Quantum sensitivity limit for detectors of weak forces between free masses

A. V. Syrtsev and F. Ya. Khalili

M. V. Lomonosov State University, 119899 Moscow, Russia
(Submitted 23 May 1994)
Zh. Eksp. Teor. Fiz. **106**, 744–752 (September 1994)

The problem of detecting a classical force acting on a quantum probe system is considered. It is shown that correlation of the noise in the position-measuring device of the quantum probe system is equivalent to a modification of its dynamic properties (in the case of frequency-independent noise this modification consists of the introduction of stiffness). Detection methods which make it possible to transcend the Standard Quantum Limit within the framework of displacement measurements are proposed.

1. INTRODUCTION

More than twenty-five years ago, Braginskiĭ[1] pointed out the existence of the so-called standard quantum limit (SQL) of sensitivity in the detection of the action of a classical force $F(t)$ on a quantum test system. SQL is a direct consequence of the Heisenberg uncertainty relation: the measuring device tracking the position of the test system perturbs its momentum, which masks the action of the force that one is trying to detect. The higher the tracking accuracy, the greater the perturbation; obviously, there exists some optimal accuracy at which the sensitivity of the system is maximum and corresponds to the SQL. For example, for a free mass m the SQL has the form

$$F_{\rm SQL}=\xi\sqrt{\frac{\hbar m}{\tau_F^3}}, \qquad (1)$$

and for a harmonic oscillator

$$F_{\rm SQL}=\frac{\xi}{\tau_F}\sqrt{\hbar m\omega_m}, \qquad (2)$$

where m and ω_m are the mass and natural frequency of the oscillator, τ_F is the time during which the force acts, and ξ is a coefficient of the order of unity which depends on the type of force.

Since the publication of Ref. 1, some methods for getting around the SQL have been proposed (see, for example, Ref. 2, a review), which can be divided into two groups. The first assumes the use of measuring devices which react not to the position of the test system, but to some integral of its motion (to the energy[3] or the quadrature component[4] of the oscillator, or to the momentum of the free mass[5]). The perturbation of the canonically conjugate observable (e.g., the position when tracking the momentum) has no effect on the output signal of the measuring device and for this reason does not affect the sensitivity. The application of such methods is hindered, however, by the complexity of the practical realization of the corresponding measuring schemes.

At the same time, there is no fundamental prohibition against obtaining a sensitivity greater than the SQL, even within the framework of ordinary position measurements. The measuring device in this case should track not the instantaneous position of the test system, but some linear functional of the coordinate, the values of which commute at different moments of time. As an example, one can cite the stroboscopic measurement scheme proposed in Ref. 6, consisting of short, periodic measurements of the position of a test oscillator with intervening interval equal to half the period of its natural oscillations.

In Ref. 7 it was shown that for an arbitrary linear test system there exists a sequence of position measurements for which the sensitivity to the external force is bounded only by dissipation in the test system, and can substantially exceed the SQL. The problem consists only in finding an explicit form of such a procedure. In principle this problem can be assumed to be solved for the case of detection of a force with nonzero mean value. As was shown in Ref. 8, when tracking the position of a free test mass or harmonic oscillator with low accuracy, but over a long period of time $\tau_m \gg \tau_F$, the sensitivity to such a force is not bounded by the values of the SQL given by formulas (1) and (2).

In the present paper we consider methods for detecting a force with zero mean value acting on a free test mass. It is specifically this formulation of the problem that is of interest in connection with the development and construction at the present time of large laser gravitational antennas.[2,9]

In Sec. 2 we present general relationships which describe the proposed detection system—the test mass and the device tracking its position—and we establish a correspondence between this scheme and a specific implementation: an optical interferometric position sensor.

In Sec. 3 we show that the noise correlation in the position-measuring device is equivalent to a small modification of the dynamic properties of the probe system (in the case of frequency-independent noise, to the introduction of additional stiffness).

