Transition radiation of moderately relativistic particles in a plasma with random inhomogeneities

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Transition radiation in particles of arbitrary energy in studied is an isotropic plasma with random inhomogeneities in the electron density. The angular distribution of the radiation is found as a function of particle energy. The transition radiation spectra are found, valid for arbitrary frequencies for particles of arbitrary energy. Convenient analytical approximations to the exact expressions are suggested. It is shown that near the plasma frequency the spectra contain a large narrow peak (the resonant transition radiation) whose height is larger by a factor of c/v_T than was previously assumed in analyzing the polarization radiation. The emission spectra are also calculated for an ensemble of particles with a power-law momentum distribution. The important role of these results in applications is discussed.

1. INTRODUCTION

The emission from charged particles of various energies in a medium differs in the variety of emission mechanisms.[1–3] One of these mechanisms in media with inhomogeneities[4] is transition radiation (TR).[2,5] Interest in the effects accompanying the generation of TR in a magnetized plasma[6,7] arises because in certain frequency intervals it can dominate over all other radiation in astrophysical sources, and in particular, in solar flares.[8] Up to now the TR spectra of ultrarelativistic particles in an isotropic plasma,[2] in a magnetized plasma including the curvature of the particle trajectories,[6] and in a gyrotropic plasma[7] have been calculated. In Ref. 6 it was shown that TR is produced efficiently even by relatively low-energy particles with

$$\gamma<\omega_p/\omega_{Be}\,, \tag{1}$$

where $\gamma=E/mc^2$ is the Lorentz factor of the particle, ω_p is the plasma frequency, and $\omega_{Be}=eB/mc$ is the electron gyrofrequency. Under natural conditions there is usually an extended charged-particle spectrum falling off with energy, e.g., $dN_e\propto N_eE^{-\xi}dE$. In this case the emission from moderately relativistic and nonrelativistic charges can turn out to be important. However, at present there are no rigorously calculated TR spectra for a particle of arbitrary energy.

Our work is devoted to calculating the TR of charged electrons (or ions) of arbitrary energy [subject to Eq. (1)] in a plasma containing an extended spectrum of random density variations:

$$|\delta N|^2_{\mathbf{k}'}=\frac{\nu-1k_0^{\nu-1}}{4\pi}\,\frac{\langle\Delta N^2\rangle}{k'^{\nu+2}}\,, \tag{2}$$

where ν is the index of the spectrum of inhomogeneities and we have written $k_0=2\pi/L_0$, where L_0 and $\langle\Delta N^2\rangle$ are the characteristic size and mean square density of the inhomogeneities. In this work we derive the spectral and angular distribution of the radiation intensity from a charge of arbitrary energy, the exact expression for the TR spectrum and its approximation, rigorously describing the exact expression for $\omega>\omega_p$, and also the emissivity of an ensemble of particles with a power-law spectrum.

2. DIFFERENTIAL TRANSITION-RADIATION SPECTRUM OF A SINGLE MOVING CHARGE

As shown by Ginzburg and Tsytovich[2] (see also Ref. 5), the source of the TR in the plasma electron current excited by the quasisteady field of a fast charged particle:

$$\mathbf{j}^m_{\omega,\mathbf{k}}=\frac{ie^2}{m\omega}\int d\mathbf{k}'\mathbf{E}^q_{\omega,\mathbf{k}-\mathbf{k}'}\delta N_{\mathbf{k}'}\,, \tag{3}$$

where e and m are the electron charge and mass, $\delta N_{\mathbf{k}'}$ is the variation in the electron density of the medium, and $\mathbf{E}^q_{\omega,\mathbf{k}-\mathbf{k}'}$ is the quasisteady electric field of the radiating particle. The field of the particle is related to its current by

$$\mathbf{j}^q_{\omega,\mathbf{k}-\mathbf{k}'}=\frac{q\mathbf{v}}{(2\pi)^3}\,\delta[\omega-(\mathbf{k}-\mathbf{k}')\mathbf{v}], \tag{4}$$

(q is the particle charge and v is its velocity) through the Green's function. In the present instance, in contrast to the ultrarelativistic limit,[6] we must take into account not only the transverse but also the longitudinal Green's function. As a result the quasisteady field $\mathbf{E}^q$ takes the form

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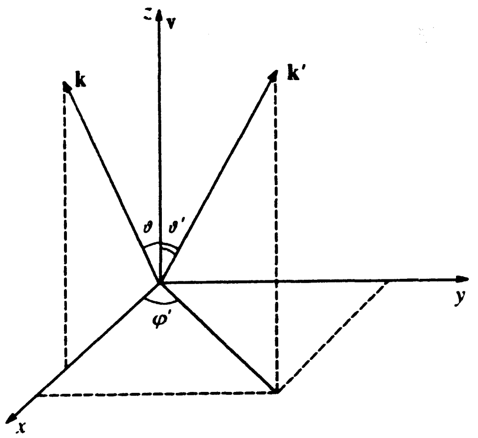

FIG. 1. The coordinate system.

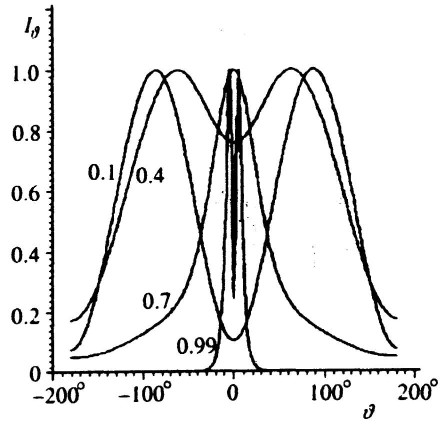

FIG. 2. Normalized angular distributions of the TR for different values of the parameter $v/v_{\rm ph}$=0.1, 0.4, 0.7, 0.99. Here ν=1.5.

$$E^{q,i}_{\omega,\mathbf{k}-\mathbf{k}'}=\frac{4\pi iq\,\delta[\omega-(\mathbf{k}-\mathbf{k}')\mathbf{v}]}{(2\pi)^3}\,\frac{\omega[v_i-(\mathbf{k}-\mathbf{k}')_i\omega/k^2]}{c^2[(\mathbf{k}-\mathbf{k}')^2-k^2]}\,. \tag{5}$$

The energy radiated by the current (3) at frequency ω in the direction $\mathbf{n}$ is described by the expression[1-6]

$$E_{\mathbf{n},\omega}=(2\pi)^6\,\frac{\omega^2}{c^3}\,\sqrt{\varepsilon}\langle|[\mathbf{n}\mathbf{j}^m_{\omega,\mathbf{k}}]|^2\rangle\,, \tag{6}$$

where $\varepsilon = 1 - \omega_{\rm p}^2/\omega^2$ is the plasma dielectric function. It is necessary to include the quantity $\sqrt{\varepsilon}$ in (6), since we are trying to describe TR rigorously at all frequencies $\omega\geqslant\omega_{\rm p}$, including also $\omega\approx\omega_{\rm p}$, where ε differs substantially from unity. The ultrarelativistic description[6] is valid for relativistic particles $\gamma\gg1$ at high frequencies $\omega\gg\omega_{\rm p}$, where $\varepsilon\approx1$.

