Observability of quantum states of the intracavity field of a micromaser under conditions of random and regular active atom injection

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The possibility is analyzed of observing the quantum states of a laser field on the basis of the well-known kinetic equations for the density matrix of the field. It is shown that the Fock states are unobservable, the Poissonian states are observable with Poissonian statistics, and some intermediate state can be observed with shot noise suppressed to the 50% level. For regular injection of the working atoms intracavity quantum states arise which correspond to complete suppression of shot noise at low frequencies. Some rules are formulated which allow a significant simplification of the calculations of the statistical characteristics of the electromagnetic field.

1. INTRODUCTION

It is well known that the states of a micromaser can be substantially quantized, including the Fock states.[1] This is interesting in itself since these states are formed automatically without any additional effort (in contrast to an optical laser, where the quantum states arise, for example, under conditions of regular pumping[2] or through a parametric interaction mechanism.[3] At the same time it should be understood that we are not talking here about an observable effect, but about intracavity states of the laser field. All the same, the question of observability still requires further study. For example, we could have said that the intracavity Fock states cannot be observed as such since the photons leaving the cavity, one after the other, in this case turn out to be completely uncorrelated due to the rapid relaxation of the photon fluctuations inside the cavity. Thus we see that the Fock states are in actual fact Poissonian. This of course does not mean in the case of a micromaser that it is totally impossible to observe quantum phenomena in the maser field. But all the same we should theoretically estimate their quantitative aspects. In particular, it will be shown that the quantum effect connected with the total suppression of shot noise is possible under conditions of regular active atom injection.

We base our theoretical description on the well-known equations for the density matrix of the maser field, which were obtained earlier by other authors in well-known papers.[1] According to Ref. 2, the equation written for random injection can be easily generalized to the case of regular injection. At the same time, we will not try here, as is customary, to solve the equation by whatever means possible, but instead will formulate some rules which will allow us to write out some final results without having to solve the kinetic equations themselves. To illustrate the possibilities of these rules, we will solve a few familiar problems. At the same time, the Appendix gives the traditional analysis of the kinetic equation using the diagonal Glauber representation.

2. RULES FOR CALCULATING CERTAIN AVERAGE VALUES

In this section we consider rules which significantly simplify calculations of the following averages:

$$\langle a^+a^+(t)a(t)a\rangle,\quad \langle a^+a(t)\rangle,\quad \langle a(t)a\rangle. \tag{1}$$

The operators $a=a(0)$, $a^+=a^+(0)$, $a(t)$, and $a^+(t)$ are the photon operators in the Heisenberg representation.

Rule A: for a stationary light flux in the most general case

$$\langle a^+a^+(t)a(t)a\rangle=\langle a^+a\rangle_{st}\langle a^+a\rangle_t,$$

where $\langle a^+a\rangle_t$ is the solution of the differential equation for the average number of photons with initial state of the special form

$$\langle a^+a\rangle_{t=0}=\frac{\langle a^+a^+aa\rangle_{st}}{\langle a^+a\rangle_{st}},$$

where $\langle a^+a\rangle_{st}$ and $\langle a^+a^+aa\rangle_{st}$ are the solutions of the corresponding stationary problems.

We will prove this assertion with the help of a generating function of the following form:

$$G(z,t)=\mathrm{Tr}\{F(t)z^{a^+a}\}. \tag{2}$$

Here $F(t)$ is the density matrix of the total real system, consisting of the light mode of interest, all other light modes, and other material subsystems, in particular, those ensuring relaxation of all subsystems and generation of the light modes. It is understood that this expression can be rewritten in terms of the density matrix of just the mode in question

$$G(z,t)=\mathrm{Tr}\{\rho(t)z^{a^+a}\},\quad \rho=\mathrm{Tr}_1\, F.$$

Since F is the density matrix of the entire system, its time evolution can be represented in terms of a unitary matrix $S(t)$:

$$F(t)=S(t)F(0)S^+(t). \tag{3}$$

Bearing this in mind, we can write down the following equalities:

$$\left[\frac{\partial G(z,t)}{\partial z}\right]_{t=1}=\mathrm{Tr}\{a^+aF(t)\}$$
$$=\mathrm{Tr}\{S^+(t)a^+S(t)S^+(t)aS(t)F(0)\}$$
$$=\mathrm{Tr}\{a^+(t)a(t)F(0)\}. \tag{4}$$

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Thus, if we require

$$F(0)=aF_{st}a^+, \tag{5}$$

then we obtain

$$\left[\frac{\partial G}{\partial z}\right]_{z=1}=g(t), \quad g(t)=\langle a^+a^+(t)a(t)a\rangle. \tag{6}$$

At the same time, taking into account $\mathrm{Tr}\,F=\langle a^+a\rangle_{st}$, according to the relations (4), we have

$$g(t)=\langle a^+a\rangle_t\langle a^+a\rangle_{st} \tag{7}$$

under the condition

$$\langle a^+a\rangle_{t=0}=\frac{\langle a^+a^+aa\rangle_{st}}{\langle a^+a\rangle_{st}}. \tag{8}$$

Thus, the theorem is proven. We emphasize again that it is formulated in the most general form since its proof makes use of the density matrix of the entire physical system without any specifics.

