}(k+1)-f^{(x)}_{n,m+1/2}(k)=\frac{T}{\Phi_0}\left\{U_{n,m+1}\left(k+\frac{1}{2}\right)\right.$$

$$\left.-U_{n,m}\left(k+\frac{1}{2}\right)\right\},$$

$$f^{(y)}_{n+1/2,m}(k+1)-f^{(y)}_{n+1/2,m}(k)=-\frac{T}{\Phi_0}\left\{U_{n+1,m}\left(k+\frac{1}{2}\right)\right.$$

$$\left.-U_{n,m}\left(k+\frac{1}{2}\right)\right\}. \tag{37}$$

We now express the number f of flux quanta in terms of the magnetic field h, express the voltage U in terms of the electric field E, and express z in terms of the current j:

$$f^{(x,y)}_{n,m}(k)=\frac{2\lambda_L a}{\Phi_0}\,h^{(x,y)}_{n,m}(k),$$

$$U_{n,m}\left(k+\frac{1}{2}\right)=-2\lambda_L E_{n,m}\left(k+\frac{1}{2}\right),$$

$$z_{n,m}(k)=\frac{j_{n,m}(k)}{j_\varphi},\quad j_\varphi=\frac{\Phi_0}{8\pi l^2\lambda_L},\quad z_c=\frac{j_M}{j_\varphi}. \tag{38}$$

Then (34) and (37) will take the form

$$h^{(y)}_{n+1/2,m}(k)-h^{(y)}_{n-1/2,m}(k)-h^{(x)}_{n,m+1/2}(k)+h^{(x)}_{n,m-1/2}(k)$$

$$=4\pi\,\frac{l^2}{a}\,[j_{n,m}(k)-j^{(1)}_{n,m}(k)],$$

$$\frac{1}{T}\{h^{(x)}_{n,m+1/2}(k+1)-h^{(x)}_{n,m+1/2}(k)\}=-\frac{1}{a}\left\{E_{n,m+1}\right.$$

$$\left.\times\left(k+\frac{1}{2}\right)-E_{n,m}\left(k+\frac{1}{2}\right)\right\},$$

$$\frac{1}{T}\{h^{(y)}_{n+1/2,m}(k+1)-h^{(y)}_{n+1/2,m}(k)\}=\frac{1}{a}\left\{E_{n+1,m}\left(k+\frac{1}{2}\right)\right.$$

$$\left.-E_{n,m}\left(k+\frac{1}{2}\right)\right\},$$

$$E_{n,m}\left(k+\frac{1}{2}\right)=E_c\{\vartheta[j_{n,m}(k)-j_M]$$

$$-\vartheta[-j_{n,m}(k)-j_M]\}$$

$$E_c=\frac{\Phi_0}{2\lambda_L T},\quad F_{n,m}=\frac{j^{(1)}_{n,m}}{j_\varphi}. \tag{39}$$

The equation for $E_{n,m}$ in (39) differs from (34) in that in (39) a further term with a threshold at $-j_M$ has been added. Up to now, this term has not appeared in the theory only because we have considered all phenomena close to one threshold, whereas, in fact, in an accurate analysis a second threshold will necessarily arise. It can be seen from (39) that our equations are indeed discrete Maxwell equations with the I-V characteristic $E_{n,m}(j_{n,m})$ of one junction. If the change in the

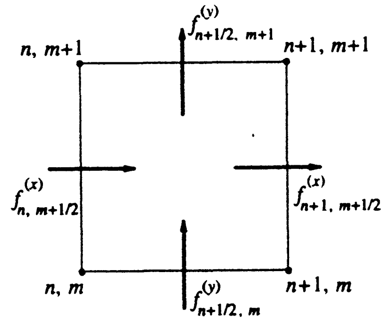

FIG. 1.