In Secs. 4 and 5 we propose two new detection methods which make it possible to transcend the SQL in position measurements. A special feature of the first of these is the use of pumping having the form of periodic short pulses, and of the second, pumping with shot noise suppressed at low frequencies. Both methods require that correlation be introduced into the noise of the measuring device.

2. MODEL OF THE DETECTION SYSTEM

The proposed model is shown in Fig. 1. The external force $F(t)$ that one wants to detect acts on the test system, as

 1063-7761/94/090409-05$10.00

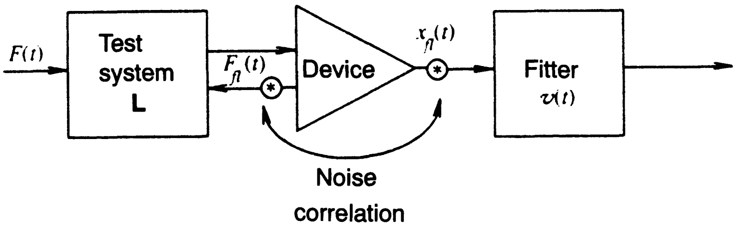

FIG. 1. Block diagram of a weak-force detector.

does a fluctuating force $F_{\rm fl}(t)$ that comes from the measuring device. The equation of motion for the position of the test system $x(t)$ in this case has the form

$$\mathbf{L}x(t)=F(t)+F_{\rm fl}(t), \tag{3}$$

where $\mathbf{L}$ is a differential operator which describes the dynamics of the test system. For example, for a free test mass m we have

$$\mathbf{L}=m\,\frac{d^2}{dt^2},$$

and for an oscillator with natural frequency ω_m

$$\mathbf{L}=m\left[\frac{d^2}{dt^2}+\omega_m^2\right].$$

The output signal of the measuring device is the sum of the position $x(t)=\mathbf{L}^{-1}F(t)$ and the fluctuating component introduced by the measuring device:

$$\tilde{x}_{\rm fl}(t)=x_{\rm fl}(t)+\mathbf{L}^{-1}F_{\rm fl}(t)=\mathbf{L}^{-1}\tilde{F}_{\rm fl}(t). \tag{4}$$

Although $x(t)$ and $\tilde{x}_{\rm fl}(t)$ are individually operator functions of time,[10] the output signal can be considered a classical observable in agreement with the ordinary rules of the classical theory of optimal detection.[11] The quantum character of the measuring device is manifested in the fact that the noises in $x_{\rm fl}(t)$ and $F_{\rm fl}(t)$ are uneliminable in principle, and their correlation functions are coupled by a relation which follows from the Heisenberg uncertainty relation.[10] In the general case the latter is quite complicated. We write out two special cases of this relation which we will have need of later on.

1) Stationary noise of the measuring device:

$$S_x(\omega)S_F(\omega)-S_{Fx}^2(\omega)\geq\frac{\hbar^2}{4}, \tag{5}$$

where $S_x(\omega)$ and $S_F(\omega)$ are the noise spectral densities of $x_{\rm fl}(t)$ and $F_{\rm fl}(t)$, and the cross-spectral density $S_{Fx}(\omega)$.

2) δ-correlated noise:

$$B_F(t,t')=B_F(t)\delta(t-t'),\quad B_x(t,t')=B_x(t)\delta(t-t'),$$

$$B_{Fx}(t,t')=B_{Fx}(t)\delta(t-t'),$$

where $B_x(t,t')$, $B_F(t,t')$, and $B_{Fx}(t,t')$ are the correlation functions of the noises; in this case