We have here considered only the simplest case, when $\langle a^+a\rangle_t$ depends only on the single initial value $\langle a^+a\rangle_{t=0}$. Of course, there are mixed situations, when this solution can depend on many initial values. But then all these initial values should be (or can be) reformulated appropriately.

Now we will give two more rules which can be just as simply proven.

Rule B: for a time-independent light flux in the most general case

$$\langle a^+a(t)\rangle=\langle a^+\rangle_{st}\langle a\rangle_t,$$

where $\langle a\rangle_t$ is the solution of the differential equation for the average complex amplitude of the field for the initial condition

$$\langle a\rangle_{t=0}=\frac{\langle a^+a\rangle_{st}}{\langle a\rangle_{st}}.$$

Here $\langle a^+a\rangle_{st}$ and $\langle a\rangle_{st}$ are solutions of the corresponding stationary problems.

Rule C: for a time-independent light flux in the most general case

$$\langle aa(t)\rangle=\langle a\rangle_{st}\langle a\rangle_t,$$

where $\langle a\rangle_t$ is the solution of the differential equation for the complex amplitude of the field for the initial condition

$$\langle a\rangle_{t=0}=\frac{\langle aa\rangle_{st}}{\langle a\rangle_{st}}.$$

Here $\langle a\rangle_{st}$ and $\langle a^+a\rangle_{st}$ are the solutions of the corresponding stationary problems.

The latter rule is given only for reference since it is needed in the analysis of compressed states, which we will not consider here. It is clear that analogous rules can be formulated for other averages.

3. A SUB-POISSONIAN LASER: THE PHOTOCURRENT SPECTRUM

The case of a sub-Poissonian laser[2] is most convenient to illustrate those advantages which obtain when one uses the above rules. The equation for the generation density matrix can be written in the form

$$\dot{\rho}=r\left(\hat{L}-\frac{1}{2}\hat{L}^2\right)\rho+\gamma\hat{R}\rho, \tag{9}$$

where the operator

$$\hat{L}=2(a^+)_\rightarrow(a)_\leftarrow\hat{Q}^{-1}-1 \tag{10}$$

with

$$\hat{Q}=(aa^+)_\rightarrow+(aa^+)_\leftarrow+\tfrac{1}{2}\beta[(aa^+)_\rightarrow-(aa^+)_\leftarrow]^2$$

defines the evolution of the field due to the interaction with the active medium, and the operator

$$\hat{R}=(a)_\rightarrow(a^+)_\leftarrow-\tfrac{1}{2}[(a^+a)_\rightarrow+(a^+a)_\leftarrow] \tag{11}$$

describes the decay of the field in the cavity. The arrows to the right of the operators indicate the direction these operators act with respect to the expressions standing to their right, r is the average rate of excitation of the working atoms to the upper laser level, γ is the average rate at which photons escape from the cavity due to its finite Q, and β is the saturation parameter (its inverse value has the physical meaning of the saturating photon number).

In writing Eq. (9) and the operator (10) (see Ref. 4), we have assumed that spontaneous decay from the upper laser level is possible only to the lower laser level, and also that excitation of the working atoms can be either regular or completely random. In the case of random excitation the operator term $-\frac{1}{2}\hat{L}^2$ must be dropped.

We first multiply Eq. (9) by the operator a^+a and take the trace and then do the same with the operator a^+a^+aa. In this way we obtain the following two equations:

$$\langle a^+\dot{a}\rangle=-\gamma\langle a^+a\rangle+r, \tag{12}$$

$$\langle a^+a^+\dot{a}a\rangle=-2\gamma\langle a^+a^+aa\rangle+2r\langle a^+a\rangle-r. \tag{13}$$

Equation (12) is exactly the same for regular and for random excitation of the atoms. For a random pump the last term on the right-hand side, $-r$, is absent. According to rule A, we should write the general solution of Eq. (12) and the stationary solutions of Eqs. (12), (13) in the form

$$\langle a^+a\rangle_t=\langle a^+a\rangle_{t=0}e^{-\gamma t}+\frac{r}{\gamma}(1-e^{-\gamma t}), \tag{14}$$

$$\langle a^+a\rangle_{st}=\frac{r}{\gamma}, \tag{15}$$

$$\langle a^+a^+aa\rangle_{st}=\left(\frac{r}{\gamma}\right)^2-\frac{r}{2\gamma}=\langle a^+a\rangle_{st}^2-\frac{1}{2}\langle a^+a\rangle_{st}. \tag{16}$$