The use of expressions (3)–(6) enables us to derive the frequency and angular distribution of the radiation intensity:

$$I_{\mathbf{n},\omega}=\frac{8\pi e^4q^2}{m^2c^3\omega^2\varepsilon^{3/2}}\int d\mathbf{k}'|\delta N|^2_{\mathbf{k}'}\,\frac{([\mathbf{n}\mathbf{v}]+[\mathbf{n}\mathbf{k}']\omega/k^2)^2}{[(\mathbf{k}-\mathbf{k}')^2/k^2-1]^2}\times\delta[\omega-(\mathbf{k}-\mathbf{k}')\mathbf{v}]\,. \tag{7}$$

The integrand in Eq. (7) depends on all three dummy variables (including the azimuthal angle φ' of the vector $\mathbf{k}'$). In performing the integration of Eq. (7) it is convenient to assume that the particle moves in the direction parallel to the z axis, while the vector $\mathbf{k}$ is in the (x,z) plane. Then the azimuthal angle φ' is equal to the angle between the projection of the vector $\mathbf{k}'$ on the (x,y) plane and the x axis (the projection of the vector $\mathbf{n}$ on the same plane); cf. Fig. 1. The radiation intensity (7) is given in the form

$$I_{\mathbf{n},\omega}=\frac{8\pi e^4q^2}{vm^2c^3\omega^2\varepsilon^{3/2}}\int_0^\infty k'dk'|\delta N|^2_{k'}\Phi, \tag{8}$$

where

$$\Phi=\int_{-1}^{1}d\cos\vartheta'\,\delta(\cos\vartheta'-\cos\vartheta_r)\times\int_0^{2\pi}\frac{k^4d\varphi'}{(a+b\cos\varphi')^2}\left\{[\mathbf{n}\mathbf{v}]^2+\frac{\omega^2k'^2}{k^4}\times(1-\cos^2\vartheta\cos^2\vartheta'-2\cos\vartheta\cos\vartheta'\sin\vartheta\sin\vartheta'\cos\varphi'-\sin^2\vartheta\sin^2\vartheta'\cos^2\varphi')+\frac{2v\omega k'}{k^2}(\cos\vartheta'\sin^2\vartheta-\cos\vartheta\sin\vartheta\sin\vartheta'\cos\varphi')\right\}, \tag{9}$$

with $\cos\vartheta_{\rm r}=-(\omega-kv\cos\vartheta)/k'v$, $a=k'^2-2kk'\cos\vartheta\cos\vartheta'$, $b=-2kk'\sin\vartheta\sin\vartheta'$. The integration over $d\cos\vartheta'$ is performed trivially using the δ function and gives rise to a θ function $\theta(1-\cos^2\vartheta_{\rm r})$, associated with the obvious requirement $\cos^2\vartheta_{\rm r}\leqslant1$. Replacing $\cos\vartheta'$ by $\cos\vartheta_{\rm r}$ in Eq. (9) and expanding the resulting expression in partial fractions, we find

$$\Phi=\theta(1-\cos^2\vartheta_{\rm r})\int_0^{2\pi}k^4d\varphi'\left\{-\frac{\omega^2}{4k^6}+\left(\frac{\omega^2k'^2}{2k^6}+\frac{\omega v}{k^3}\cos\vartheta\right)\frac{1}{a+b\cos\varphi'}+\left[v^2\sin^2\vartheta+\frac{\omega v}{k}\times\left(2-\frac{k'^2}{k^2}\right)\cos\vartheta-\frac{\omega^2k'^4}{4k^6}+\frac{\omega^2k'^2}{k^4}-\frac{2\omega^2}{k^2}\right]\times\frac{1}{(a+b\cos\varphi')^2}\right\}. \tag{10}$$

Performing the integrals with respect to φ' in (10) (which are tabulated) and simplifying the results in terms of their power in $\cos\vartheta$ in the last term and combining common terms, we find the following:

$$\Phi=\frac{\pi\omega^2}{2k^2}\,\theta(1-\cos^2\vartheta_r)\left\{-1+\frac{\beta^{-2}\cos^2\vartheta+\beta^{-1}\cos\vartheta+(1-\beta^{-2})(1-y^2/2+y^4/4)+y^2/2}{\sqrt{\cos^2\vartheta+\beta(y^2-2)\cos\vartheta+\beta^2-y^2+y^4/4}}\right.$$
$$\left.+(1-\beta^{-2})\left(1-\frac{y^2}{2}\right)^2\frac{\beta(1-y^2)\cos\vartheta-\beta^2+y^2/2-y^4/4}{[\cos^2\vartheta+\beta(y^2-2)\cos\vartheta+\beta^2-y^2+y^4/4]^{3/2}}\right\}, \tag{11}$$

where $\beta=\omega/kv$, $y=k'/k$. Thus, the frequency and angular distribution of the TR intensity reduces to a single integration over the spectrum of the plasma density inhomogeneities:

$$I_{\mathbf{n},\omega}=\frac{4\pi e^4q^2}{vm^2c^3\varepsilon^{3/2}}\int_{\beta-\cos\vartheta}^{\infty}ydy|\delta N|^2_{\mathbf{k}'}F(y,\beta,\cos\vartheta), \tag{12}$$

where $F(y,\beta,\cos\vartheta)=2k^2\Phi/\pi\omega^2\theta(1-\cos^2\vartheta_r)$. However, it is impossible to integrate (12) analytically with the power-law spectrum (2). Figure 2 displays the normalized angular distribution of the TR intensity at different values of the parameter kv/ω, obtained by numerical integration of (12). In the nonrelativistic region the radiation is similar to that from a dipole and its maximum occurs at angles of order 90° with respect to the particle velocity. As kv/ω increases the radiation maximum approaches the direction in which the particle is moving, and in the ultrarelativistic limit a strong directionality is observed for the radiation in the direction of the velocity vector. Note that in the calculating the TR in the ultrarelativistic limit[6] the radiation intensity in the forward direction $\vartheta=0$ vanishes. In Fig. 2 we see $I(\vartheta=0)>0$, which is related to the inclusion of the longitudinal self-field of the relativistic particle; this was discarded in Ref. 6.

