Small-angle multiple scattering of light in a random medium

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Small-angle multiple scattering of unpolarized light in a random medium with large-scale irregularities is considered. The intensity and degree of polarization of the scattered radiation are calculated. Their sensitivity to the angular dependence of the cross section for an individual scattering event are studied. For the case of a scattering phase function that falls off according to the power law $d\sigma/d\Omega \propto \gamma^{-\alpha}$, where γ is the individual scattering angle, it is shown that both the angular distribution of the radiation as a whole and the degree of polarization at relatively small scattering angles depend substantially on the index α. This dependence is analyzed both in the case of purely elastic scattering and for strongly absorbent media.

1. INTRODUCTION

One has to deal with small-angle multiple scattering of different kinds of radiation (light, electrons, neutrons, etc.) in many physical problems,[1–5] and these have been studied in considerable detail in scalar theory.[1,6–14] In the case of scattering of electromagnetic radiation, however, the scalar approximation[1,7,9–14] is inadequate, since it does not take into account the vector nature of the waves.

Up until now the polarization of multiply scattered light has been analyzed, generally speaking, for the case of photons undergoing spatial diffusion, when the angular spectrum of the radiation is essentially isotropic (see, e.g., Refs. 15–17). At the present time there are essentially no solutions to the problem of the polarization properties associated with small-angle scattering of electromagnetic waves. The results currently available on this problem[18–22] relate only to certain limiting cases and do not enable one to draw any sort of general conclusion about the magnitude of the polarization of multiply scattered radiation in media with large-scale irregularities.

Note that knowledge of light polarization in connection with small-angle multiple scattering is of interest for many problems: optical studies of random systems with long-range correlations in the dielectric fluctuations,[4,23] analysis of coherent inverse scattering,[24–27] and fluctuations of the polarization[28] of multiply scattered light in materials with large-scale irregularities.

Estimates reveal that the depolarization associated with propagation of initially polarized light through a medium with large-scale scatterers is an effect of higher order than the polarization of an initially unpolarized beam. The intensity of the cross-polarized component of a polarized beam is proportional to the fourth power θ^4 of the scattering angle,[18,19] while the degree of polarization P of an initially unpolarized beam only goes as the square of the scattering angle ($P \propto \theta^2$).

Below we will generalize the results of Refs. 6–14 and consider small-angle multiple scattering of an unpolarized light beam in a random medium with large-scale irregularities (we will not treat the propagation of light in optically anisotropic media). We will calculate the intensity and degree of polarization of the scattered radiation and study their sensitivity to the angular dependence of the cross section for single scattering. The propagation of light will be studied in detail both for the case of purely elastic scattering and when absorption in the medium takes place. The calculations are performed for cross sections that fall off with scattering angle γ according to a power law $\gamma^{-\alpha}$. It is shown that both the angular distribution of the radiation as a whole and the degree of polarization at relatively small scattering angles depend sensitively on the index α. Moreover, we find a number of qualitative features in the behavior of the degree of polarization as the inhomogeneity scale increases and when the Born approximation for single scattering is inapplicable.

These results may prove useful for optical studies of random media with large scatterers or long-range correlations of fluctuations in the dielectric function.[2,4,23,29,30]

2. SMALL-ANGLE TRANSPORT EQUATION

Assume that a broad beam of unpolarized light is incident normally on a layer of thickness L $(0<z<L)$ consisting of large spherical irregularities (dimension a much larger than the wavelength λ, $a \gg \lambda$). We assume that the mean free path l is much longer than the wavelength λ and the characteristic size a of the irregularities,

$$l \gg \lambda, \quad l \gg a. \tag{1}$$

We assume initially that the phase shift $\Delta\phi$ of a wave undergoing a single scattering is small:

$$|\Delta\phi| \sim k_0 a |n-1| \ll 1, \tag{2}$$

where n is the index of refraction of a scattering particle and $k_0 = 2\pi/\lambda$. Expression (2) enables us to describe an individual collision using the Born approximation.[31]