According to the rule, it is necessary to choose the initial condition in the form (8), which gives

$$\langle a^+a\rangle_{t=0}=\langle a^+a\rangle_{st}-\tfrac{1}{2}. \tag{17}$$

Thus, as a result of applying rule A we obtain

$$g(t)=\langle a^+a\rangle_{\rm st}^2-\frac{1}{2}\langle a^+a\rangle_{\rm st}e^{-\gamma t}. \qquad (18)$$

Bearing in mind that the photocurrent spectrum is given by the formula[2]

$$i_\omega^{(2)}=i_{\rm shot}^{(2)}\left\{1+2q\gamma\langle a^+a\rangle_{\rm st}^{-1}\ {\rm Re}\int_0^\infty g(t)e^{i\omega t}dt\right\}, \qquad (19)$$

we obtain the well-known result for the noise spectrum of a sub-Poissonian laser

$$i_\omega^{(2)}=i_{\rm shot}^{(2)}\left\{1-q\,\frac{\gamma^2}{\gamma^2+\omega^2}\right\} \qquad (20)$$

(q is the quantum efficiency of the photocathode).

4. THE OPTICAL SPECTRUM OF THE RADIATION OF A SUB-POISSONIAN LASER

To find the optical spectrum it is necessary to use rule B since

$$I_\omega={\rm Re}\int_0^\infty\langle a^+a(t)\rangle e^{i\omega t}dt. \qquad (21)$$

It is clear that when the photon fluctuations are small the main part of the optical spectrum is formed by the phase fluctuations. Indeed, employing the coherent-state representation, we can write

$$\langle a^+a(t)\rangle=\overline{\alpha^*\alpha(t)},$$

where the bar above denotes averaging with the two-point weight $P(\alpha_1,t_1;\alpha_2,t_2)$ in the form

$$\overline{\alpha^*\alpha(t)}=\int\int d^2\alpha_1 d^2\alpha_2\alpha_1^*\alpha_2P(\alpha_1t=0;\alpha_2,t).$$

Going over to the amplitude and phase

$$\alpha=\sqrt{U}e^{i\varphi} \qquad (22)$$

and assuming that the photon fluctuations are small

$$U=n_0+\varepsilon,\quad \varepsilon\ll n_0, \qquad (23)$$

we find that everything reduces to phase averages alone:

$$\langle a^+a(t)\rangle\approx n_0\overline{e^{i\varphi(t)-i\varphi(0)}}.$$

In order to apply rule B, we need the two equations

$$\dot{\langle a^+a\rangle}=-\gamma\langle a^+a\rangle+r=0, \qquad (24)$$

$$\langle\dot a\rangle=-\tfrac{1}{2}\gamma\langle a\rangle+r(1-\tfrac{1}{2}\beta)\langle a\hat Q^{-1}\rangle. \qquad (25)$$

We apply the averaging procedure in the language of coherent states (see the Appendix), taking account of the approximation of small photon fluctuations (23):

$$\langle a\hat Q^{-1}\rangle=\frac{1}{2}\int\int dUd\varphi\,\frac{1}{\sqrt{U}}\times e^{i\varphi}\frac{1}{1-\dfrac{\partial}{\partial U}-\dfrac{\beta}{2U}\dfrac{\partial^2}{\partial\varphi^2}}P(U,\varphi)\approx\left(\frac{1}{2n_0}-\frac{1}{2n_0^2}\right)\langle a\rangle.$$

Allowing for the condition of stationary generation $n_0=r/\gamma$, we obtain the following equation for the complex amplitude:

$$\langle\dot a\rangle=-\tfrac{1}{2}\Gamma\langle a\rangle,\quad \Gamma=\frac{\gamma}{2n_0}\left(1+\frac{1}{2}\beta n_0\right). \qquad (26)$$

Applying rule B gives

$$\langle a^+a(t)\rangle=\frac{r}{\gamma}e^{-(1/2)\Gamma t}. \qquad (27)$$

Substituting this result in Eq. (21), we obtain the Lorentzian optical spectrum with linewidth equal to Γ, which coincides with the known results.

5. A POISSONIAN LASER

The model of a Poissonian laser is more complicated in a mathematical sense, and a consideration of this question will allow us to track down some typical difficulties that one is likely to encounter in the solution of nonlinear problems, and to work out recipes for overcoming them.