$$B_F(t)B_x(t)-B_{Fx}^2(t)\geq\frac{\hbar^2}{4}. \tag{6}$$

It is not hard to establish a correspondence between the considered noise parameters of an abstract coordinate-measuring device and the parameters of the optical interferometric sensor used in laser gravitational antennas. The force $F_{\rm fl}(t)$ is the fluctuating component of the pressure force of the probe light wave on the mirror whose position is being measured. The noise $x_{\rm fl}(t)$ corresponds to fluctuations of the in-phase (with the reference wave) quadrature amplitude of the probe wave. As was shown in Ref. 12, by varying the phase shift θ between the probe wave and the reference wave, it is possible to arbitrarily vary the correlation coefficient between the noises $x_{\rm fl}(t)$ and $F_{\rm fl}(t)$. In the simplest case, when the probe wave is in a coherent state and its power W is constant in time, the noise is stationary, and its spectral density does not depend on frequency:

$$S_x=\frac{\hbar c^2}{16\omega_0 W\sin^2(\theta)},\quad S_F=\frac{4\hbar\omega_0 W}{c^2},\quad S_{Fx}=\frac{\hbar}{2}\cot\theta, \tag{7}$$

where ω_0 is the frequency of the light. In Ref. 8 it was shown that the sensitivity of a measuring device with stationary frequency-independent noise, for a force with zero mean value, corresponds to the SQL (1).

3. NOISE CORRELATION IN THE MEASURING DEVICE

In Ref. 13 it was noted that a small correlation between the noises $x_{\rm fl}(t)$ and $F_{\rm fl}(t)$ when tracking the position of a test oscillator has the same effect on the sensitivity as does a small shift in the natural frequency of the latter. This effect is a manifestation of the general rule that correlation is equivalent to some modification of the dynamic properties of the probe system.

Indeed, in the presence of correlation the fluctuating force can be represented as

$$F_{\rm fl}(t)=F_{\rm fl}^{(0)}(t)+\mathbf{k}x_{\rm fl}(t),$$

where $F_{\rm fl}^{(0)}(t)$ is the component that is not correlated with $x_{\rm fl}(t)$, and $\mathbf{k}$ is a linear operator that describes the correlation. Its kernel $K(t,t')$ is the solution of the equation

$$\int_{-\infty}^{\infty}K(t,t')B_x(t,t'')dt'=B_{Fx}(t,t'').$$

Here the problem reduces to detection of the force $F(t)$ against the noise background

$$\tilde{F}_{\rm fl}(t)=F_{\rm fl}^{(0)}(t)+(\mathbf{L}+\mathbf{k})x_{\rm fl}(t). \tag{8}$$

Consequently, the sensitivity of a measuring device with correlated noise coincides with the sensitivity of a measuring device with uncorrelated noise that is connected to a probe system described by the linear operator $\mathbf{L}+\mathbf{k}$.

The simplest form that the operator $\mathbf{k}$ can take is multiplication by some constant k. It has just precisely this form when the noise of the measuring device is stationary white noise (for which $k=S_{Fx}/S_x$), and it is to this form that the method of introducing correlation in the noise of an optical interferometric sensor (7), proposed in Ref. 12, leads. Such a form of the operator **k** is equivalent to introducing additional stiffness into the probe system, where this stiffness, depending on the sign of the correlation, can be negative as well as positive.

Note that for $k=-m\omega_m^2$, the oscillator is, as it were transformed into a free mass whose sensitivity when measuring a force with nonzero mean value is not bounded by the SQL.[8] In this way the sensitivity of the oscillator to a force with nonzero mean value, for such a value of the anticorrelation, is also not bounded by the SQL.

4. STROBOSCOPIC MEASUREMENTS OF THE POSITION OF A FREE MASS

Positive correlation of the noises $x_{fl}(t)$ and $F_{fl}(t)$, equivalent to introducing stiffness into the probe system, allows one to use a stroboscopic measurement procedure not only for an oscillator, but also for a free test mass. In the case of an optical interferometric sensor, such a procedure can be realized by pulsed modulation of a probe wave.