3. INTEGRATED SPECTRUM OF THE TRANSITION RADIATION FROM A SINGLE MOVING CHARGE

We proceed now to the evaluation of the TR intensity over the full solid angle:

$$I_\omega=2\pi\int_{-1}^{1}I_{\mathbf{n},\omega}d\cos\vartheta. \tag{13}$$

Changing the order of integration with respect to y and $\cos\vartheta$ we find

$$I_\omega=\frac{8\pi^3e^4q^2}{vm^2c^3\varepsilon^{3/2}}\left\{\int_{\beta-1}^{\beta+1}ydy\int_{\beta-y}^{1}d\cos\vartheta+\int_{\beta+1}^{\infty}ydy\int_{-1}^{1}d\cos\vartheta\right\}|\delta N|^2_{\mathbf{k}'}F(y,\beta,\cos\vartheta). \tag{14}$$

Since the quantity $|\delta N|^2_{\mathbf{k}'}$ does not depend on the angle ϑ for isotropically distributed inhomogeneities, we need only integrate the function F over angle. We introduce two new functions F_1 and F_2:

$$F_1=\int_{\beta-y}^{1}Fd\cos\vartheta;\quad F_2=\int_{-1}^{1}Fd\cos\vartheta. \tag{15}$$

Taking into account the variation of y in each of the integrals, we find by evaluating (15)

$$F_1=\frac{(y+1-\beta)(y^2-2)^2[1+\beta(y+1)]}{2\beta^2y(y+2)}+(y+1-\beta)$$
$$\times\left(-\frac{3}{4\beta}y^3+\frac{3\beta-1}{4\beta^2}y^2+\frac{6\beta+1}{2\beta^2}y-\frac{1}{2\beta^2}-\frac{5}{2\beta}-1\right)$$
$$+\left(\frac{5\beta^2-3}{8\beta^2}y^4-\frac{2\beta^2-1}{\beta^2}y^2+\frac{3\beta^2-1}{\beta^2}\right)$$
$$\times\ln\frac{y(\beta+1)}{(y+2)(\beta-1)}, \tag{16}$$

$$F_2=\frac{3-5\beta^2}{2\beta^2}y^2+\frac{3\beta^2-1}{\beta^2}+\frac{\beta^2-1}{\beta^2(y^2-4)}$$
$$-\left(\frac{5\beta^2-3}{8\beta^2}y^4-\frac{2\beta^2-1}{\beta^2}y^2+\frac{3\beta^2-1}{\beta^2}\right)$$
$$\times\ln\left(1-\frac{4}{y^2}\right). \tag{17}$$

If we substitute the spectrum of the equilibrium thermal fluctuations $|\delta N|^2_{\mathbf{k}'}$ in (14) and use (16) and (17), we obtain as a result the intensity of the polarized bremsstrahlung. This problem has been analyzed by Akapyan and Tsytovich.[9] Unfortunately, the integration cannot be completely carried out in terms of elementary functions.

Consider the emission that occurs when superthermal noise is present in the plasma. After expressing the spectrum (2) of the inhomogeneities in terms of the dimensionless variable y,

$$|\delta N|^2_{\mathbf{k}'}=\frac{\nu-1}{4\pi}\frac{k_0^{\nu-1}\langle\Delta N^2\rangle}{k^{\nu+2}}y^{-\nu-2}, \tag{18}$$

we can represent the radiation intensity in the form

$$I_\omega=\frac{2\pi^2(\nu-1)e^4q^2\langle\Delta N^2\rangle k_0^{\nu-1}}{vm^2c^3\varepsilon^{3/2}k^{\nu+2}}\left\{\int_{\beta-1}^{\beta+1}F_1y^{-\nu-1}dy+\int_{\beta+1}^{\infty}F_2y^{-\nu-1}dy\right\}. \tag{19}$$

Expression (19) is evaluated by expanding the functions F_1 and F_2 in partial fractions and integrating by parts the terms which contain logarithms. After applying some identities we find

$$I_\omega=\frac{8\pi^2(\nu-1)}{(4-\nu)(2-\nu)}\frac{ve^4q^2\langle\Delta N^2\rangle k_0^{\nu-1}}{\omega^2m^2c^3\varepsilon^{3/2}k^\nu}$$
$$\times\sum_{\sigma=\pm1}\left\{(\beta+\sigma)^{-\nu}\left[\frac{2(\beta\sigma)^3}{\nu(\nu-1)}+\frac{2(\beta\sigma)^2}{\nu-1}\right.\right.$$
$$\left.+\frac{\nu^2+2\nu+16}{4(\nu+1)}\left(\frac{\nu+2}{\nu}\beta\sigma+1\right)\right]+(\nu^2-2\nu+8)$$
$$\left.\times\frac{(\nu+6)\beta^2-(\nu+2)}{4\nu}\sigma\int_{\beta-\sigma}^{\infty}\frac{y^{-\nu-1}dy}{y+2\sigma}\right\}. \qquad (20)$$