We write the master equation for the density matrix of the laser field again in the form

$$\dot\rho=r\hat L\rho+\gamma\hat R\rho. \qquad (28)$$

Here $\hat R$ coincides with its value as given by Eq. (11), and $\hat L$ has the form[4]

$$\hat L=\tfrac{1}{2}\beta[(a^+)_\to(a)_\leftarrow-\tfrac{1}{2}[(aa^+)_\to+(aa^+)_\leftarrow]-\tfrac{1}{8}\beta[(aa^+)_\to-(aa^+)_\leftarrow]^2][1+\tfrac{1}{2}[(aa^+)_\to+(aa^+)_\leftarrow]+\tfrac{1}{16}\beta^2[(aa^+)_\to-(aa^+)_\leftarrow]^2]^{-1}. \qquad (29)$$

This formula exactly corresponds to Lamb and Scully's model of the laser[5] in the case $\gamma_a=\gamma_b$ (the equation is given in more general form in Ref. 4). Again we obtain the equations

$$\dot{\langle a^+a\rangle}=-\gamma\langle a^+a\rangle+\frac{1}{2}r\beta\left\langle\frac{aa^+}{1+\beta aa^+}\right\rangle, \qquad (30)$$

$$\dot{\langle a^+a^+aa\rangle}=-2\gamma\langle a^+a^+aa\rangle+r\beta\left\langle\frac{a^+aaa^+}{1+\beta aa^+}\right\rangle. \qquad (31)$$

As can be seen, here, in contrast to the case of a sub-Poissonian laser, we do not obtain closed equations. In place of the particle number operator a^+a, we may now introduce the operator $\hat\varepsilon$ which describes the photon fluctuations

$$a^+a=n_0+\hat\varepsilon, \qquad (32)$$

where n_0 is the solution of the semiclassical problem in the stationary generation regime. An expression for n_0 is obtained from Eq. (30) by replacing the photon number operators with n_0 and taking the time derivative, which is equal to zero:

$$n_0=\frac{r}{2\gamma}\frac{\beta n_0}{1+\beta n_0}. \tag{33}$$

From Eq. (30) upon linearizing in $\hat{\varepsilon}$ it follows that

$$\langle\dot{\hat{\varepsilon}}\rangle=-\Gamma\langle\hat{\varepsilon}\rangle,\quad \Gamma=\gamma\frac{\beta n_0}{1+\beta n_0}. \tag{34}$$

It is clear that

$$\langle\hat{\varepsilon}\rangle_t=\langle\hat{\varepsilon}\rangle_{t=0}e^{-\Gamma t}$$

and, consequently, in accordance with Eq. (32)

$$\langle a^+a\rangle_t=n_0+(\langle a^+a\rangle_{t=0}-n_0)e^{-\Gamma t}. \tag{35}$$

From Eq. (31) in the stationary regime we obtain

$$\langle\hat{\varepsilon}^2\rangle_{\rm st}=n_0\frac{1+\beta n_0}{\beta n_0}. \tag{36}$$

After allowing for relation (32)

$$\langle a^+a^+aa\rangle_{\rm st}=n_0^2+\xi n_0,\quad \xi=\frac{1}{\beta n_0}. \tag{37}$$

Here ξ is the Mandel statistical parameter. And now it only remains to apply rule A and we obtain

$$g(t)=n_0^2+\xi n_0e^{-\Gamma t}. \tag{38}$$

The form of the photocurrent spectrum, which determines the spectral makeup of the noise, is given in the case of the Poissonian laser by the formula

$$i_\omega^{(2)}=i_{\rm shot}\left\{1+2q\xi\frac{\gamma\Gamma}{\Gamma^2+\omega^2}\right\}. \tag{39}$$

As can be seen, in the case of saturation ($\beta n_0\to\infty$) it satisfies $\xi\to 0$, i.e., the excess noise disappears and the laser radiation becomes Poissonian.

6. THE PHOTOCURRENT SPECTRUM WHEN MICROMASER RADIATION IS DETECTED

We write the master equation for the density matrix in the form

$$\dot{\rho}=r(\hat{M}-\tfrac{1}{2}\hat{M}^2)\rho+\gamma\hat{R}\rho. \tag{40}$$

The operator

$$\hat{M}=(a^+)_\to(a)_\leftarrow\hat{f}_\leftarrow\hat{f}_\to+\hat{g}_\leftarrow\hat{g}_\to-1 \tag{41}$$

with

$$\hat{f}=\frac{\sin\sqrt{\beta aa^+}}{\sqrt{aa^+}},\quad \hat{g}=\cos\sqrt{\beta aa^+} \tag{42}$$

describes the interaction of the micromaser mode with the active medium.[1] In writing down the operator $\hat{R}$ now it is necessary to take into account thermal phenomena in the cavity (n_b is the average number of thermal photons in the lasing mode):

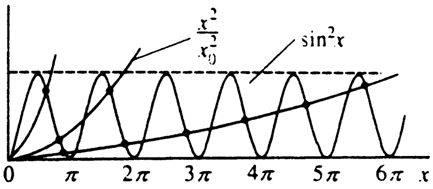

FIG. 1. $x=\sqrt{\beta n_0}$, $x_0=\sqrt{\beta r/\gamma}$, $x^2/x_0^2=\sin^2x$.