Under these conditions it follows from azimuthal symmetry that the third Stokes parameter U vanishes. The fourth Stokes parameter V, corresponding to the elliptically polarized component, also vanishes, since the incident beam is unpolarized and the medium is assumed to be optically isotropic. The occurrence of electrical polarization is related to

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higher orders of $\sqrt{|n-1|}$ in the single-scattering amplitude, i.e., it lies outside the Born approximation. However, even in that case there will be no elliptical polarization as long as the scattering geometry has azimuthal symmetry. Thus, for a complete description of the polarization properties of light undergoing multiple scattering it suffices to calculate the intensity I and the second Stokes parameter Q [here $I=I_\parallel+I_\perp$, $Q=I_\parallel-I_\perp$, where $I_\parallel$ and $I_\perp$ are the intensities of the components polarized in the directions of the unit vectors $\mathbf{e}_\perp=(-\sin\varphi,\cos\varphi,0)$ and $\mathbf{e}_\parallel=(\cos\theta\cos\varphi,\cos\theta\sin\varphi,-\sin\theta)$; the vector $\mathbf{e}_\parallel$ lies in the scattering plane formed by the z axis and the direction in which the photons are propagating, while the vector $\mathbf{e}_\perp$ is perpendicular to the scattering plane]. In this case I and Q obey the following system of equations:[1]

$$\left\{\mu\frac{\partial}{\partial z}+\sigma_{\text{tot}}\right\}\begin{pmatrix} I(z,\mu) \\ Q(z,\mu)\end{pmatrix} = \int d\mathbf{n}'\mathscr{D}(\mathbf{nn}')\hat{P}(\mathbf{nn}')\begin{pmatrix} I(z,\mu') \\ Q(z,\mu')\end{pmatrix}, \quad (3)$$

where $\hat{P}=\hat{P}^{(0)}+\hat{P}^{(1)}+\hat{P}^{(2)}$, with

$$\hat{P}^{(0)}=\frac{3}{8}\begin{pmatrix} 3+3\mu^2\mu'^2-\mu^2-\mu'^2 & 1+3\mu^2\mu'^2-3\mu^2-\mu'^2 \\ 1+3\mu^2\mu'^2-\mu^2-3\mu'^2 & 3+3\mu^2\mu'^2-3\mu^2-3\mu'^2\end{pmatrix},$$

$$\hat{P}^{(1)}=\frac{3}{2}\sqrt{(1-\mu^2)(1-\mu'^2)}\mu\mu'\cos\psi\begin{pmatrix} 1 & 1 \\ 1 & 1\end{pmatrix},$$

$$\hat{P}^{(2)}=\frac{3}{8}\cos 2\psi\begin{pmatrix} (1-\mu^2)(1-\mu'^2) & -(1-\mu^2)(1+\mu'^2) \\ -(1+\mu^2)(1-\mu'^2) & (1+\mu^2)(1+\mu'^2)\end{pmatrix}, \quad (4)$$

$$\mu=\cos\theta,\quad \mu'=\cos\theta',\quad \psi=\varphi-\varphi'.$$

The quantity σ_{tot} in Eq. (3) is the total attenuation coefficient $\sigma_{\text{tot}}=\sigma+\sigma_{\text{a}}$, where σ and σ_{a} are the scattering and absorption coefficients.

The boundary condition for Eq. (3) when the incident light has unit intensity takes the form

$$\begin{pmatrix} I(z=0,\mu>0) \\ Q(z=0,\mu>0)\end{pmatrix}=\begin{pmatrix} 1 \\ 0\end{pmatrix}\delta(\mathbf{n}). \quad (5)$$