The summation over $\sigma=\pm1$ has been introduced in Eq. (20) to make it more compact; the integral that remains can be expressed in terms of hypergeometric functions.[10] For a charged particle of arbitrary energy moving rectilinearly Eq. (20) is valid at frequencies $\omega>\omega_p$, except for a small region near ω_p, $\omega\lesssim\omega_p(1+v_T^2/v^2)$. In order to describe the intensity I_ω rigorously in this region we must take into account the spatial dispersion of the plasma. The analysis presented in the Appendix shows that the role of spatial dispersion reduces to the substitution

$$\varepsilon^{-3/2}\to F(\alpha), \qquad (21)$$

where

$$F(\alpha)=2\varepsilon^{-3/2}\left\{\frac{1}{\alpha}\left[\frac{1}{1-\alpha}+2+\frac{\alpha}{2}+\frac{3}{\alpha}\ln(1-\alpha)\right]\right.$$
$$\times\theta(\omega_1-\omega)+\frac{1}{\alpha^2}\frac{c}{2\sqrt{3}v_T}\left(1-\frac{6\sqrt{3}v_T}{c}\right.$$
$$\left.\times\ln\frac{c}{2\sqrt{3}v_T}\right)\theta(\omega-\omega_1)\theta(\omega_2-\omega)$$
$$+\left[\frac{1}{\alpha^2}\frac{c}{\sqrt{3}v_T}+\frac{1}{\alpha}\left(\frac{1}{1-\alpha}+2+\frac{\alpha}{2}+\frac{3}{\alpha}\right.\right.$$
$$\left.\left.\left.\times\ln(\alpha-1)\right)\right]\theta(\omega-\omega_2)\right\}, \qquad (22)$$

and we have written $\alpha=(\varepsilon/3)(v/v_T)^2$, where v_T is the thermal velocity of the plasma electrons and

$$\omega_{1,2}=\omega_p\left[1+\frac{3}{2}\left(\frac{v_T}{v}\right)^2\left(1\mp\frac{2\sqrt{3}v_T}{c}\right)\right]. \qquad (23)$$

The function (22) describes a strong narrow peak in a spectrum near the frequencies $\omega_{1,2}$. We will refer to this part of the emission as resonant TR and label it with the subscript "R." Under the condition $(\omega-\omega_p)/\omega_p\gg(v_T/v)^2$ we have $F(\alpha)\approx\varepsilon^{-3/2}$ and expression (20) is valid. In the opposite limit $\omega\to\omega_p$, $\alpha\ll1$ we have

$$F(\alpha)\approx\frac{1}{18}\sqrt{\varepsilon}\left(\frac{v}{v_T}\right)^4.$$

If we note that $\beta\gg1$ holds here we find

$$I_\omega(\omega\to\omega_p)=\frac{16\pi^2(\nu-1)}{27(\nu+2)}$$
$$\times\sqrt{\varepsilon}\frac{e^4q^2\langle\Delta N^2\rangle k_0^{\nu-1}}{c^3m^2}\left(\frac{v}{v_T}\right)^4\frac{v^{\nu+1}}{\omega_p^{\nu+2}}. \qquad (24)$$

Similar asymptotic expressions in the neighborhood of the plasma frequency were given in Ref. 3 (p. 59) for the polarized emission in a plasma. For $\omega\sim\omega_0=(\omega_1+\omega_2)/2$ both asymptotic forms (20) and (24) give rise to quantities of the same magnitude for the radiation intensity. On this basis Amus'ya *et al.*[3] reached the conclusion that the height of the peak at the frequency ω_0 was equal to $(v/v_T)^3$. However, as can be seen from (22), the peak is higher by approximately a factor of c/v_T, which is related to the complicated shape of the spectrum for $\omega\approx\omega_0$. Resonant polarized bremsstrahlung is discussed in Ref. 11.

We proceed to find asymptotic representations of the expression

$$S=\Sigma_{\sigma=\pm1}\cdots,$$

which depends on β, for small and large values of $\beta-1$. For $\beta-1\ll1$ we take $\beta=1$ everywhere except in terms of the form $\beta-1$, and in the denominator of the integrand we neglect y in comparison with 2. Then we find

$$S=\frac{(4-\nu)(2-\nu)}{2\nu^2(\nu+1)}(\beta-1)^{-\nu}. \qquad (25)$$

Substituting (25) in (20) and setting $v=c$, $\varepsilon=1$ in the coefficient, we find the ultrarelativistic limit of TR, which agrees with the familiar result[6]

$$I_\omega^{\text{UR}}=\frac{8\pi^2(\nu-1)}{\nu^2(\nu+1)}\frac{e^4q^2\langle\Delta N^2\rangle}{cm^2\omega^3}\frac{(2k_0c/\omega)^{\nu-1}}{[\gamma^{-2}+\omega_p^2/\omega^2]^\nu} \qquad (26)$$

and is correct in the limit $\gamma\gg1$, $\omega\gg\omega_p$.

In the opposite limiting case $\beta-1\gg1$ the quantities $(\beta\pm1)^{-\nu}$ must be expanded in powers of $1/\beta$, where the first nonvanishing terms of this expansion arise only in fourth order. Evaluating the integrals to the same accuracy in $1/\beta$ we find

$$S=\frac{4(4-\nu)(2-\nu)}{3(\nu+2)}\beta^{-\nu}. \qquad (27)$$

As a result, for $\beta-1\gg1$ we have

$$I_\omega^{\text{NR}}=\frac{32\pi^2(\nu-1)}{3(\nu+2)}\frac{e^4q^2\langle\Delta N^2\rangle k_0^{\nu-1}}{c^3m^2\omega^{\nu+2}}F(\alpha)v^{\nu+1}. \qquad (28)$$

Note that expression (28) is valid also for the TR of relativistic particles in the limit $\omega\to\omega_p$, since in this case we have $\beta-1\gg1$.

Using the asymptotic forms (25) and (27) and carrying out the usual procedure for approximating the results of a numerical calculation of the function S by an analytical formula we arrive at the result:

$$S=\frac{(4-\nu)(2-\nu)}{2\nu^2(\nu+1)}\{A_1+A_2\}, \qquad (29)$$

where

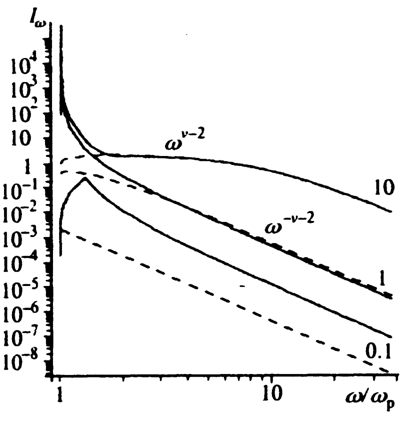

FIG. 3. Set of TR spectra for different values of the dimensionless charged-particle momentum $x=p/mc=0.1, 1, 10$. The broken traces represent spectra calculated using the relativistic asymptotic forms; $\nu=1.7$.