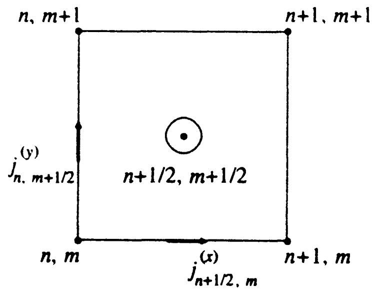

FIG. 2.

number of flux quanta is large in our processes, i.e., we rapidly change the injection currents, in (39) we can go over to the continuum limit; in doing this, we must take into account in E_c the dependence of T on j in (19), and replace $\sqrt{2j_c(j-j_c)}$ by $\sqrt{j^2-j_c^2}$ in the expression obtained from (19) in order that the formula apply near both thresholds. Then, replacing $\mathbf{h}_{n,m}$ by $\mathbf{h}(x,y)$, $\mathbf{E}_{n,m}$ by $\mathbf{E}(x,y)$, and $j_{n,m}$ by $(a^2/l^2)\mathbf{j}(x,y)$, we obtain

$$\frac{\partial \mathbf{h}}{\partial t}=-\operatorname{curl}\mathbf{E},\quad \operatorname{curl}\mathbf{h}=4\pi(\mathbf{j}-\mathbf{j}^{(1)}), \tag{40a}$$

$$\mathbf{E}=E\mathbf{e}_z,\quad \mathbf{h}=h_x\mathbf{e}_x+h_y\mathbf{e}_y,\quad \mathbf{j}=j\mathbf{e}_z,$$

$$E(j)=\rho\sqrt{j^2-j_c^2}[\vartheta(j-j_c)-\vartheta(-j-j_c)],$$

$$\rho=\frac{\rho_0 a^2}{2\lambda_L l^2},\quad j_c=j_M\frac{l^2}{a^2}, \tag{40b}$$

$$\frac{\partial \mathbf{j}}{\partial t}=\frac{\partial \mathbf{j}^{(1)}}{\partial t}+\frac{1}{4\pi}\Delta\mathbf{E}(\mathbf{j}). \tag{40c}$$

Equation (40c) corresponds to (36), if we take into account the dependences $T(j)$ and $E(j)$.

To conclude this section, we note the following. If we are working near threshold, i.e., with the condition

$$|j_{n,m}-j_M|\sim j_\varphi, \tag{41}$$

in (39) we must use for T its value in (28). But the continuum limit corresponds to the condition

$$j_\varphi \ll |j_{n,m}-j_M| \ll j_M, \tag{42}$$

and then the dependence $T(\alpha)$ reduces to (40).

Instead of the condition (41), the following condition arises quite often:

$$|j_{n,m}\pm j_M|\sim j_\varphi, \tag{43}$$

i.e., it is necessary to take both thresholds into account, and the continuum condition is not fulfilled. In this case it is necessary simply to use Eq. (36) but, instead of (34), to write

$$U_{n,m}\left(k+\frac{1}{2}\right)=-\frac{\Phi_0}{T}\{\vartheta[z_{n,m}(k)-z_c]-\vartheta[-z_{n,m}(k)-z_c]\}. \tag{44}$$

We note one further, extremely important fact. In the discrete equations (30), (34), (36), (37), (39), and (44), and the continuum equations (40), the phase has vanished from the theory, and only gauge-invariant quantities (the currents and fields) remain. Therefore, it is very easy to generalize these equations to more-complicated systems.

4. TWO-DIMENSIONAL MODEL OF A GRANULATED SUPERCONDUCTOR

We shall consider the following very simple model of a granulated superconductor. Suppose that we have a hollow superconducting system that is infinite along the z axis and, in the xy plane, is a square lattice with lattice constant a. The edges of this lattice have thickness l. At the middle of each edge is a junction with critical-current density j_M.

It is clear that in such a system the magnetic field $\mathbf{h}$ is along the z axis, and we shall denote the number of flux quanta across one cell by $f_{n+1/2,m+1/2}$. The current and voltage are vectors along the lattice edges. We denote them by

$$j^{(x)}_{n+1/2,m}=j_\varphi z^{(x)}_{n+1/2,m},\quad j^{(y)}_{n,m+1/2}=j_\varphi z^{(y)}_{n,m+1/2},$$

$$j_\varphi=\frac{\Phi_0}{4\pi l a^2},\quad U^{(x)}_{n+1/2,m},\quad U^{(y)}_{n,m+1/2}, \tag{45}$$

where the notation is clear from Fig. 2. In the middle of the square in Fig. 2 the current $f_{n+1/2,m+1/2}$ is depicted. We note that j_φ defined in (45) differs from j_φ in (3).