For the case of discrete scatterers[1,31] the function $\mathscr{D}$ is equal to

$$\mathscr{D}(\mathbf{nn}')=\frac{\mathscr{N}k_0^4}{16\pi^2}\left|\int d\mathbf{r}(\varepsilon(r)-1)\exp(ik_0r\mathbf{nn}')\right|^2,$$

where $\mathscr{N}$ is the number of scatterers per unit volume and $\varepsilon(r)=n^2$ is the dielectric function of the scattering particle. The integration is performed over the volume of the scatterer. For a continuous random medium $\mathscr{D}$ is determined by[1,32]

$$\mathscr{D}(\mathbf{nn}')=\frac{k_0^4}{16\pi^2}\int d\mathbf{r}B_\varepsilon(r)\exp(ik_0r\mathbf{nn}'),$$

where $B_\varepsilon(|\mathbf{r}-\mathbf{r}'|)=\langle(\varepsilon(\mathbf{r})-\langle\varepsilon\rangle)(\varepsilon(\mathbf{r}')-\langle\varepsilon\rangle)\rangle$ is the correlation function of the fluctuations in the dielectric function. The scattering coefficient σ is related to $\mathscr{D}(\mathbf{nn}')$ by

$$\sigma=\int d\mathbf{n}'\mathscr{D}(\mathbf{nn}')P_{11}(\mathbf{nn}'). \quad (6)$$

When radiation propagates in a medium with fine-scale ($\lambda\gg a$) irregularities the function $\mathscr{D}$ does not depend on the angle, and Eq. (3) describes multiple Rayleigh scattering.[1,15]

For the case of scattering by large weakly refracting irregularities ($\lambda\ll a, |n-1|\ll 1$) the photon changes its direction of motion by a small amount in a single scattering; the characteristic deflection angle is $\gamma_0\ll 1$ (Ref. 31). As for the spread in the directions that results from multiple scattering, it is determined both by the thickness of the scattering layer and by the absorption properties of the medium. If the absorption in the medium is weak (the absorption length $l_{\text{a}}=\sigma_{\text{a}}^{-1}$ is longer than the transport length $l_{\text{tr}}=\sigma_{\text{tr}}^{-1}$, where $\sigma_{\text{tr}}=\sigma(1-\langle\cos\gamma\rangle)$ is the momentum transfer scattering coefficient and $\langle\cos\gamma\rangle$ is the mean value of the cosine of the angle for a single scattering), then the radiation passes through relatively thin ($L<l_{\text{tr}}$) scattering layers the small-angle propagation regime also develops.[1,3,8] In thick ($L>l_{\text{tr}}$) layers at depth z greater than the transport length l_{tr} ($z\gg l_{\text{tr}}$), the beam isotropizes and a regime in which the radiation undergoes spatial diffusion sets in.[1,3,8] In strongly absorbing media ($l_{\text{a}}<l_{\text{tr}}$), by virtue of the preferential absorption of photons deflected through large angles, isotropization in general does not take place and the small-angle propagation regime obtains at all depths.[9–11] In what follows we will consistently assume that the conditions for the applicability of the small-angle approximation always holds.

We expand the terms appearing in Eq. (3) to terms of order θ^2, θ'^2. As a result, to leading order in the small parameter $\theta_z\ll 1$ (θ_z is the characteristic angle for multiple scattering at depth z) we find the following system:

$$\left\{\frac{\partial}{\partial z}+\sigma+\sigma_{\text{a}}\frac{\theta^2}{2}\right\}\begin{pmatrix} \tilde{I}(z,\theta) \\ \tilde{Q}(z,\theta)\end{pmatrix} = \sigma\int d\boldsymbol{\theta}'\hat{d}(\boldsymbol{\theta},\boldsymbol{\theta}')\begin{pmatrix} \tilde{I}(z,\theta') \\ \tilde{Q}(z,\theta')\end{pmatrix}, \quad (7)$$

where we have written $\tilde{I}'=I\exp(\sigma_{\text{a}}z)$, $\tilde{Q}=Q\exp(\sigma_{\text{a}}z)$; $\boldsymbol{\theta}=\{\theta\cos\varphi,\theta\sin\varphi\}$ is a two-dimensional vector, and

$$d_{11}=\chi(|\boldsymbol{\theta}-\boldsymbol{\theta}'|),\quad d_{22}=\chi(|\boldsymbol{\theta}-\boldsymbol{\theta}'|)\cos 2\psi, \tag{8}$$