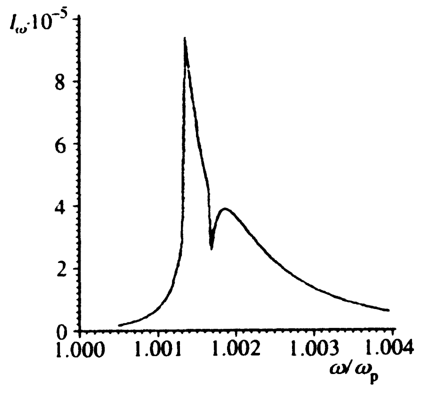

FIG. 4. Spectrum of resonant transition radiation near the plasma frequency. Parameters: $x=10$, $v_T/c=0.032$, $\nu=1.7$.

$$A_1=(\beta-1)^{-\nu}+\frac{8\nu^3+8\nu^2-3\nu-6}{3(\nu+2)}\beta^{-\nu}, \tag{30}$$

$$A_2=-\frac{400(1.18\nu^2-2.17\nu+1.18)}{3(\nu+2)}\beta^{-3.03\nu-1.14}. \tag{31}$$

Retaining only the quantities A_1 [consisting of the asymptotic forms (25) and (26)] in (29) yields the correct order of magnitude (the error is less than 100%), and the correction A_2 has an average accuracy $\sim$10% and a maximum error $\sim$20%. Thus, the intensity of the transition radiation of a charged particle of arbitrary energy at all frequencies $\omega \geqslant \omega_p$ is given by the expression

$$I_\omega=\frac{4\pi^2(\nu-1)}{\nu^2(\nu+1)}\frac{e^4q^2\langle\Delta N^2\rangle k_0^{\nu-1}}{m^2c^3}\frac{v}{k^\nu\omega^2}F(\alpha)$$
$$\times\left\{\left(\frac{\omega}{kv}-1\right)^{-\nu}+\frac{8\nu^3+8\nu^2-3\nu-6}{3(\nu+2)}\left(\frac{kv}{\omega}\right)^{\nu}\right.$$
$$\left.-\frac{400(1.18\nu^2-2.17\nu+1.18)}{3(\nu+2)}\left(\frac{kv}{\omega}\right)^{3.03\nu+1.14}\right\}. \tag{32}$$

Spectra of TR calculated according to Eq. (32) and the relativistic asymptotic forms for various values of the particle momentum are displayed in Fig. 3. It can be seen that at high frequencies the relativistic expressions provide a good description of the emission from high-energy particles, while near the plasma frequency a strong narrow peak develops which exceeds the level of the "substrate" by several orders of magnitude. Figure 4 shows this peak in more detail. Note that the spectrum has two local maxima: the small declivity on the right slope is related to "impoverishment" of the quasisteady field of the charge with respect to harmonics due to the emission of protons at the corresponding frequencies (these frequencies correspond to propagating Cherenkov plasmons, not the quasisteady field). However, this declivity plays a minor energetic role; for example, it disappears entirely when we integrate over the spectrum of the emitting particles or over the volume of the nonuniform plasma.

Now let us find the total energy emitted by the resonant transition mechanism. Integrating the spectrum (28) over frequency, which is valid near ω_p for particles of arbitrary energy, we find

$$I_{\text{tot}}^{\text{R}}=\frac{16\pi^2(\nu-1)}{45}\frac{e^4q^2\langle\Delta N^2\rangle k_0^{\nu-1}}{c^3m^2}\left(\frac{v}{\omega_p}\right)^{\nu+1}\frac{vc}{v_T^2}. \tag{33}$$

Comparing (33) with the energy emitted in the ultrarelativistic region,[5] we find that for particles with $\gamma<c^2/v_T^2$ the resonant TR is more substantial than ordinary TR. For $\gamma>c^2/v_T^2$ most of the energy is emitted at frequencies $\omega_{\max}\sim\omega_p\gamma>\omega_pc^2/v_T^2$.

If the charged particle is moving in a plasma of continuously varying density [with some distribution $\Phi(\omega_p)$ over the plasma frequencies], then the principal contribution to the observed emission at frequency ω comes from the regions of the plasma with $\omega_p\approx\omega$. To estimate the TR from these (large-scale) inhomogeneous sources it is convenient to use the approximation

$$I_\omega^{\text{R}}=\frac{16\pi^2(\nu-1)}{45}\frac{e^4q^2\langle\Delta N^2\rangle k_0^{\nu-1}}{c^3m^2}\left(\frac{v}{\omega_p}\right)^{\nu+1}$$
$$\times\frac{vc}{v_T^2}\delta(\omega-\omega_p), \tag{34}$$

which should be integrated over the function $\Phi(\omega_p)$.

These results can be used under laboratory conditions to analyze experiments with monoenergetic particle beams. For astrophysical applications it is often necessary to average as well over the energy distribution of the emitting particles.

4. TRANSITION RADIATION FROM AN ENSEMBLE OF PARTICLES WITH A POWER-LAW SPECTRUM

Usually the spectrum of emitting particles under astrophysical conditions can be represented in the form

$$dN_e=(\xi-1)N_e(x>x_0)\frac{x_0^{\xi-1}dx}{x^\xi},\quad x_0<x<x_1, \tag{35}$$

where $x=p/mc$ is the dimensionless particle momentum. Then the emission from an ensemble of particles with the spectrum (35) can be written as follows:

$$P_\omega=\frac{4\pi^2(\nu-1)(\xi-1)}{\nu^2(\nu+1)}\frac{e^4q^2\langle\Delta N^2\rangle k_0^{\nu-1}}{m^2c^2}\times N_e(x>x_0)\frac{G(\omega)}{k^\nu\omega^2}, \tag{36}$$

where

$$G(\omega)=\int_{x_0}^{x_1}\frac{F(\alpha)x_0^{\xi-1}dx}{\sqrt{1+x^2}x^{\xi-1}}\left\{(\beta-1)^{-\nu}+\frac{8\nu^3+8\nu^2-3\nu-6}{3(\nu+2)}\beta^{-\nu}-\frac{400(1.18\nu^2-2.17\nu+1.18)}{3(\nu+2)}\beta^{-3.03\nu-1.14}\right\}, \tag{37}$$

and the magnitude of the velocity v which enters into the definition of α should be expressed in terms of the dimensionless momentum:

$$\frac{v}{c}=\frac{x}{\sqrt{1+x^2}}. \tag{38}$$

The most important thing to consider is the asymptotic behavior of the radiation spectrum in the region $\omega\gg\omega_p$, when the peak at $\omega\to\omega_p$ is unimportant. As will become clear, the shape of the corresponding asymptotic forms varies in three different cases: 1) $\xi>2\nu+1$, 2) $\nu+2<\xi<2\nu+1$, 3) $\xi<\nu+2$. If the number of particles as a function of x falls off sufficiently rapidly ($\xi>2\nu+1$), then the main contribution to the emission comes from the nonrelativistic particles. Then we can set $x=v/c$ and integrate (37) with respect to v from v_0 to infinity. Since it is necessary here to evaluate the integral

$$\int_{v_0}^{\omega}x^{\nu+1-\xi}dx,$$

[cf. the asymptotic form (25)], it is clear that it will converge for

$$\xi>\nu+2, \tag{39}$$

which always holds in this region, since we have $\nu+2<2\nu+1$ for $\nu>1$ and the radiation spectrum can be written in the form

$$P_\omega=\frac{32\pi^2(\nu-1)}{3(\nu+2)(\xi-\nu-2)}\frac{e^4q^2\langle\Delta N^2\rangle k_0^{\nu-1}}{m^2c^3}N_e(v>v_0)\times\frac{v_0^{\nu+1}}{\omega^{\nu+2}}. \tag{40}$$

In the other limiting case $\xi<\nu+2$, the main contribution to the radiation at these frequencies is associated with the ultrarelativistic particles. In accordance with the known results,[8] the integration of (37) for $\omega<\min\{\omega_p/\omega_{Be},\omega_p\gamma_1\}$, where $\gamma_1=\sqrt{1+x_1}$ is the maximum Lorentz factor in the spectrum (35), yields

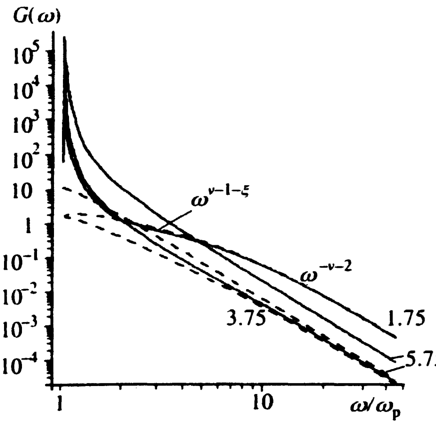

FIG. 5. Emission from an ensemble of charged particles with a power-law momentum distribution for different values of the spectral index ξ=1.75, 3.75, 5.75. The broken traces indicated spectra calculated using the relativistic asymptotic forms; ν=1.5.

$$P_\omega=\frac{8\pi^2(\nu-1)\Gamma[(\xi+1)/2]\Gamma[(2\nu-\xi+1)/2]}{\nu^2(\nu+1)\Gamma(\nu)}x_0^{\xi-1}\times N_e\frac{e^4q^2\langle\Delta N^2\rangle}{cm^2\omega_p^3}\left(\frac{2k_0c}{\omega_p}\right)^{\nu-1}\left(\frac{\omega_p}{\omega}\right)^{\xi+1-\nu}. \tag{41}$$

Finally, for

$$\nu+2<\xi<2\nu+1 \tag{42}$$

both expressions (40) and (41) hold, and the total spectrum is given by their sum. Since the emission spectrum in the relativistic case $\omega^{\nu-1-\xi}$ is more gentle than in the nonrelativistic spectrum $\omega^{-\nu-2}$, at lower frequencies the nonrelativistic emission is observed, while at higher frequencies the emission comes from the relativistic particles, and the contribution of particles with $E_{kin}\sim mc^2$ is negligible.

Figure 5 displays the behavior of the function $G(\omega)$ as a function of frequency for three different values of ξ, occurring in the three intervals described (we set ν=1.5 in generating the plots). The solid traces indicate the results of calculations according to the exact formula (37), while the broken traces correspond to the relativistic asymptotic forms. From the figure it is clear that the estimates given above agree well with the asymptotic forms of the exact calculations. The value of the TR intensity at the peak that remains after integration over the particle spectrum can be several orders of magnitude greater than the emission from the "substrate." These peaks are shown in a form which is convenient for interpretation in Fig. 6. It is clear that now the declivity in the neighborhood of the maximum of the spectrum is gone. For soft distributions (large values of ξ) the peaks are lower and smeared out. The reason for this is easily discerned from the structure of the singularity (22).

In order to analyze the TR from large-scale inhomogeneous plasmas we should integrate expression (34) over the spectrum (35):

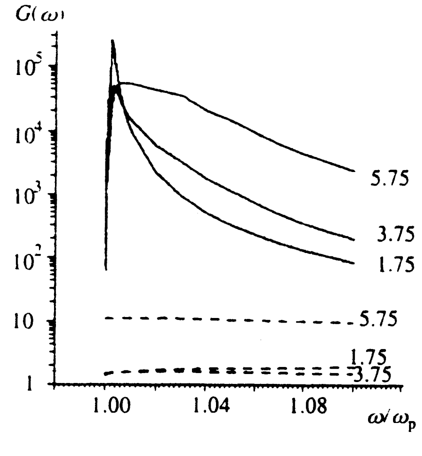

FIG. 6. Blowup of the peak in the radiation shown in Fig. 5.

$$P^{\mathrm{R}}_{\omega}=\frac{\nu-1}{90}\,\frac{(\xi-1)\Gamma((\xi-1)/2)\Gamma(\nu-\xi+3/2)}{\Gamma((\nu+2)/2)}\,\frac{e^2}{c}\,x_0^{\xi-1}$$
$$\times N_e\omega_{\mathrm{p}}^2\,\frac{\langle\Delta N_e^2\rangle}{n_e^2}\left(\frac{k_0c}{\omega_{\mathrm{p}}}\right)^{\nu-1}\frac{c^2}{v_T^2}\,\delta(\omega-\omega_{\mathrm{p}}). \tag{43}$$

Here n_e is the density of the warm electrons. In this expression the main contribution comes from particles of moderate relativistic energy with $E_{\mathrm{kin}}\sim mc^2$.

5. CONCLUSION

We have considered transition radiation from charged particles of arbitrary energy in a plasma with random density variations in the absence of external fields. We have found and analyzed the frequency and angular distribution and the spectral distribution of the radiation. In the latter case we have derived a simple approximate formula in place of the exact expression (20), which is accurate to about 10% and has a maximum error of order 20%. We have also studied the emission from an ensemble of particles with a power-law momentum distribution.