In the analysis of a sequence of "almost instantaneous" measurements, it is convenient to go from the continuous time variable t to the discrete time variable j, $t=j\tau$, where τ is the interval between measurements. In this case the expression for the signal-to-noise ratio[11] takes the form

$$\frac{s}{n}=\frac{(\Sigma_f v_j x_j)^2}{\Sigma_{jl} B_{jl} v_j v_l}, \tag{9}$$

where v_j is a weighting function defined by the relation

$$\sum_l B_{jl} v_l = x_j, \tag{10}$$

where

$$x_j=\int_{-\infty}^{j\tau} dt' \int_{-\infty}^{t'} dt'' \frac{F(t'')}{m}$$

is the value of the response of the test mass to the force to be detected at the instant of the jth measurement,

$$B_{jl}=b_x\delta_{jl}+\frac{b_{Fx}\tau}{m}|j-l|+\frac{b_F\tau^2}{m^2}\sum_{j'=-\infty}^{\min(j,l)}(j-j')(l-j')$$

is the correlation matrix of the noise at the input of the measuring device. The parameters b_x, b_F, and b_{xF} must satisfy the uncertainty relation

$$b_F b_x - b_{Fx}^2 \geq \frac{\hbar^2}{4}. \tag{11}$$

In the case of an optical interferometric sensor with pulsed modulation, they are equal to (compare with formulas (7))

$$b_x=\frac{\hbar c^2}{16\omega_0 E \sin^2\theta}, \quad b_F=\frac{4\hbar\omega_0 E}{c^2}, \quad b_{Fx}=\frac{\hbar}{2}\cot\theta, \tag{12}$$

where E is the energy of each pulse.

Calculating the signal-to-noise ratio (9) is simplest in the spectral representation. Setting

$$x_j=\frac{1}{2\pi}\int_{-\pi}^{\pi} x(\nu)e^{ij\nu}d\nu, \quad v_j=\frac{1}{2\pi}\int_{-\pi}^{\pi} v(\nu)e^{ij\nu}d\nu$$

and substituting these values into formulas (9) and (10), we obtain

$$\frac{s}{n}=\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{|x(\nu)|^2 d\nu}{b_x-\frac{b_{Fx}\tau}{2m}\frac{1}{\sin^2(\nu/2)}+\frac{b_F\tau^2}{16m}\frac{1}{\sin^4(\nu/2)}}. \tag{13}$$

Expression (13) depends on the type of force that is to be detected. Let us consider the case that is typical of a gravitational-wave experiment, that is, a "memoryless" force,[2] which obtains when there are no aftereffects in the response of the test system, either in the position or in the momentum. The sensitivity is maximum when the duration of the force τ_F is close to τ. In this case we can set

$$|x(\nu)|\cong\frac{F\tau^2}{2m}=\text{const}.$$

The signal-to-noise ratio in this case is equal to

$$\frac{s}{n}=\left(\frac{s}{n}\right)_{\text{SQL}}\beta\left[\sqrt{1-\rho^2}+\sqrt{\frac{\beta}{2}\left(\sqrt{1-2\beta\rho+\beta^2}+\rho-\beta\right)}\times\left(\frac{2\rho-\beta}{\sqrt{1-2\beta\rho+\beta^2}}-1\right)\right], \tag{14}$$

where

$$\left(\frac{s}{n}\right)_{\text{SQL}}=\frac{F^2\tau_F^3}{\hbar m}$$

is the signal-to-noise ratio corresponding to the SQL (1),

$$\rho\equiv\frac{b_{Fx}}{\sqrt{b_F b_x}}.$$

is the correlation coefficient of the noise of the measuring device, and the parameter β is equal to

$$\beta\equiv\frac{\tau}{4m}\sqrt{\frac{b_F}{b_x}}.$$

In the case of an optical interferometric sensor, referring to formulas (12) we see that

$$\rho=\cos\theta, \quad \beta=\frac{2\omega_0\tau E}{mc^2}|\sin\theta| = \frac{2\omega_0\tau E}{mc^2}\sqrt{1-\rho^2}, \tag{15}$$

i.e., β is proportional to the energy of the optical pulses E.