$$d_{12}=-\Pi(|\boldsymbol{\theta}-\boldsymbol{\theta}'|)[\theta^2(1-\cos 2\psi)-(\boldsymbol{\theta}-\boldsymbol{\theta}')^2], \tag{9}$$

$$d_{21}(\boldsymbol{\theta},\boldsymbol{\theta}')=d_{12}(\boldsymbol{\theta}',\boldsymbol{\theta}), \tag{10}$$

where $\chi(\gamma)=D(\gamma)P_{11}(\gamma)/\sigma$ is the scattering phase function for single scattering. For Born scatterers [cf. Eq. (2)] we have

$$\Pi(\gamma)\approx-\frac{\chi(\gamma)}{2}. \tag{11}$$

If the phase shift associated with single scattering cannot be regarded as small and Eq. (2) does not hold, Eqs. (8)–(10) still remain correct; only the relation (11) changes (for more detail see below, Sec. 6).

In connection with the transition from Eq. (3) to Eq. (7) it is necessary to take into account the following circumstance. From the energy conservation law it follows that the total contribution from the second Stokes parameter to the transport of the intensity should vanish identically,

$$\int d\mathbf{n}\int d\mathbf{n}'\mathscr{D}(\mathbf{nn}')P_{12}(\mathbf{n},\mathbf{n}')Q(z,\theta')\equiv 0. \tag{12}$$

Condition (12) is exact and should hold to all orders in the small-angle expansion of the corresponding term on the right-hand side of Eq. (3). Direct calculations show that the right-hand side of Eq. (7) satisfies the identity (12) (see Appendix A).

The system of equations (7) can be reduced to differential form if we carry out a Bessel transformation with the function $J_0(\omega\theta)$ in the equation for $\tilde{I}$ and in the equation for $\tilde{Q}$ transform with $J_2(\omega\theta)$. As a result we obtain the following system of differential equations (cf. Appendix B):

$$\left\{\frac{\partial}{\partial z}+\sigma[1-\chi(\omega)]-\frac{\sigma_a}{2}\Delta_\omega\right\}\tilde{I}(z,\omega)=\sigma\omega\frac{\partial}{\partial\omega}\left[\frac{1}{\omega}\frac{\partial\Pi(\omega)}{\partial\omega}\right]\tilde{Q}(z,\omega), \tag{13}$$

$$\left\{\frac{\partial}{\partial z}+\sigma[1-\chi(\omega)]-\frac{\sigma_a}{2}\left(\Delta_\omega-\frac{4}{\omega^2}\right)\right\}\tilde{Q}(z,\omega)=\sigma\omega\frac{\partial}{\partial\omega}\left[\frac{1}{\omega}\frac{\partial\Pi(\omega)}{\partial\omega}\right]\tilde{I}(z,\omega), \tag{14}$$

where

$$\chi(\omega)=2\pi\int_0^\infty\gamma d\gamma J_0(\omega\gamma)\chi(\gamma),$$

$$\Pi(\omega)=2\pi\int_0^\infty\gamma d\gamma J_0(\omega\gamma)\Pi(\gamma),$$

$$\tilde{I}(z,\omega)=2\pi\int_0^\infty\theta d\theta J_0(\omega\theta)\tilde{I}(z,\theta),\quad \tilde{Q}(z,\omega)=2\pi\int_0^\infty\theta d\theta J_2(\omega\theta)\tilde{Q}(z,\theta),$$

$$\Delta_\omega=\frac{\partial^2}{\partial\omega^2}+\frac{1}{\omega}\frac{\partial}{\partial\omega}.$$

The boundary conditions for Eqs. (13) and (14) take the form

$$\tilde{I}(z=0,\omega)=1,\quad \tilde{Q}(z=0,\omega)=0. \tag{15}$$

By virtue of the condition (15) a nonzero value for the polarization $\tilde{Q}$ arises only due to the "source term" on the right-hand side of Eq. (14). This source is determined by the intensity $\tilde{I}$. In turn, the quantity on the right-hand side of Eq. (13) is the source of polarization corrections to the intensity. If we neglect this quantity, Eq. (13) reduces to a scalar small-angle transport equation written so as to include the effect of absorption on the angular distribution of the radiation (this effect corresponds to the term $\sigma_a\Delta_\omega\tilde{I}/2$) (Ref. 10).