$$\hat{R}=(1-n_b)[(a)_\to(a^+)_\leftarrow-\tfrac{1}{2}[(a^+a)_\to+(a^+a)_\leftarrow]]$$
$$+n_b[(a^+)_\to(a)_\leftarrow-\tfrac{1}{2}[(aa^+)_\to-(aa^+)_\leftarrow]]. \tag{43}$$

First let us consider the case of completely random pumping of the working atoms to the upper laser level. In this case, it is not necessary to include the operator term $-\frac{1}{2}\hat{M}^2$ in Eq. (40). Then from Eq. (40) we have

$$\langle\dot{a^+a}\rangle=-\gamma\ +\ ((a^+a\rangle-n_b)-r\langle\sin^2\sqrt{\beta aa^+}\rangle, \tag{44}$$

and

$$\langle\dot{a^+a^+aa}\rangle=-2\gamma\langle a^+a^+aa\rangle+2r\langle a^+a\ \sin^2\sqrt{\beta aa^+}\rangle$$
$$+4\gamma n_b\langle a^+a\rangle. \tag{45}$$

These two expressions are written in exactly. However, if we consider the case of regular pumping, then additional terms arise in both equations. All of them, for the most part, will be small under the condition

$$\sqrt{\frac{\beta}{\langle a^+a\rangle}}\ll 1, \tag{46}$$

except for one term in Eq. (45) of the form $-r/2\langle\sin^4\sqrt{\beta aa^+}\rangle$.

We will again assume that the photon number in the stationary regime weakly fluctuates about its semiclassical solution n_0 (32), for which in the case of a micromaser Eq. (30) leads to the equality

$$\gamma n_0=r\ \sin^2\sqrt{\beta n_0}. \tag{47}$$

Here account has been taken of the inequality $n_b\ll n_0$. We have not considered a different situation since no other situation could satisfy the requirement of small photon fluctuations. In Fig. 1 the points represent the various sets of stationary solutions of Eq. (47), which depend on the physical parameter set. Now let us convince ourselves that $\cot\sqrt{\beta n_0}<0$ holds for these solutions. We linearize Eq. (44) in the new operator $\hat{\varepsilon}$:

$$\langle\dot{\hat{\varepsilon}}\rangle=-\Gamma\langle\hat{\varepsilon}\rangle,\quad \Gamma=\gamma(1-\sqrt{\beta n_0}\ \cot\sqrt{\beta n_0}). \tag{48}$$

The above requirement follows from the positive definiteness of Γ. We may rewrite the resulting constant Γ in the form

$$\Gamma=\gamma(1+\sqrt{\beta n_0}\ |\cot\sqrt{\beta n_0}|). \tag{49}$$

The solution of Eq. (48) is written in the form

$$\langle\hat{\varepsilon}\rangle_t=\langle\hat{\varepsilon}\rangle_{t=0}e^{-\Gamma t}. \tag{50}$$

From Eq. (45) after linearization we obtain

$$\langle a^+a^+aa\rangle_{\rm st}=n_0^2+\xi n_0, \tag{51}$$

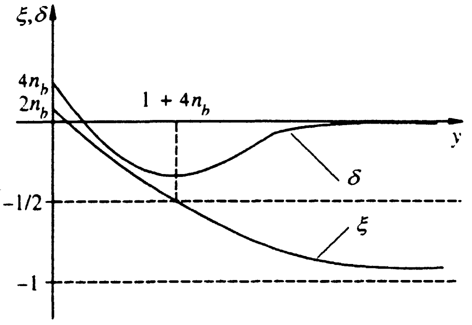

FIG. 2. $y = \sqrt{\beta n_0}|\cot\sqrt{\beta n_0}|$.

where ξ is the Mandel statistical parameter, which is represented explicitly as follows:

$$\xi=\frac{2n_b-\sqrt{\beta n_0}|\cot\sqrt{\beta n_0}|}{1-\sqrt{\beta n_0}|\cot\sqrt{\beta n_0}|}. \tag{52}$$

Using rule A, we obtain

$$g(t)=n_0^2+\xi n_0 e^{-\Gamma t}. \tag{53}$$

As a result, the photocurrent spectrum has the same form as (39) for a Poissonian laser, but, of course, with its own coefficients.