It is easy to show that if V, defined in (6) in terms of the j_φ in (45), is large, then, repeating almost verbatim all the calculations of Sec. 2, we obtain the following equations, which again are discrete Maxwell equations, as one can convince oneself by looking at Fig. 2:

$$\begin{aligned}
&f_{n+1/2,m+1/2}(k+1)-f_{n+1/2,m+1/2}(k)=\\
&\quad -\frac{T}{\Phi_0}\left\{U^{(x)}_{n+1/2,m+1}\left(k+\frac{1}{2}\right)+U^{(y)}_{n,m+1/2}\left(k+\frac{1}{2}\right)\right.\\
&\quad \left.-U^{(x)}_{n+1/2,m}\left(k+\frac{1}{2}\right)-U^{(y)}_{n+1,m+1/2}\left(k+\frac{1}{2}\right)\right\},\\
&f_{n+1/2,m+1/2}(k)-f_{n+1/2,m-1/2}(k)=z^{(x)}_{n+1/2,m}(k)\\
&\quad -F^{(x)}_{n+1/2,m}(k),\\
&f_{n+1/2,m+1/2}(k)-f_{n-1/2,m+1/2}(k)=-z^{(y)}_{n,m+1/2}(k)\\
&\quad +F^{(y)}_{n,m+1/2}(k),
\end{aligned} \tag{46}$$

while $U^{(x)}_{n+1/2,m}$ and $U^{(y)}_{n,m+1/2}$ are expressed in terms of $z^{(x)}_{n+1/2,m}$ and $z^{(y)}_{n,m+1/2}$ by the complete formula (44) or the

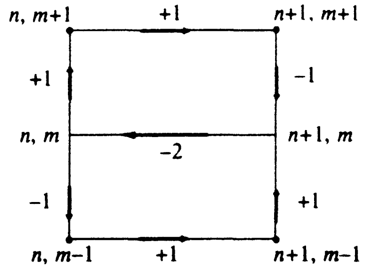

FIG. 3.

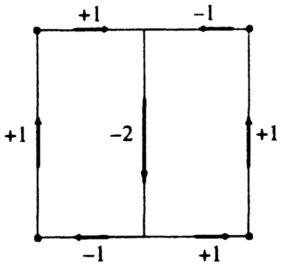

FIG. 4.

abridged formula (34). To go over to the fields h and E it is necessary to use in place of (38) the expressions

$$f_{n,m}=\frac{a^2}{\Phi_0}h_{n,m},\quad U_{n,m}^{(\alpha)}=-aE_{n,m}^{(\alpha)}. \tag{47}$$

The physical meaning of these changes is obvious. The flux now extends over an area a^2 instead of the area $2\lambda_L a$ in (38), while the potential drops over a length a, and not $2\lambda_L$. Substituting (47) into (46) and passing to the continuum limit, we obtain equations analogous to (40). The equations (40a) and (40c) are reproduced completely, but instead of (40b) we obtain

$$\mathbf{E}=E_x\mathbf{e}_x+E_y\mathbf{e}_y,\quad \mathbf{j}+j_x\mathbf{e}_x+j_y\mathbf{e}_y,\quad \mathbf{h}=h\mathbf{e}_z,$$
$$E_\alpha(j_\alpha)=\rho\sqrt{j_\alpha^2-j_c^2}\{\vartheta(j_\alpha-j_c)-\vartheta(-j_\alpha-j_c)\},$$
$$\alpha=x,y,\quad \rho=\frac{\rho_0}{l},\quad j_c=j_M\frac{l}{a}. \tag{48}$$

We now write out the diffusion equation for z in the region in which the condition (41) is fulfilled. Then from (34) and (46) we obtain, in place of (30),

$$\begin{aligned} z_{n+1/2,m}^{(x)}(k+1)-z_{n+1/2,m}^{(x)}(k)&=\xi_{n+1/2,m}^{(x)}(k)\\ &+\vartheta\{z_{n+1/2,m+1}^{(x)}(k)-z_c\}+\vartheta\{z_{n+1/2,m-1}^{(x)}(k)-z_c\}\\ &-2\vartheta\{z_{n+1/2,m}^{(x)}(k)-z_c\}+\vartheta\{z_{n,m+1/2}^{(y)}(k)-z_c\}\\ &+\vartheta\{z_{n+1,m-1/2}^{(y)}(k)-z_c\}-\vartheta\{z_{n+1,m+1/2}^{(y)}(k)-z_c\}\\ &-\vartheta\{z_{n,m-1/2}^{(y)}(k)-z_c\}, \end{aligned} \tag{49}$$

and an analogous expression for $z^{(y)}$, which is obtained from (49) by replacing $z_{n+1/2,m}^{(x)}$ by $z_{n,m+1/2}^{(y)}$ and vice versa. Equation (49) differs strongly from (30). It is easy to see that the right-hand side of (49) is the discrete analog not of the Laplacian but of the operator curlcurl $\mathbf{E}$. It is easy to see that in (30) grad div $\mathbf{E}=0$ holds because of the symmetry of the problem.

