Properties of the ferromagnetic Fermi liquid

É. G. Batyev

Institute of Semiconductor Physics, Siberian Branch, Russian Academy of Sciences, 630090 Novosibirsk, Russia

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A simple model of a ferromagnetic Fermi liquid is employed to investigate the interaction of spin and spatial degrees of freedom. The results are applied to discuss the relaxation of a current-carrying state due to spin oscillation build-up. © *1994 American Institute of Physics.*

1. INTRODUCTION

In a ferromagnetic Fermi liquid, in addition to ordinary Fermi excitations there are also Bose type quasiparticles, magnons (the oscillation quanta of spin degrees of freedom with quadratic dispersion; see Ref. 1). An interesting problem to be addressed is a description of the interaction between these types of elementary excitation. It is the solution of this problem which is the subject of the present work.

The problem was solved previously for a Hubbard ferromagnet both on an elementary level[2] and for spin waves of finite amplitude.[3] As was shown,[3] the effect of the spin subsystem on the carrier motion resembles that of an electromagnetic field and is described by a scalar and a vector potential, while the carriers affect the spin subsystem via the density and velocity (assuming long-wavelength spin oscillations).

It would appear that these features are specific to the Hubbard model. Actually, the same can be expected for the general case of a free-carrier ferromagnet. In fact, in the presence of a current an obvious change must occur in the magnetic moment equation of motion, namely, in the Landau-Lifshitz equations a contribution of the form $(\mathbf{V}\nabla)\mathbf{S}$ must appear, where $\mathbf{V}$ is the average carrier velocity and $\mathbf{S}$ is the spin momentum. The appearance of the carrier velocity in the spin equation of motion is one of manifestations of the interaction in question, and the fact that this interaction depends on the velocity requires that a vector potential be introduced. The above argument is general, has no relation to the Hubbard model as such and thus provides justification for the expectation mentioned above. Needless to say, the concrete results to be obtained in the general case are not immediately obvious, and it is this question which will be examined here.

This is done in the next section, where a simple model we propose is first discussed as it stands and then extended. Here the basic equations and relations are given. In Sec. 3 we explore how a current-carrying state relaxes if deceleration is only possible due to the build-up of spin oscillations. This question is of interest, in particular, in connection with its possible relevance to high-temperature superconductors.[4]

2. EQUATIONS

There are two points underlying our approach which must be emphasized. First, the principle of least action is used. Second, the spin and coordinate degrees of freedom are separated.

We start by noting that the Schrödinger equation may be obtained from the principle of least action by writing the Lagrangian density $\mathscr{L}$ in the form

$$\mathscr{L}=-\left\{\frac{1}{2m}\left|\nabla\Phi\right|^2+W\left|\Phi\right|^2-\frac{i}{2}\left[\Phi^*\frac{\partial\Phi}{\partial t}-\Phi\frac{\partial\Phi^*}{\partial t}\right]\right\}, \quad (1)$$

where Φ is the wave function and W is the potential energy of the particle. Following well-known rules[5] and varying the action with respect, for example, to Φ^* we obtain the Schrödinger equation for Φ, which is the justification for this form of $\mathscr{L}$.

Now to the model. We adopt the simplest version possible and consider a one-component Fermi system. This means that the spin part of the wave function is the same for all particles. The wave function Φ is understood to have the product form

$$\Phi=\chi\Psi \quad (2)$$

with the spinor

$$\chi=\begin{pmatrix}u\\ v\end{pmatrix}$$

the same for all particles; u and v are functions of position and time in the general case. The orthogonality of the states is ensured by the spatial part Ψ. For example, in the ground state we have $u=1$, $v=0$, and Ψ is a set of plane waves, with momenta ranging all the way to the Fermi value.

The meaning of this approach is clear: the exchange interaction "holds" all the spins parallel at each given point in space, and this is modeled in just the manner above. In this way one separates the spin and coordinate degrees of freedom. To obtain the equations of motion we vary with respect to χ and Ψ independently.