7. DISCUSSION OF THE PHOTOCURRENT SPECTRUM WHEN MICROMASER RADIATION IS DETECTED

Thus, the observed photocurrent spectrum has the form

$$i_\omega^2=i_{\text{shot}}^2\left\{1+2q\xi\frac{\gamma\Gamma}{\Gamma^2+\omega^2}\right\}. \tag{54}$$

To start with, we assume that the working atoms are only injected into the micromaser cavity in a completely random fashion, i.e., the spectral width of the noise Γ and the Mandel parameter ξ are given by formulas (49) and (52). In order to estimate the magnitude of the observed quantum effect, it is convenient to introduce one more parameter into the discussion:

$$\delta=2q\xi\frac{\gamma}{\Gamma}=2q\frac{2n_b-\sqrt{\beta n_0}|\cot\sqrt{\beta n_0}|}{(1+\sqrt{\beta n_0}|\cot\sqrt{\beta n_0}|)^2}. \tag{55}$$

This parameter gives the magnitude of the dip (for $\xi<0$) or peak (for $\xi>0$) in the photocurrent spectrum (54). Thus, when it is negative it quantitatively characterizes the quantum manifestations in the process of detecting radiation. At the same time, the Mandel parameter ξ characterizes the intracavity state of the field oscillator. Comparing ξ and δ makes it possible for us to judge the relation between the observed phenomena and the intracavity states and, in particular, the fundamental observability of the intracavity quantum states.

Figure 2 gives a qualitative picture of the dependence of the parameters ξ and δ on $y = \sqrt{\beta n_0}|\cot\sqrt{\beta n_0}|$, which can, in principle, run through any values from zero to infinity with variation of the stationary solutions n_0 for micromaser generation. We will take the quantum efficiency of the photocathode to be equal to unity ($q=1$) in all that follows.

As can be seen, the parameter ξ falls as y rises from the value n_b (associated, as we should recall, with the cavity temperature) to -1. This means that inside the cavity, at least in principle, both Poissonian and sub-Poissonian states occur, as well as states with positive excess noise (slightly super-Poissonian), all the way to the Fock states at large y.

The parameter δ behaves differently. At first, like ξ, it falls smoothly, reflecting the decrease of the super-Poissonian noise, but beyond the point $y=1+4n_b$ it begins to grow, tending finally toward zero. Thus, in the initial region there is, as it were, a qualitative correspondence between observation and the intracavity states: the positive excess noise in the photocurrent corresponds to the super-Poissonian state, and the negative excess noise with, in this case, maximum possible depth of the dip (about 50%) corresponds to the sub-Poissonian state with $\xi=-1/2$. However, as one goes beyond the initial region this correspondence breaks down and in fact the closer the intracavity state is to a Fock state (i.e., to the most quantum-like of all the possible states), the less observable is the quantum effect and the closer are the observed generation statistics to Poissonian. This phenomenon becomes completely comprehensible if we recall that everything about the photoregistration noise that is non-Poissonian is due to correlations between each photon of the photon pairs that arrive at the photocathode. But such correlations are simply impossible in the case of a Fock intracavity state. The point is that in this case any fluctuation inside the cavity decays with an infinitely fast decay-rate, ensuring the existence of a Fock (and this means nonfluctuating) state, and therefore the second photon leaving the cavity, right behind the first, is already completely uncorrelated with it.

Now let us discuss the case of regular injection of the working atoms into the intracavity space of the micromaser. It is specifically in this case that the additional term $-1/2\hat{M}^2$ appears on the right in the master kinetic equation (41) in comparison with the case of completely random injection. As a consequence, there appears an additional term on the right in Eq. (45) in the form $-r/2\langle\sin^4\sqrt{\beta a a^+}\rangle$, and the formulas for ξ and δ are now, naturally, different:

$$\xi=\frac{2n_b-y-\frac{1}{2}\sin^2\sqrt{\beta n_0}}{1+y}, \tag{56}$$

$$\delta=2\frac{2n_b-y-\frac{1}{2}\sin^2\sqrt{\beta n_0}}{(1+y)^2}. \tag{57}$$

The change in these formulas consists only in the additional term in the numerators of these expressions in the form $-1/2\ \sin^2\sqrt{\beta n_0}$.

For our purposes, when we try to observe stationary states most strongly evidenced by their quantum manifestations, the most interesting stationary micromaser states are those for which $\sin^2\sqrt{\beta n_0}\ =\ 1$ (or is close to 1). Figure 3 gives a qualitative picture of just such a case with the addi-

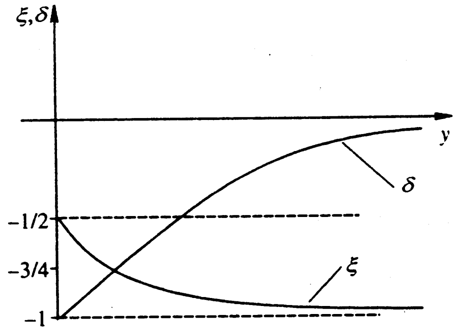

FIG. 3. $y = \sqrt{\beta n_0}|\cot\sqrt{\beta n_0}|$.

tional condition $n_b \ll 1$. As can be seen, now all of the chosen stationary intracavity states are sub-Poissonian (all the way to the Fock states, as before). All these states are substantially quantum-like. In this regard, note that the state $\xi = -1/2$ corresponds to the most observable quantum effect since in this case $\delta = -1$ holds, i.e., here we have the same situation as for a sub-Poissonian laser.[2] Thus, in the case of regular injection complete suppression of shot noise is possible, which substantially distinguishes this situation from the situation with random injection, where complete suppression is never achieved. As for the Fock states, here the picture is the same as before: these states are unobservable.