Before going on to solve Eqs. (13) and (14) we specify the form of the single-scattering scattering phase function $\chi(\gamma)$. We consider scattering phase functions of algebraic form[33]

$$\chi(\gamma)=\frac{\chi_0\gamma_0^{\alpha-2}}{(\gamma_0^2+\gamma^2)^{\alpha/2}}, \tag{16}$$

where γ_0 signifies the characteristic angle for single scattering, $\gamma_0\ll 1$; χ_0 is a normalization constant [we have $\chi_0=(\alpha-2)/2\pi$ for $\alpha-2>[\ln(2/\gamma_0)]^{-1}$, and $\chi_0=[2\pi\ln(2/\gamma_0)]^{-1}$ for $|\alpha-2|<[\ln(2/\gamma_0)]^{-1}$].

For the scattering phase function (16) the momentum-transfer scattering coefficient σ_{tr} introduced above is equal to

$$\sigma_{tr}=\sigma(1-\langle\cos\gamma\rangle)=\frac{\sigma\gamma_0^2}{2}\left\{\frac{\alpha-2}{\alpha-4}\frac{1-(\gamma_0/2)^{\alpha-4}}{1-(\gamma_0/2)^{\alpha-2}}-1\right\},$$

whence we obtain

$$\sigma_{tr}=\begin{cases}\dfrac{\sigma\gamma_0^2}{2(\alpha-4)}, & \alpha>4,\\ \sigma\gamma_0^2\ln(2/\gamma_0), & \alpha=4,\\ 2\sigma(\gamma_0/2)^{\alpha-2}(\alpha-2)/(4-\alpha), & 2<\alpha<4,\\ \sigma/\ln(2/\gamma_0), & \alpha=2.\end{cases}$$

The algebraic scattering phase functions (16) with index $\alpha\approx 2$–4 describe realistic small-angle scattering in different media with large irregularities.[1,4,8] In particular, the index $\alpha=11/3$ corresponds to light scattering in a turbulent medium (the Kolmogorov–Obukhov spectrum),[1,7,32] while for $\alpha=3$ Eq. (16) determines the Henyey–Grinstein scattering phase function, which is often used to describe light scattering in aerosols and an aqueous medium.[1,33] When light is scattered in material near phase-transition points, the index α is equal to $\alpha=2-\eta$, where $\eta\geqslant 0$ (see, e.g., Refs. 4 and 29). In the case of fractal media α coincides with the fractal dimensionality and, depending upon the conditions of formation, the properties of the subparticles, etc., can be either larger or smaller than $\alpha=2$ (Refs. 23, 30, 34).

The most interesting case is that of light scattering at depth $z>l$, where multiple scattering plays an important role and the effective deflection angles θ are greater than the characteristic angle γ_0 for single scattering, $\theta>\gamma_0$.

In this range of angles the values of the Stokes parameters $\tilde{I}(z,\omega)$ and $\tilde{Q}(z,\omega)$ entering into Eqs. (13) and (14) are determined by the behavior of the function $\chi(\omega)$ for relatively small values of ω, $1<\omega<1/\gamma_0$. For these values of ω the function $\chi(\omega)$ can be written as a power series. Then in the general case for scattering phase functions of the form (16) it is necessary to retain four terms in the expansion of $\chi(\omega)$:

$$\chi(\omega)=\tilde{\chi}_0\{1+a_\alpha\omega^2-b_\alpha\omega^{\alpha-2}+c_\alpha\omega^4\}, \tag{17}$$

where

$$a_\alpha=\frac{\gamma_0^2}{2(4-\alpha)},\quad b_\alpha=\left(\frac{\gamma_0}{2}\right)^{\alpha-2}\frac{\Gamma(2-\alpha/2)}{\Gamma(\alpha/2)},$$