In Ref. 8 it was shown that some radio flares occurring on the Sun can be explained by the mechanism of transition radiation if the plasma has density irregularities of order

$$\frac{\langle\Delta N^2\rangle}{N^2}\sim10^{-3}-10^{-4}. \tag{44}$$

The peak found in this work, which is several orders of magnitude greater than the emission calculated in the relativistic limit, can, generally speaking, reduce the level of the observed density irregularities in the plasma in solar flares to values

$$\frac{\langle\Delta N^2\rangle}{N^2}\sim10^{-7}-10^{-8} \tag{45}$$

exceeding the level of equilibrium (thermal) fluctuations by no more than one or two orders of magnitude. It should, however, be kept in mind that under the conditions of large-scale inhomogeneity in the solar corona, these peaks will be smeared out and will be reduced (in comparison with a perfectly uniform source of the same volume). Nonetheless, since the total energy of the transition radiation including the peak is found to be larger by about c^2/v_T^2 than in the ultrarelativistic case, the role of this mechanism can increase considerably. Note that the similar intensity enhancement factor indicated in Ref. 3 for polarized radiation, was smaller by a factor c/v_T (Ref. 11).

APPENDIX: CALCULATION OF RESONANT TRANSITION RADIATION

The transition radiation generated near the plasma frequency may be termed resonant by analogy with resonant polarized radiation.[3]

Since the phase velocity of transverse waves near ω_p is considerably greater than the velocity of light and $v/v_{\mathrm{ph}}\ll1$ holds for arbitrary $v<c$, when we calculate resonant transition radiation to an accuracy of $(v/v_{\mathrm{ph}})^2$ we can restrict ourselves to the contribution of the longitudinal field of the fast particle (the nonrelativistic approximation). However, the dielectric function that enters the expression for this field should be written so as to retain the spatial dispersion

$$\varepsilon(\omega,\mathbf{k})=\varepsilon(\omega)-3k^2d^2+i\varepsilon'',$$

since for $\omega\sim\omega_{\mathrm{p}}$ we have $\varepsilon(\omega)\sim k^2d^2$. Then the intensity of the resonant transition radiation can be represented in the form

$$I^{\mathrm{R}}_{\mathbf{n},\omega}=\frac{8\pi e^4q^2\varepsilon^{1/2}}{m^2c^3}\int k'^2dk'$$
$$\times\frac{[\mathbf{n}\mathbf{k}']^2\delta[\omega-(\mathbf{k}-\mathbf{k}')\mathbf{v}]\,|\delta N|^2_{\mathbf{k}'}d\varphi d\cos\vartheta}{(\mathbf{k}-\mathbf{k}')^4[(\varepsilon(\omega)-3(\mathbf{k}-\mathbf{k}')^2d^2)^2+\varepsilon''^2]}, \tag{A1}$$

where $d=v_T/\omega_{\mathrm{p}}$ is the Debye radius and the imaginary part ε'' of the dielectric function is written so as to eliminate the divergence when (A1) is integrated. Note that at the frequencies in question we have $\varepsilon(\omega)\ll1$ and $k\ll k'$. This enables us to neglect $\mathbf{k}$ in comparison with $\mathbf{k}'$ everywhere except in the resonant denominator. Then it is convenient to integrate (A1) over the angles of the vector $\mathbf{n}$, i.e., find the energy emitted over the complete solid angle (the directionality diagram in this case corresponds to that of a dipole):

$$I^{\mathrm{R}}_{\omega}=\frac{32\pi^3e^4q^2\sqrt{\varepsilon}}{vm^2c^3}\int_{\omega/v}^{\infty}\frac{dk'}{k'}|\delta N|^2_{\mathbf{k}'}\int_{-1}^{1}$$
$$\times\frac{\sin^2\vartheta d\cos\vartheta}{(\varepsilon(\omega)+6kk'd^2\cos\vartheta-3k'^2d^2)^2+\varepsilon''^2}. \tag{A2}$$

In (A2) we have also performed the trivial integration over the azimuthal angle, $\int d\varphi\ldots=2\pi$, ϑ is the angle between the vector $\mathbf{k}'$ and the particle velocity $\mathbf{v}$. Breaking up the integrand into partial fractions and integrating over the angle ϑ, we arrive at the result

$$I^{\mathrm{R}}_{\omega}=\frac{32\pi^3e^4q^2\sqrt{\varepsilon}}{vm^2c^3}\int_{\omega/v}^{\infty}\frac{dk'}{k'}|\delta N|^2_{\mathbf{k}'}\,\frac{J_\vartheta}{36k^2k'^2d^4}, \tag{A3}$$

where

$$J_\vartheta=a\ \ln\frac{(a+1)^2+b^2}{(a-1)^2+b^2}-2+\frac{1+b^2-a^2}{b}$$
$$\times\left(\pi\theta(1-a^2)+\mathrm{arctg}\,\frac{2b}{a^2+b^2-1}\right), \qquad (A4)$$

and we have written $a=3k'^2d^2-\varepsilon(\omega)/6kk'd^2$, $b=\varepsilon''/6kk'd^2$. Let us say a bit more about the analysis of the expression for J_ϑ in Eq. (A4). The case of a nonabsorbent medium corresponds to the limit $b\to 0$. Then for $a^2\leqslant 1$ we have $J_\vartheta\to\infty$ as π/b. This divergence has a simple physical origin. It happens that for $a^2\leqslant 1$ the condition for Cherenkov radiation holds for arbitrary (plasma) waves. Hence the particle field for the corresponding values of ω, $\mathbf{k}$, $\mathbf{k}'$ is not quasisteady but propagates, and its interaction with the plasma inhomogeneities corresponds to scattering of already emitted quanta, not the production of new ones. The mean free path of the Cherenkov plasmons in an infinite nonabsorbent medium is infinite, which is the reason for this divergence. In order to calculate the intensity of the transition radiation, viewed as the result of transforming the quasisteady field of the particle into propagating waves,[5] we must exclude values $a^2\leqslant 1$ from the range of integration over k'. In this case the function J_ϑ can be simplified. Discarding the term $\pi\theta(1-a^2)$ and expanding arctan x in a series in its small argument for $a^2>1$, we find

$$J_\vartheta=\left\{a\ \ln\frac{(a+1)^2+b^2}{(a-1)^2+b^2}-4\right\}\theta(a^2-1). \qquad (A5)$$