We substitute (2) into (1) and drop the term in W as unimportant for further discussion. As a result, in place of (1) we obtain

$$-\mathscr{L}\rightarrow\frac{1}{2m}|(-i\nabla-\mathbf{A})\Psi|^2-\frac{i}{2}\left[\Psi^*\frac{\partial\Psi}{\partial t}-\Psi\frac{\partial\Psi^*}{\partial t}\right]$$
$$+\left\{\frac{1}{2m}[(\nabla\chi^*,\nabla\chi)-\mathbf{A}^2]-\frac{i}{2}\left[\left(\chi^*,\frac{\partial\chi}{\partial t}\right)\right.\right.$$
$$\left.\left.-\left(\frac{\partial\chi^*}{\partial t},\chi\right)\right]\right\}|\Psi|^2. \quad (3)$$

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Here we have introduced the vector potential **A** defined by

$$\mathbf{A}=\frac{i}{2}[(\chi^*,\nabla\chi)-(\nabla\chi^*,\chi)]. \tag{4}$$

The "scalar" product of spinors is understood as usual, for example,

$$(\chi^*,\nabla\chi)=u^*\nabla u+v^*\nabla v. \tag{5}$$

In the first two terms in (3) the spinor normalization is included:

$$(\chi^*,\chi)=1. \tag{6}$$

Note that in expression (3), in addition to the vector potential **A** there is also a scalar potential, which appears as the coefficient of $|\Psi|^2$. Both enter the equation for Ψ in the usual way.

The Lagrangian density (3) has so far been written for a single particle with a wave function Ψ in the spinor field χ. Clearly, in the derivation of the expression for χ all particles must be accounted for. More on this later; for the moment, possible ways of generalizing Eq. (3) will be discussed.

In the general case, changes may only occur in the coefficients[3] of, for example, the product

$$[(\nabla\chi^*,\nabla\chi)-\mathbf{A}^2]|\Psi|^2.$$

As will be seen later, this coefficient determines the "rigidity" of the spin subsystem and need not generally be equal to $1/2m$, as in the simple model. If we denote its generalized value by $1/2M$, the scalar potential U becomes

$$U=\frac{1}{2M}[(\nabla\chi^*,\nabla\chi)-\mathbf{A}^2]-\frac{i}{2}\left[\left(\chi^*,\frac{\partial\chi}{\partial t}\right)-\left(\frac{\partial\chi^*}{\partial t},\chi\right)\right]. \tag{7}$$

As a result, for the Lagrangian density we have

$$-\mathscr{L}=\frac{1}{2m}|(-i\nabla-\mathbf{A})\Psi|^2+U|\Psi|^2-\frac{i}{2}\left[\Psi^*\frac{\partial\Psi}{\partial t}-\Psi\frac{\partial\Psi^*}{\partial t}\right], \tag{8}$$

where the vector **A** and scalar U potentials are given by the Eqs. (4) and (7).

One could make one further modification in Eq. (8), by introducing a "charge" (i.e., a coefficient of **A** and U) different from unity. This, however, would render the scheme inconsistent because, e.g., the energy of the system would change while the spin equations of motion remain intact.

By varying the action with the Lagrangian density (8) for fixed χ, we obtain an equation for Ψ in the usual form:

$$i\frac{\partial\Psi}{\partial t}=\frac{1}{2m}(-i\nabla-\mathbf{A})^2\Psi+U\Psi. \tag{9}$$

This equation is for a particle which interacts only with the spinor field.

Now to the derivation of the equation for χ. First of all, all particles must be accounted for. This can be achieved by replacing the bilinear combination $\Psi^*(\mathbf{r}')\Psi(\mathbf{r})$ in Eq. (8) by the one-particle density matrix. The same result can be obtained in a different way, however.

The point is that all of the above development applies only to long-wavelength spin oscillations; though not evident formally, this is implied by the meaning of the procedure employed. This means that the particles act on χ as a continuous medium, that is, only through the density and velocity. But these characteristics are also captured in the classical description, and this we can apply. The Lagrangian L for classical particles can (by analogy with electrodynamics) be written in the form

$$L=\sum_k\left\{\frac{m\dot{\mathbf{r}}_k^2}{2}+\dot{\mathbf{r}}_k\mathbf{A}(\mathbf{r}_k)-U(\mathbf{r}_k)\right\}, \tag{10}$$

where the summation runs over all particles. In deriving the equation for χ we may change, in Eq. (10), from a sum to an integral (in which the particles behave as a continuum); the corresponding Lagrangian density $\mathscr{L}(\chi)$, as a function of χ, has the form

$$\mathscr{L}(\chi)=n\left\{\mathbf{VA}-\frac{1}{2M}[(\nabla\chi^*,\nabla\chi)-\mathbf{A}^2]+\frac{i}{2}\left[\left(\chi^*,\frac{\partial\chi}{\partial t}\right)-\left(\frac{\partial\chi^*}{\partial t},\chi\right)\right]\right\}, \tag{11}$$

where n and **V** are respectively the particle density and velocity (in general, dependent on position and time).