Whereas the expression (30) corresponded to the change algorithm (20), the expression (49) corresponds to the algorithm depicted in Fig. 3. In the latter we have drawn two current loops adjacent to the link along which the current $z_{n+1/2,m}^{(x)}$ flows, and in all other links of these loops the current changes in such a way that the total current is conserved. In the link $n+1/2,m$ itself the current changes by two units (the arrow along the negative x direction), while in the other links it increases by unity (+1, with the arrow along the positive axis) or decreases by unity (−1, with the arrow along the negative axis). Thus, two units of current from the link $n+1/2,m$ are dumped into two loops. Figure 4 depicts the same picture for a vertical link.

We see that, whereas in the preceding problem four units of current were dumped on to the four nearest neighbors, in the present case two units of current are dumped into the two nearest loops.

5. THREE-DIMENSIONAL MODEL OF A GRANULATED SUPERCONDUCTOR

Suppose that we have a three-dimensional superconducting network, forming a simple cubic lattice with lattice constant a. A section of each edge of this lattice is an $l\times l$ square, and in the middle of each edge there is a Josephson junction with critical-current density j_M.

All the results of the preceding section can be generalized without difficulty. First of all, we write out the expression for the I-V characteristic in the continuum approximation:

$$\mathbf{E}=\sum_{\alpha=x,y,z}E_\alpha(j_\alpha)\mathbf{e}_\alpha,\quad \mathbf{j}=\sum_{\alpha=x,y,z}j_\alpha\mathbf{e}_\alpha,$$
$$E_\alpha(j_\alpha)=\rho\sqrt{j_\alpha^2-j_c^2}\{\vartheta(j_\alpha-j_c)-\vartheta(-j_\alpha-j_c)\},$$
$$\rho=\frac{\rho_0 a}{l^2},\quad j_c=j_M\left(\frac{l}{a}\right)^2. \tag{50}$$

Equations (50), together with the Maxwell equations (40a), form a complete system of equations.

Near the threshold (41) we have, analogously to (45),

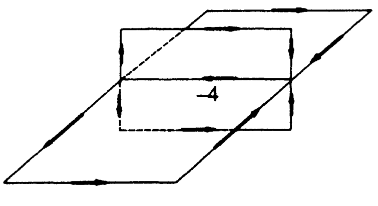

FIG. 5.

$$j^{(x)}_{n+1/2,m,l}=j_\varphi z^{(x)}_{n+1/2,m,l},$$
$$j^{(y)}_{n,m+1/2,l}=j_\varphi z^{(x)}_{n,m+1/2,l},$$
$$j^{(z)}_{n,m,l+1/2}=j_\varphi z^{(z)}_{n,m,l+1/2},$$
$$j_\varphi=\frac{\Phi_0}{4\pi l^2 a}. \tag{51}$$

Then for $z^{(\alpha)}_{n,l,m}$ we have a diffusion equation analogous to (49):

$$\begin{aligned}z^{(x)}_{n+1/2,m,l}(k+1)-z^{(x)}_{n+1/2,m,l}=\xi^{(x)}_{n+1/2,m,l}-4\vartheta\{z^{(x)}_{n+1/2,m,l}\\ \times(k)-z_c\}+\vartheta\{z^{(x)}_{n+1/2,m+1,l}(k)-z_c\}\\ +\vartheta\{z^{(x)}_{n+1/2,m-1,l}(k)-z_c\}+\vartheta\{z^{(x)}_{n+1/2,m,l+1}(k)\\ -z_c\}+\vartheta\{z^{(x)}_{n+1/2,m,l-1}(k)-z_c\}+\vartheta\{z^{(y)}_{n,m+1/2,l}(k)\\ -z_c\}+\vartheta\{z^{(y)}_{n+1,m-1/2,l}(k)-z_c\}-\vartheta\{z^{(y)}_{n+1,m+1/2,l}\\ \times(k)-z_c\}-\vartheta\{z^{(y)}_{n,m-1/2,l}(k)-z_c\}+\vartheta\{z^{(z)}_{n,m,l+1/2}\\ \times(k)-z_c\}+\vartheta\{z^{(z)}_{n+1,m,l-1/2}(k)-z_c\}\\ -\vartheta\{z^{(z)}_{n+1,m,l+1/2}(k)-z_c\}-\vartheta\{z^{(z)}_{n,m,l-1/2}(k)-z_c\}.\end{aligned} \tag{52}$$

The change of the currents that corresponds to (52) is depicted in Fig. 5, in which an increase of current is shown by an arrow parallel to the corresponding axis while a decrease of current is shown by an arrow antiparallel to the corresponding axis. Here, only on the central link is there a decrease by four units of current (this is indicated), while on the other links the change is equal to ± 1, and, in order not to encumber the figure, has not been indicated.