$$c_\alpha=\frac{\gamma_0^4}{8(\alpha-4)(\alpha-6)},$$

the constant $\tilde{\chi}_0$ for $\alpha>2$ is equal to unity, while near the value $\alpha=2$ it is necessary to use for $\tilde{\chi}_0$ the more accurate expression $\tilde{\chi}_0=(1-(\gamma_0/2)^{\alpha-2})^{-1}$; here $\Gamma(x)$ is the gamma function.[35] For $\alpha>4$ the magnitude $|a_\alpha|$ is proportional to the mean square of the single-scattering angle, $|a_\alpha|=\langle\gamma^2\rangle/4$, where $\langle\gamma^2\rangle=2\pi\int_0^\infty\gamma d\gamma\gamma^2\chi(\gamma)$. Accordingly, for $\alpha>6$ the magnitude $|c_\alpha|$ is proportional to the fourth moment $|c_\alpha|=\langle\gamma^4\rangle/64$ of the scattering phase function, where $\langle\gamma^4\rangle=2\pi\int_0^\infty\gamma d\gamma\gamma^4\chi(\gamma)$.

Two factors are involved in the need to treat all the terms[1)] in (17). First, for α close to values $\alpha\approx2n$ ($n=1,2,3$), the terms in the expansion (17) proportional to $\omega^{2(n-1)}$ and $\omega^{\alpha-2}$ are of the same order and should be retained together. Secondly, as can easily be seen by direct differentiation, the first two terms in (17) yield a vanishing contribution to the derivatives of the function $\Pi(\omega)$ defined in (11) on the right-hand sides of Eqs. (13) and (14).

3. SOLUTION OF THE TRANSPORT EQUATION; GENERAL RELATIONS

For small-angle light scattering in weakly absorbent ($l_a>l_{tr}$) media the effect of the absorption on the angular and spatial divergence of the radiation can be neglected and in Eqs. (13) and (14) we can set $\sigma_a=0$. Under these conditions Eqs. (13) and (14) admit an exact solution for a single-scattering phase function of arbitrary form:

$$\tilde{I}(z,\omega)=\cosh\left\{\sigma z\omega\frac{\partial}{\partial\omega}\left[\frac{1}{\omega}\frac{\partial\Pi(\omega)}{\partial\omega}\right]\right\}\times\exp(-\sigma z(1-\chi(\omega))), \tag{18}$$

$$\tilde{Q}(z,\omega)=\sinh\left\{\sigma z\omega\frac{\partial}{\partial\omega}\left[\frac{1}{\omega}\frac{\partial\Pi(\omega)}{\partial\omega}\right]\right\}\times\exp(-\sigma z(1-\chi(\omega))). \tag{19}$$

For small-angle scattering the argument of the hyperbolic functions in Eqs. (18) and (19) is small. Expanding the hyperbolic functions in power series we find

$$\tilde{I}(z,\theta)=\int_0^\infty\frac{\omega d\omega}{2\pi}J_0(\omega\theta)\exp(-\sigma z(1-\chi(\omega)))+\frac{\sigma^2z^2}{2}\int_0^\infty\frac{\omega^3d\omega}{2\pi}J_0(\omega\theta)\left[\frac{\partial}{\partial\omega}\left[\frac{1}{\omega}\frac{\partial\Pi(\omega)}{\partial\omega}\right]\right]^2\times\exp(-\sigma z(1-\chi(\omega)))+\dots, \tag{20}$$

$$\tilde{Q}(z,\theta)=\sigma z\int_0^\infty\frac{\omega^2d\omega}{2\pi}J_2(\omega\theta)\frac{\partial}{\partial\omega}\left[\frac{1}{\omega}\frac{\partial\Pi(\omega)}{\partial\omega}\right]\times\exp(-\sigma z(1-\chi(\omega)))+\dots. \tag{21}$$

Relations (18)–(21) are an immediate generalization of the results for scalar small-angle multiple-scattering theory.[1,6–8,12–14]