8. APPENDIX: THE MASTER KINETIC EQUATION OF A MICROMASER IN THE DIAGONAL GLAUBER REPRESENTATION

We represent the operator functions $\hat{f}$ and $\hat{g}$ in the form of an expansion in the small parameter

$$\hat{\varepsilon} = aa^+ - n_0. \tag{A1}$$

We keep terms to second order in $\hat{\varepsilon}$:

$$\hat{f} = \frac{\sin\sqrt{\beta n_0}}{\sqrt{n_0}} \left\{ 1 - \frac{1}{2}(y+1)\frac{\hat{\varepsilon}}{n_0} + \frac{1}{8}(3y+3-\beta n_0)\left(\frac{\hat{\varepsilon}}{n_0}\right)^2 \right\}, \tag{A2}$$

$$\hat{g} = \cos\sqrt{\beta n_0} \left\{ 1 + \frac{\beta n_0}{2y}\frac{\hat{\varepsilon}}{n_0} - \beta n_0 \frac{y+1}{8y}\left(\frac{\hat{\varepsilon}}{n_0}\right)^2 \right\}, \tag{A3}$$

here $y = \sqrt{\beta n_0}|\cot\sqrt{\beta n_0}|$ and allowance is made for the fact that $\cot\sqrt{\beta n_0} < 0$ for stable stationary solutions.

Now we can write the operators $\hat{M}$ and $\hat{M}^2$ in the same approximation, and after that go over to the diagonal Glauber representation, which is defined by the following integral relation

$$\rho(t) = \int d^2\alpha P(\alpha,t)|\alpha\rangle\langle\alpha|. \tag{A4}$$

Here α and $|\alpha\rangle$ are the eigenvalue and eigenfunction of the photon annihilation operator:

$$a|\alpha\rangle = \alpha|\alpha\rangle, \quad \alpha = \sqrt{U}e^{i\varphi}.$$

As is well known, the transformation to the diagonal representation can be carried out with the help of the following rules:

$$(a)_{\rightarrow} \rightarrow \alpha, \quad (a^+)_{\rightarrow} \rightarrow \alpha^* - \frac{\partial}{\partial\alpha}, \quad (a^+)_{\leftarrow} \rightarrow \alpha^*,$$
$$(a)_{\leftarrow} \rightarrow \alpha - \frac{\partial}{\partial\alpha^*}. \tag{A5}$$

Bearing this in mind, we obtain

$$(aa^+)_{\rightarrow} \rightarrow U - U\frac{\partial}{\partial U} + \frac{i}{2}\frac{\partial}{\partial\varphi},$$
$$(aa^+)_{\leftarrow} \rightarrow U - U\frac{\partial}{\partial U} - \frac{i}{2}\frac{\partial}{\partial\varphi},$$
$$(a^+)_{\rightarrow}(a)_{\leftarrow} \rightarrow U - U\frac{\partial}{\partial U} - \frac{\partial}{\partial U}U + \frac{\partial}{\partial U}U\frac{\partial}{\partial U} + \frac{1}{4U}\frac{\partial^2}{\partial\varphi^2}, \quad (a)_{\rightarrow}(a^+)_{\leftarrow} \rightarrow U. \tag{A6}$$

Two more useful relations:

$$\frac{1}{2}[(aa^+)_{\rightarrow} + (aa^+)_{\leftarrow}] \rightarrow U - U\frac{\partial}{\partial U},$$
$$(aa^+)_{\rightarrow} - (aa^+)_{\rightarrow} \rightarrow i\frac{\partial}{\partial\varphi}. \tag{A7}$$

We can now write down explicit expressions for the operators $\hat{\varepsilon}_{\leftarrow}$ and $\hat{\varepsilon}_{\rightarrow}$:

$$\hat{\varepsilon}_{\leftarrow} \rightarrow \varepsilon - U\frac{\partial}{\partial U} + \frac{i}{2}\frac{\partial}{\partial\varphi}, \quad \hat{\varepsilon}_{\rightarrow} \rightarrow \varepsilon - U\frac{\partial}{\partial U} - \frac{i}{2}\frac{\partial}{\partial\varphi}, \tag{A8}$$

where

$$U = n_0 + \varepsilon, \quad \varepsilon \ll n_0. \tag{A9}$$

Using all these formulas, we can finally write down the equation for the density matrix in diagonal form:

$$\frac{\partial P(\alpha,t)}{\partial t} = \Gamma\frac{\partial}{\partial\varepsilon}(\varepsilon P) + D_a\frac{\partial^2 P}{\partial\varepsilon^2} + \frac{1}{2}D_\phi\frac{\partial^2 P}{\partial\varphi^2} + \{\cdots\}. \tag{A10}$$