The quantity J_ϑ has a singularity in the limit $b\to 0$, $a^2\to 1$, but this singularity is integrable. This can easily be shown if we expand J_ϑ in powers of $1/a$, the expansion converges within $1/|a|<1$. Retaining the first nonvanishing term of this expansion

$$J_\vartheta\simeq\frac{4}{3a^2}\,\theta(a^2-1) \qquad (A6)$$

ensures an accuracy of better than 30%. After substituting (A6) in (A3) and going over to the dimensionless variable $\mu=k'v/\omega$, we can write the resonant transition radiation spectrum in the form

$$I_\omega^{\mathrm{R}}=\frac{32\pi^2(\nu-1)}{27}\,\frac{e^4q^2\sqrt{\varepsilon}}{vm^2c^3}\,k_0^{\nu-1}\langle\Delta N^2\rangle$$
$$\times\left(\frac{v}{\omega}\right)^{\nu+2}\left(\frac{v}{\omega d}\right)^4\int_1^\infty\frac{d\mu\,\theta(a^2-1)}{\mu^{\nu+3}(\mu^2-\alpha)^2}, \qquad (A7)$$

where $\alpha=(\varepsilon/3)(v/\omega d)^2\approx(\varepsilon/3)(v/v_T)^2$ and v_T is the thermal velocity of the plasma electrons. For arbitrary values of the spectral index ν the integral in (A7) can be expressed in terms of hypergeometric functions. In fact, the shape of the peak has only a weak dependence on the spectral index, so it is convenient to use the result of integrating (A7) for $\nu=2$, which can be expressed in terms of elementary functions:

$$\Phi(\alpha)\equiv\int_1^\infty\frac{d\mu\,\theta(a^2-1)}{\mu^5(\mu^2-\alpha)^2}=\frac{1}{\alpha^3}\left[\frac{1}{1-\alpha}+2+\frac{\alpha}{2}\right.$$
$$\left.+\frac{3}{\alpha}\ln(1-\alpha)\right]\theta(\omega_1-\omega)+\frac{1}{\alpha^4}\frac{c}{2\sqrt{3}v_T}$$
$$\times\left[1-\frac{6\sqrt{3}v_T}{c}\ln\frac{c}{2\sqrt{3}v_T}\right]\theta(\omega-\omega_1)$$
$$\times\theta(\omega_2-\omega)+\left\{\frac{1}{\alpha^4}\frac{c}{\sqrt{3}v_T}+\frac{1}{\alpha^3}\left[\frac{1}{1-\alpha}+2+\frac{\alpha}{2}\right.\right.$$
$$\left.\left.+\frac{3}{\alpha}\ln(\alpha-1)\right]\right\}\theta(\omega-\omega_2), \qquad (A8)$$

where

$$\omega_{1,2}=\omega_{\mathrm{p}}\left[1+\frac{3}{2}\left(\frac{v_T}{v}\right)^2\left(1\mp\frac{2\sqrt{3}v_T}{c}\right)\right]. \qquad (A9)$$

Next it is convenient to transform from the function $\Phi(\alpha)$ to another function

$$F(\alpha)=\frac{2\sqrt{\varepsilon}}{9}\left(\frac{v}{v_T}\right)^4\Phi(\alpha), \qquad (A10)$$

since for large frequencies $\omega\gg\omega_{\mathrm{p}}$, $\alpha\gg 1$ we have $F(\alpha)\approx\varepsilon-3/2(\omega)$, and the corresponding expression matches with the overall transition radiation spectrum (20). Thus, the TR spectrum which is correct at all frequencies $\omega\geqslant\omega_p$ is found if we note the substitution $\varepsilon^{-3/2}\to F(\alpha)$ in Eq. (20). It is easy to see that at high frequencies $\omega\gg\omega_{\mathrm{p}}$ the role of the spatial dispersion is minor, i.e.,

$$F(\alpha)\approx\varepsilon^{-3/2\approx 1},$$

at low frequencies, $\alpha\ll 1$,

$$F(\alpha)\approx(\sqrt{\varepsilon}/18)(v/v_T)^4\propto(\omega-\omega_{\mathrm{p}})^{1/2},$$

while near the maximum of the spectrum $\alpha\approx 1$,

$$F(\alpha)\sim v^3c/v_T^4,$$

i.e., larger by a factor c/v_T, which follows from the estimates (see Ref. 3, p. 59).

In conclusion, we comment on the estimate of the contribution to the spectrum of the radiation produced in a randomly varying plasma as a result of the scattering of Cherenkov plasmons. The spectral intensity of Cherenkov plasma generation is given by the well-known expression

$$I_\omega^{\mathrm{p}}=\frac{e^2\omega}{v}\ln\frac{v}{v_T}. \qquad (A11)$$

Accordingly, the total energy emitted per unit time is equal to

$$I_{\mathrm{tot}}^{\mathrm{p}}=\frac{e^2\omega_{\mathrm{p}}^2}{v}\left(\frac{v_T}{v}\right)^2\ln\left(\frac{v}{v_T}\right),$$

and the fraction of plasmons $I_{\mathrm{tot}}^{\mathrm{ps}}$ scattered per unit time is proportional to the dimensionless factor l_1/l_2, equal to the ratio of the plasmon free path l_1 to the length l_2 for conversion into transverse waves. Under realistic conditions this ratio is much less than unity, and as a rule we have $l_1/l_2 \lesssim \langle \Delta N^2 \rangle / N^2$. Thus, the ratio of the energy I_{tot}^{ps} of the scattered Cherenkov radiation to the resonant energy TR I_{tot}^{R} in Eq. (33) is equal in order of magnitude to

$$\frac{I_{tot}^{ps}}{I_{tot}^{R}} \sim \frac{v_T^4}{v^3 c} \ln \frac{v}{v_T} \ll 1. \tag{A12}$$

Accordingly, in this case the effect of Cherenkov plasmon scattering from irregularities plays no important role, and the expressions obtained here in fact describe the total emission spectrum from charged particles in a randomly inhomogeneous plasma. It should, however, be noted that an analogous effect can be important if the conditions for Cherenkov emission of transverse waves are satisfied.[12]

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