Varying the action with respect to χ^* yields

$$in\frac{\partial\chi}{\partial t}=-\frac{i}{2}\frac{\partial n}{\partial t}\chi-\frac{n}{2M}(M\mathbf{V}+\mathbf{A})^2\chi+\frac{1}{2M}\times[i\nabla-(M\mathbf{V}+\mathbf{A})]\{n[i\nabla-(M\mathbf{V}+\mathbf{A})]\chi\}. \tag{12}$$

Note that among the solutions of this equation there are solutions with the normalization (6), of interest here (see Ref. 3).

Making use of this equation one can obtain the equation for **S**, the average value of spin,

$$\mathbf{S}=(\chi^*,\hat{\mathbf{S}}\chi),$$

where $\hat{\mathbf{S}}$ is the operator of spin 1/2. The corresponding derivation is like that in Ref. 3, so we just present the result:

$$n\left[\frac{\partial\mathbf{S}_k}{\partial t}+(\mathbf{V}\nabla)\mathbf{S}_k\right]-\mathrm{div}\left[\frac{n}{M}e_{klm}\mathbf{S}_l\nabla\mathbf{S}_m\right]=0. \tag{13}$$

Here the particle continuity equation has been used; e_{klm} is the completely asymmetric tensor ($e_{123}=1$); summation over repeated indices is assumed. Note that, for constant density, we simply recover the Landau–Lifshitz equation for momentum oscillations—but including the fluid motion, as it must. This may be viewed as an argument in favor of the approach we have chosen.

One goal is thus achieved: we have obtained the equations of motion, Eqs. (9), (12), (13), illustrating the interaction of particles and spins. In the next section, some properties of such a ferromagnet are discussed.

3. PROPERTIES

In this section we are dealing with the deceleration of the system, assuming it is only possible via spin oscillation build-up. Suppose, for example, that the Cooper effect occurs in the system, so that there is a gap in the Fermi spectrum and hence no relaxation due to Fermi excitations. In this case no other deceleration mechanism is available.

To proceed further we need expressions for the energy and momentum of the system. In the usual way,[5] we find for the energy

$$\mathscr{E}=\sum_k \left\{ \frac{m\dot{\mathbf{r}}_k^2}{2}+\frac{1}{2M}\left[(\nabla\chi^*,\nabla\chi)-\mathbf{A}^2\right]_k\right\}, \tag{14}$$

using the Lagrangian in its classical version (10) (which suffices for our purposes). Or, for long-wavelength changes, the energy density can be expressed as

$$\mathscr{E}_1=\mathscr{E}_{10}+n\left\{ \frac{m\mathbf{V}^2}{2}+\frac{1}{2M}\left[(\nabla\chi^*,\nabla\chi)-\mathbf{A}^2\right]\right\}, \tag{15}$$

where $\mathscr{E}_{10}$ is the ground state energy, of no significance here. This expression clearly displays different contributions to the energy of the system, for example, the one explicitly dependent on χ is simply the spin-wave energy density; this contribution looks as usual, i.e., is proportional to $\partial S_i/\partial x_k$ $\partial S_i/\partial x_k$.

In a similar way, one obtains an expression for the momentum density,

$$\mathscr{P}_1=nm\mathbf{V}. \tag{16}$$

It might seem strange that this has no explicit dependence on the spin variable, since a momentum may be ascribed to the magnon; in fact, spin wave contributions will enter just through the velocity $\mathbf{V}$, as will be seen later. If one remembers that initially we had particles with spin rather than particles and spins separately, then Eq. (16) is in fact what one should expect.

From the equations of motion (13) it is not difficult to find the oscillation frequency ω at a given wave vector $\mathbf{k}$ (Ref. 6); for oscillations with a constant value of the spin projection S_3 we have

$$\omega=\mathbf{Vk}+\frac{1}{M}S_3k^2. \tag{17}$$

For spin 1/2 we have $S=|\mathbf{S}|=1/2$ and $S_3=1/2\cos\theta$. The azimuthal angle φ of the vector $\mathbf{S}$ varies according to the law

$$\varphi=\mathbf{kr}-\omega t. \tag{18}$$

The energy of the elementary excitations (small oscillations, $\theta\to 0$) is negative for certain $\mathbf{k}$, which will cause these excitations to be created and the moving system to decelerate (since the Landau superfluidity criterion breaks down).