The signal-to-noise ratio (14) grows without bound for $\beta=1$ as $\rho\to1$ (see Fig. 2). For these values of the parameters, formula (14) simplifies:

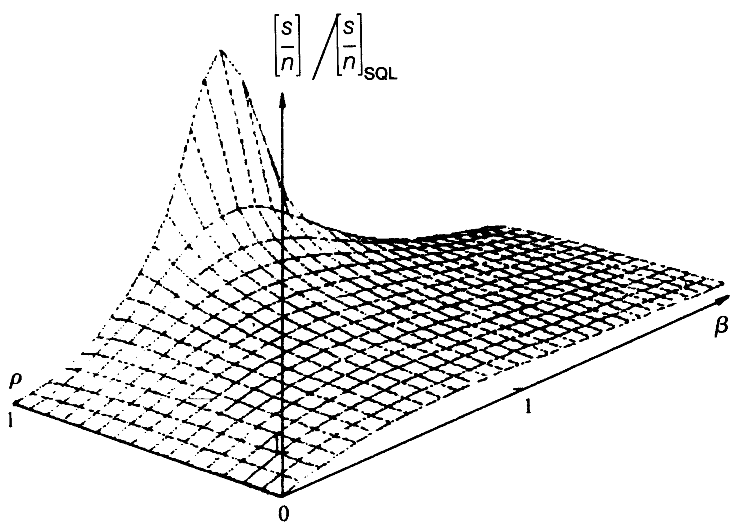

FIG. 2. Dependence of the sensitivity of the weak-force detector (14) on the parameters ρ and β.

$$\frac{s}{n}=\left(\frac{s}{n}\right)_{\rm SQL}\frac{1}{2}\sqrt[4]{\frac{2}{1-\rho}}=\left(\frac{s}{n}\right)_{\rm SQL}\sqrt{\frac{\omega_0\tau E}{mc^2}}. \qquad (16)$$

Thus, the proposed method can provide sensitivity exceeding the SQL. A necessary condition of this is strong correlation in the noise of the measuring device and sufficiently high-energy sounding light pulses.

5. MEASURING DEVICE WITH COLORED NOISE[14]

The progress that has been achieved in recent years in the preparation of nonclassical states of electromagnetic radiation affords the possibility of using another method of raising the sensitivity, based on the optimum choice of the shape of the spectral density of the noise of the measuring device.

Let us consider some expressions for the signal-to-noise ratio for the case of stationary noise of the measuring device:

$$\frac{s}{n}=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{|F(\omega)|^2d\omega}{m^2\omega^4S_x(\omega)-2m\omega^2S_{Fx}(\omega)+S_F(\omega)}. \qquad (17)$$

Let the spectral densities of the noise of the measuring device have the form

$$S_x(\omega)=\frac{\hbar}{2m\omega^2\sqrt{l-\rho^2}},\qquad S_F(\omega)=\frac{\hbar m\omega^2}{2\sqrt{1-\rho^2}},$$

$$S_{Fx}(\omega)=\frac{\hbar\rho}{2\sqrt{1-\rho^2}}, \qquad (18)$$

i.e., the spectral density of $F_{\rm fl}(t)$ is reduced at lower frequencies, and the spectral density of $x_{\rm fl}(t)$ is consequently enhanced. The signal-to-noise ratio in this case is

$$\frac{s}{n}=\frac{1}{\pi\hbar m}\sqrt{\frac{1+\rho}{1-\rho}}\int_{-\infty}^{\infty}\frac{|F(\omega)|^2d\omega}{\omega^2}$$

$$=\left(\frac{s}{n}\right)_{\rm SQL}\sqrt{\frac{1+\rho}{1-\rho}}. \qquad (19)$$

In other words, in the presence of strong correlation of the noise in the amplifier ($\rho\to 1$), the signal-to-noise ratio can substantially exceed $(s/n)_{\rm SQL}$.