Thus, we see that in a three-dimensional system four units of current are dumped into four loops adjacent to our link. The same will be true for the currents $z^{(y)}_{n,m+1/2,l}$ and $z^{(z)}_{n,m,l+1/2}$ in the y and z links.

6. BOUNDARY CONDITIONS

It is well known[1–5] that in the problem of self-organized criticality a very important role is played by the boundary conditions. We shall consider various boundary conditions and their physical meaning for the simple example of the Josephson two-dimensional lattice considered in Sec. 2. There, m and n ran over values in the range $1\leqslant m,n\leqslant N$. We now add junctions with $m,n=0,N+1$, and assume that at these junctions the critical-current density is equal to infinity, i.e., instead of a Josephson junction we include at the points with $m,n=0,N+1$ a normal superconductor whose critical current is several orders of magnitude greater than the critical current of a junction. Then, from (6) and (10), it can be seen that

$$z_c=\infty,\quad n,m=0,N+1. \tag{53}$$

Then it can be seen from (30) that $z_{n,m}$ for n, $m=0$, $N+1$ can only increase and can never decrease. Furthermore, for sites on the edge we have, e.g., for $n=1$, $m\neq 1,N$ and for $n,m=1$,

$$\begin{aligned}z_{1,m}(k+1)=z_{1,m}(k)+\vartheta\{z_{2,m}(k)-z_c\}\\ +\vartheta\{z_{1,m+1}(k)-z_c\}+\vartheta\{z_{1,m-1}(k)-z_c\}\\ -4\vartheta\{z_{1,m}(k)-z_c\}+\xi_{1,m}(k),\\ z_{1,1}(k+1)=z_{1,1}(k)+\vartheta\{z_{2,1}(k)-z_c\}+\vartheta\{z_{1,2}(k)-z_c\}\\ -4\vartheta\{z_{1,1}(k)-z_c\}+\xi_{1,1}(k).\end{aligned} \tag{54}$$

In (54) the current is not conserved, since current is dumped on to the site $0,m$, and current does not return from this site. It is easy to see that exactly the same situation will be obtained if instead of the condition (53) we use the condition

$$z_{n,m}=0,\quad n,m=0,N+1. \tag{55}$$

In this case, from (30) we again obtain Eq. (54). However, the condition (53) is more physical, since the current fails to be conserved only on the sublattice $1\leqslant m,n\leqslant N$, but if we consider the complete lattice $0\leqslant m,n\leqslant N+1$ the total current is conserved. The condition (53) corresponds simply to shunting of our sublattice by a normal superconductor.

These boundary conditions (53) or (55) will be called open, since current can emerge from the sublattice with $1\leqslant m,n\leqslant N$.

Those boundary conditions under which current cannot emerge from the system will be called closed or reflecting conditions. Then, instead of (54), we obtain for $n=1$, $m\neq 1,N$ and for $n,m=1$

$$\begin{aligned}z_{1,m}(k+1)=z_{1,m}(k)+\vartheta\{z_{2,m}(k)-z_c\}+\vartheta\{z_{1,m+1}(k)\\ -z_c\}+\vartheta\{z_{1,m-1}(k)-z_c\}-3\vartheta\{z_{1,m}(k)\\ -z_c\}+\xi_{1,m}(k),\\ z_{1,1}(k+1)=z_{1,1}(k)+\vartheta\{z_{2,1}(k)-z_c\}+\vartheta\{z_{1,2}(k)-z_c\}\\ -2\vartheta\{z_{1,1}(k)-z_c\}+\xi_{1,1}(k),\end{aligned} \tag{56}$$

and analogously for $z_{1,N}$, $z_{N,1}$, and $z_{N,N}$. Unlike Eqs. (54), Eqs. (56) conserve current.