The first term on the right-hand side of (20) is identical with the familiar result of the scalar approach.[1,6–8,12–14] The second term in (20) describes the polarization correction $\delta\tilde{I}_{pol}$ to the intensity. This correction is determined by the term in the right-hand side of Eq. (13) containing the second Stokes parameter Q. The ratio of (21) to (20) yields the degree of polarization $P=\tilde{Q}/\tilde{I}$ of the scattered radiation. Since the quantity $\delta\tilde{I}_{pol}$ is small, $\delta\tilde{I}_{pol}/\tilde{I}\ll1$, the correction $\delta\tilde{I}_{pol}$ in (20) can be neglected to lowest order in calculating the degree of polarization.

In strongly absorbent ($l_a<l_{tr}$) media, beginning at some depth z, it is no longer permissible to neglect the effect of absorption on the angular divergence of the beam. It is significant that this occurs while still in the small-angle regime of multiple scattering (i.e., for $z\ll l_{tr}$). Absorption suppresses the contribution of the photons scattered through relatively large angles and enables the small-angle light propagation regime to persist at all depths.[9–11] For $l_a<l_{tr}$ the asymptotic value of the mean square angle for multiple scattering is always small, $\langle\theta^2\rangle_\infty<1$ (Refs. 9–11; see also the numerical calculations presented in Refs. 36 and 37). In this connection Eqs. (13) and (14) can be used[2)] to describe light propagation at arbitrary values of z. For small z, so long as absorption does not effect the angular distribution of the radiation, expressions (20) and (21) for $\tilde{I}$ and $\tilde{Q}$ follow from (13) and (14). At larger depths Eqs. (13) and (14) must now be solved including terms proportional to σ_a, which begin to be of the same order as the other terms of these equations.

There is no difficulty in transforming Eqs. (13) and (14) together with the boundary conditions (15) into two independent equations for $\tilde{I}$ and $\tilde{Q}$ separately:

$$\tilde{I}(z,\omega)=\tilde{I}^{(0)}(z,\omega)+\int_0^z dz'\int d\boldsymbol{\omega}'\Gamma_{\tilde{I}}\times(z-z'|\omega,\omega')\Xi(\omega')\int_0^{z'}dz''\int d\boldsymbol{\omega}''\Gamma_{\tilde{Q}}\times(z'-z''|\omega',\omega'')\Xi(\omega'')\tilde{I}(z,\omega''), \tag{22}$$

$$\tilde{Q}(z,\omega)=\int_0^z dz'\int d\boldsymbol{\omega}'\Gamma_{\tilde{Q}}(z-z'|\omega,\omega')\Xi(\omega')$$
$$\times\left(\tilde{I}^{(0)}(z,\omega')+\int_0^{z'}dz''\int d\boldsymbol{\omega}''\Gamma_{\tilde{I}}\right.$$
$$\left.\times(z'-z''|\omega',\omega'')\Xi(\omega'')\tilde{Q}(z,\omega'')\right),\qquad(23)$$

where we have written $\Xi(\omega)=\sigma\omega(\partial/\partial\omega)[(1/\omega)(\partial\Pi(\omega)/\partial\omega)]$, $\Gamma_{\tilde{I}}$ and $\Gamma_{\tilde{Q}}$ are the Green's functions of Eqs. (13) and (14) respectively, i.e., the solutions of the equations which follow from (13) and (14) if we replace the terms on their right-hand sides by sources of the form $\tilde{\Phi}(z,\omega)=\delta(z)\delta(\omega-\omega')/\omega$, and the function $\tilde{I}^{(0)}(z,\omega)$ is the solution of the scalar transport equation in the small-angle approximation,

$$\left\{\frac{\partial}{\partial z}+\sigma(1-\chi(\omega))-\frac{\sigma_a}{2}\Delta_\omega\right\}\tilde{I}^{(0)}(z,\omega)=0,\qquad(24)$$

with the boundary condition (15).