This equation as written takes account of the fact that n_0 coincides with the semiclassical solution given by Eq. (47). The coefficients are given by

$$\Gamma = \gamma(1+y), \tag{A11}$$

$$D_a = \Gamma n_0 \xi, \tag{A12}$$

where the Mandel parameter for regular injection has the form

$$D_\varphi = \frac{\gamma}{4n_0}\left(1 + 2n_b + \frac{\beta n_0}{\sin^2\sqrt{\beta n_0}}\right). \tag{A13}$$

For random injection it is necessary to drop the term $-1/2\ \sin^2\sqrt{\beta n_0}$,

$$\xi = \frac{2n_b - y - \frac{1}{2}\sin^2\sqrt{\beta n_0}}{1+y}. \tag{A14}$$

The terms in (A10) not written out explicitly have the following form:

$$\{\cdots\}=\left[D_{03}\frac{\partial^3}{\partial\varepsilon^3}+D_{04}\frac{\partial^4}{\partial\varepsilon^4}+D_{40}\frac{\partial^4}{\partial\varphi^4}+D_{12}\frac{\partial^3}{\partial\varepsilon\partial\varphi^2}+D_{22}\frac{\partial^4}{\partial\varepsilon^2\partial\varphi^2}\right]P(\varepsilon,\varphi,t). \tag{A15}$$

Below we will show that they do not contribute to the observed signal. Thus, we could have written the equation in the diagonal form in the diffusion approximation although, strictly speaking, for quantum fields this is invalid.

What we do next depends on what we want to calculate next. If we are interested in the photocurrent spectrum, then we need to calculate the average $\langle\varepsilon\varepsilon(t)\rangle$. This being the case, we can limit ourselves to the equation for the amplitude density matrix:

$$R(\varepsilon,t)=\int P(\sqrt{n_0+\varepsilon}e^{i\varphi},t)d\varphi. \tag{A16}$$

Integrating (A10) over φ, we obtain

$$\frac{\partial R}{\partial t}=\Gamma\frac{\partial}{\partial\varepsilon}(\varepsilon R)+D_a\frac{\partial^2 R}{\partial\varepsilon^2}+\cdots. \tag{A17}$$

Now we must take into consideration the relation

$$\langle\varepsilon\varepsilon(t)\rangle=\int d\varepsilon_1 d\varepsilon_2\varepsilon_1\varepsilon_2 R(\varepsilon_1,t=0)G(\varepsilon_1,t=0|\varepsilon_2,t), \tag{A18}$$

where $R(...)$ and $G(...)$ are solutions of the same equation (A17) for the physical initial condition and for the special initial condition

$$G(\varepsilon_1,0|\varepsilon_2,0)=\delta(\varepsilon_1-\varepsilon_2). \tag{A19}$$

From Eq. (17) we now get the equation

$$\frac{d}{dt}\langle\varepsilon\varepsilon(t)\rangle=-\Gamma\langle\varepsilon\varepsilon(t)\rangle. \tag{A20}$$

The solution of this equation has the form

$$\langle\varepsilon\varepsilon(t)\rangle=\langle\varepsilon^2\rangle e^{-\Gamma t}. \tag{A21}$$

Thus, we still need to know the quantity $\langle\varepsilon^2\rangle$, an equation for which is also not difficult to obtain from Eq. (17):

$$\frac{d}{dt}\langle\varepsilon^2\rangle=-2\Gamma\langle\varepsilon^2\rangle+2D_a=0. \tag{A22}$$

We have set the derivative equal to zero since we are interested only in the stationary solution. These formulas make it possible to bring this problem to a conclusion. And we see that higher derivatives indeed do not contribute to the desired quantity.

In exactly the same way we could work out the optical spectrum, which, as is well known, is defined by the expression

$$\langle e^{i\varphi(t)-i\varphi(0)}\rangle. \tag{A23}$$

To start with, it is necessary to write out the equation for the phase density matrix

$$\Phi(\varphi,t)=\int d\varepsilon P(\sqrt{n_0+\varepsilon}e^{i\varphi},t), \tag{A24}$$

and then construct an equation for the unknown quantity, which is just as simply solved. Strictly speaking, the equation for Φ contains the fourth derivative with respect to φ besides the usual second. However, because the coefficient of the second derivative is positive and in this case there is no "quantum" unpleasantness, the fourth derivative can with good accuracy be dropped.

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