It is unclear whether a complete solution of the problem is possible. We therefore restrict ourselves to the single-mode regime, i.e., we will consider the oscillation (17)–(18), with fixed $\mathbf{k}$, and find out what will occur as the mode builds up (the angle θ increases). It is natural to choose $\mathbf{k}$ to be at the minimum of the spectrum (17) for $\theta=0$, that is, in what follows we take

$$\mathbf{k}=-M\mathbf{V}_0 \tag{19}$$

($\mathbf{V}_0$ is the velocity at $\theta=0$).

As the mode is built up, the energy decreases (the spin part increases and the kinetic energy decreases) and it is found that in some cases it may reach a minimum, after which the deceleration ceases. To see all this, it is necessary that all of the quantities involved be expressed in terms of the angle θ.

We begin with the spinor χ. It can easily be seen using Eq. (12) that for the oscillation (17)–(18) the spinor components are of the form

$$u=e^{-i\varphi}\cos\frac{\theta}{2},\quad U=\sin\frac{\theta}{2} \tag{20}$$

(n and $\mathbf{V}$ are constant for the oscillation considered). The vector potential is then

$$A=\mathbf{k}\cos^2\frac{\theta}{2}. \tag{21}$$

As the oscillation builds up, the vector potential changes in time and hence the velocity does also, in accordance with the equation of motion

$$m\dot{\mathbf{V}}=-\dot{\mathbf{A}}, \tag{22}$$

where on the right we have an "electric field" (the potential gradient and the "magnetic field" are zero for this oscillation). Incidentally, Eq. (22) answers the question suggested by the momentum expression (16).

From (22), with (21), we obtain the velocity expression

$$\mathbf{V}=\mathbf{V}_0+\frac{\mathbf{k}}{m}\sin^2\frac{\theta}{2}. \tag{23}$$

From the relations obtained, and making use of the condition (19), we find the dependence of the energy (15) on the spin wave amplitude,

$$\mathscr{E}_1-\mathscr{E}_{10}=n\,\frac{mV_0^2}{2}\left\{\left[1-\frac{M}{m}\sin^2\frac{\theta}{2}\right]^2+\frac{M}{4m}\sin^2\theta\right\}. \tag{24}$$

This expression is easy to analyze. Let us present the results. Under the condition

$$\frac{M}{m}>\frac{3}{2} \tag{25}$$

there is a minimum. At the minimum,

$$\cos\theta=\frac{M/m-2}{M/m-1},$$

$$\mathbf{V}=\frac{1}{2}\mathbf{V}_0\frac{M/m-2}{M/m-1}, \tag{26}$$

$$\mathscr{E}_1-\mathscr{E}_{10}=n\,\frac{mV_0^2}{2}\left\{1-\frac{1}{4}\frac{M/m}{M/m-1}\right\}.$$

The oscillation frequency (17) at the minimum goes to zero, $\omega=0$, that is, in the laboratory system we have a frozen spin wave, as it were.

For $M/m>2$, the deceleration reaches the value given in Eq. (26), after which the deceleration stops. At $M/m=2$ the deceleration is complete (the velocity vanishes), and in the interval $3/2<M<2$ the velocity even becomes negative.

As to the case $M/m<3/2$, here the process of relaxation is accompanied by a complete remagnetization of the system (the angle θ changes from 0 to π), after which the whole story repeats itself from the beginning, with the spin wave building up again—though with a different **k** corresponding to a new (slower) velocity—and so until complete deceleration is achieved. At the end of each cycle, the velocity is related to the initial one by

$$\mathbf{V}=\mathbf{V}_0\left(1-\frac{M}{m}\right),$$

so that for $1<M/m<3/2$ the velocity changes not only its magnitude but also its sign.

Such is the picture of what goes on in the single-mode regime. The actual situation is difficult to predict and would require a complete solution of the problem; or one must at least convince oneself that in the steady-state regime assumed, spin excitations—the new ones obtained even when both the motion and the frozen spin wave are already present—have positive energy. This remains an open question.

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