Note that the shape of the spectral densities (18) can hardly be reproduced in experiment, since $S_F(\omega)\to\infty$ as $\omega\to\infty$, which corresponds to unbounded growth of the fluctuational feedback of the measuring device at high frequencies. However, it is obvious that a dependence of the form (19) is necessary only over the limited frequency range in which the spectrum of the forced to be detected, $F(\omega)$, substantially differs from zero. For example, the spectral densities of the noises can have the form

$$S_F(\omega)=\frac{\hbar m\omega_0^2}{2\sqrt{1-\rho^2}}\,\frac{\omega^2}{\omega^2+\omega_0^2},$$

$$S_x(\omega)=\frac{\hbar}{2m\omega_0^2\sqrt{1-\rho^2}}\,\frac{\omega^2+\omega_0^2}{\omega^2},$$

$$S_{Fx}(\omega)=\frac{\hbar\rho}{2\sqrt{1-\rho^2}},$$

where ω_0 is some critical frequency. Calculation shows that if $\omega_0\tau_F\geqslant 1$, then the signal-to-noise ratio, as before, is given by (19).

Practical realization of the given method in the case of an optical sensor requires the use of a probe wave with suppression at low frequencies of photon shot noise. Reference 15 demonstrates a method of preparing light in such a state, based on the use of a high-efficiency current-pumped injection laser with suppressed shot-noise fluctuations.

[1] V. B. Braginskiĭ, Zh. Eksp. Teor. Fiz. **53**, 1434 (1967) [Sov. Phys. JETP **26**, 831 (1968)].
[2] V. B. Braginskiĭ, Usp. Fiz. Nauk **156**, 93 (1988) [Sov. Phys. Usp. **31**, 836 (1988)].
[3] V. B. Braginskiĭ, Yu. I. Vorontsov, and F. Ya. Khalili, Zh. Eksp. Teor. Fiz. **73**, 1340 (1977) [Sov. Phys. JETP **46**, 705 (1977)].
[4] K. S. Thorne *et al.*, Phys. Rev. Lett. **40**, 667 (1978).
[5] V. B. Braginskiĭ and F. Ja. Khalili, Phys. Lett. A **147**, 251 (1990).
[6] V. B. Braginskiĭ, Yu. I. Vorontsov, and F. Ya. Khalili, Pis'ma Zh. Eksp. Teor. Fiz. **27**, 296 (1978) [JETP Lett. **27**, 276 (1978)].

[7] F. Ya. Khalili, Dokl. Akad. Nauk SSSR **294**, 602 (1987) [Sov. Phys. Dokl. **32**, 409 (1987)].
[8] Yu. I. Vorontsov, *Theory and Methods of Macroscopic Measurements* [in Russian], Nauka, Moscow (1989).
[9] D. G. Blair (ed.), *The Detection of Gravitational Waves*, Cambridge Univ. Press, Cambridge (1992).
[10] V. B. Braginskiĭ and F. Ya. Khalili, *Quantum Measurement*, Cambridge Univ. Press, Cambridge (1992).
[11] B. R. Levin, *Theoretical Principles of Statistical Radiophysics* [in Russian], Sovetskoe Radio, Moscow (1974).
[12] S. P. Vyatchanin and A. B. Madko, Zh. Eksp. Teor. Fiz. **104**, 2668 (1993) [JETP **77**, 218 (1993)].
[13] A. V. Gusev and A. V. Tsyganov, Vestnik Moskovskogo Universiteta, Ser. 3, Fizika. Astronomiya **34**, No. 4 (1993).
[14] D. A. Pobedrya, *Diploma Work*, Lomonosov State Univ., MFFI Department, Moscow (1993).
[15] Y. Yamamoto, S. Machida, and O. Nisson, in *Proceedings of the Conference on Amplification and Quantum Effects of Semiconductor Lasers*.

Translated by Paul F. Schippnick