The boundary conditions can be different on different boundaries. It is not difficult to show that, if the conditions are closed or reflecting on all boundaries, i.e., of the form (56), then self-organized criticality cannot exist, and there is an ordinary phase transition in which the parameter is the total current. For self-organized criticality to exist it should be possible for current to escape from the system, i.e., the condition (53) or (55) should be fulfilled somewhere. This implies that a Josephson junction should be shunted at some point by a normal superconductor.

The conditions (55) and (56) have already been considered previously in other papers devoted to self-organized criticality. We now consider another condition that has not arisen previously, inasmuch as this condition does not appear in the analysis of a sandpile. We consider the situation when the current in the junction is excited not by an injection current but by an external magnetic field.

Suppose that the left, upper, and lower boundaries have open boundary conditions (55), while the right boundary has closed boundary conditions. We apply an external magnetic field $h_{ex}^{(y)}$ to the system. Since open boundary conditions correspond to shunting by an ordinary superconductor, which excludes a magnetic field, the field penetrates only to the right boundary.

From the first equation (37), taking into account that the tangential component of the magnetic field is continuous, we obtain

$$f_{ex}^{(y)}(k)-f_{N-1/2,m}^{(y)}(k)-f_{N,m+1/2}^{(x)}(k)+f_{N,m-1/2}^{(x)}(k) = z_{N,m}(k),$$

$$f_{ex}^{(y)}(k)=\frac{2\lambda_L a}{\Phi_0} h_{ex}^{(y)}(k). \tag{57}$$

Equation (57) differs from (37) in that we have set $F_{n,m}=0$ and replaced $f_{N+1/2,m}^{(y)}(k)$ by the external field $f_{ex}^{(y)}(k)$. Substituting Eq. (57) now into the second equation (37), we obtain [taking (34) into account] Eqs. (56), but on the right rather than the left boundary, i.e., for $n=N$, $m\neq 1$, N and for $n,m=N$ and $n=N$, $m=1$. The difference from (56) is that $\xi_{n,m}(k)$ must be replaced by the analogous quantity

$$\xi_{N,m}(k)=f_{ex}^{(y)}(k+1)-f_{ex}^{(y)}(k), \tag{58}$$

which does not depend on m. In all other cases, $\xi_{n,m}=0$. Thus, we have obtained on this boundary the usual closed boundary conditions, and the external magnetic field has been reduced to a surface injection current.

We now consider another, very interesting case. We have already said that with current injection, in the case of reflecting boundaries, self-organized criticality does not exist. However, when a magnetic field is applied to a system with reflecting boundaries self-organized criticality is realized. Suppose that we apply a magnetic field $h_{ex}^{(y)}(k)$ to a system with reflecting boundaries. Then on the upper and lower boundary conditions of the type (56) hold, on the right boundary we have the conditions (57), and on the left boundary we have the equation

$$f_{3/2,m}^{(y)}(k)-f_{ex}^{(y)}-f_{1,m+1/2}^{(x)}(k)+f_{1,m-1/2}^{(x)}(k)=z_{1,m}(k). \tag{59}$$

Comparing (57) and (58), we see one very important difference between them. Whereas in (57) $f_{ex}^{(y)}(k)$ plays the role of a positive injection current, in (59) it plays the role of a negative one.

Thus, on the right the magnetic field injects a positive current, while on the left it injects exactly the same negative current. As a result, the total current is conserved and is equal to zero. We note that for $U_{k+1/2}$ it is absolutely necessary to use Eq. (44) in order to take both thresholds into account.

This formulation of the problem is completely equivalent to the usual equation of the critical state in superconductors.[10,11] As a result, in one half of the superconductor we have $j=j_c$, while in the other we have $j=-j_c$. In our case, on this is imposed a fluctuation spectrum that leads to self-organization. A detailed study of this case will be carried out in a separate publication. We note only that since the avalanches that accompany self-organized criticality cannot slide anywhere, because an avalanche current is reflected from the boundaries, they will be annihilated, and this annihilation will lead to very strong noise.

The boundary conditions for granulated superconductors can also be constructed in an analogous way, but we shall not do this here.

7. CONCLUSION

In the present paper we have shown that the equations describing Josephson lattices coincide with the standard equations of self-organized criticality. Furthermore, we have derived the equations describing granulated superconductors, and these equations, to all appearances, also possess self-organization properties, which we subsequently investigate. In addition, we have written down equations describing systems with reflecting boundaries in a magnetic field, which define an entirely new class of systems with self-organized criticality in which instead of current discharge current annihilation occurs.

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