It is easy to see by direct substitution that the functions $\Gamma_{\tilde{I}}$ and $\Gamma_{\tilde{Q}}$ are the zeroth and second harmonics in the azimuthal angle of the solution $\Gamma_{\tilde{I}}(z|\boldsymbol{\omega},\boldsymbol{\omega}')$ of the scalar transport equation (24) with the boundary condition $\Gamma_{\tilde{I}}(z=0|\boldsymbol{\omega},\boldsymbol{\omega}')=\delta(\boldsymbol{\omega}-\boldsymbol{\omega}')$,

$$\Gamma_{\tilde{I}}(z|\omega,\omega')=\int_{-\pi}^{\pi}d\zeta\Gamma_{\tilde{I}}(z|\boldsymbol{\omega},\boldsymbol{\omega}'),\quad \Gamma_{\tilde{Q}}(z|\omega,\omega')$$
$$=\int_{-\pi}^{\pi}d\zeta\Gamma_{\tilde{I}}(z|\boldsymbol{\omega},\boldsymbol{\omega}')\cos(2\zeta),\qquad(25)$$

where $\boldsymbol{\omega}\boldsymbol{\omega}'=\omega\omega'\cos\zeta$.

Under small-angle scattering conditions ($\theta\ll1,\omega\sim1/\theta\gg1$) analysis shows that the estimate $|\Xi|\sim\sigma(1-\chi)/\omega^2$ holds. Ultimately the quantity ω^{-4} enters in Eqs. (22) and (23) as a small parameter and they can be solved by iteration. In particular, if the functions $\Gamma_{\tilde{I}}$ and $\Gamma_{\tilde{Q}}$ are calculated neglecting absorption, then from (22) and (23) we immediately obtain expressions (20) and (21). But in the general case the problem of calculating the polarization properties of an initially unpolarized beam reduces to solving the scalar transport equation (24) in the small-angle approximation.

Equation (24) cannot be solved analytically for an arbitrary scattering phase function. The only case in which Eq. (24) can be solved explicitly is that in which the function $\chi(\gamma)$ falls off rapidly [as in Eq. (17) with $\alpha>4$], for which $\chi(\omega)$ can be approximated by $\chi(\omega)\approx1-\langle\gamma^2\rangle\omega^2/4$. In this approximation Eq. (24) is equivalent to the diffusive transport equation in the small-angle approximation[9,10]

$$\left\{\frac{\partial}{\partial z}+\frac{\sigma_a}{2}\theta^2\right\}\tilde{I}^{(0)}(z,\theta)=D\frac{1}{\theta}\frac{\partial}{\partial\theta}\theta\frac{\partial}{\partial\theta}\tilde{I}^{(0)}(z,\theta),\qquad(26)$$

in which the quantity

$$D=\frac{1}{4}\sigma\langle\gamma^2\rangle=\frac{\pi}{2}\sigma\int_0^\infty\gamma^3d\gamma\chi(\gamma)\qquad(27)$$

is the photon angular diffusion coefficient $D=\sigma_{tr}/2$. Using the results of Refs. 9 and 10 we find

$$\tilde{I}^{(0)}(z,\omega)=\frac{1}{A_0(z)}\exp\left(\frac{\omega^2A_1(z)}{4}\right),\qquad(28)$$

$$\Gamma_{\tilde{I}}(z|\omega,\omega')=\frac{2\langle\theta^2\rangle_\infty^2}{A_0(z)A_1(z)}\exp\left\{-\frac{\langle\theta^2\rangle_\infty^2}{4A_1(z)}\right.$$
$$\left.\times(\omega^2+\omega'^2)\right\}I_0\left\{\frac{\langle\theta^2\rangle_\infty^2\omega\omega'}{2A_0(z)A_1(z)}\right\},\qquad(29)$$

where $I_n(x)$ is the modified Bessel function of order n;[35] the expression for $\Gamma_{\tilde{Q}}$ differs from (29) only in the replacement of I_0 by I_2. The functions which appear in Eqs. (28) and (29) are defined by

$$A_0(z)=\cosh(z\sqrt{2D\sigma_a}),$$
$$A_1(z)=\langle\theta^2\rangle_\infty\tanh(z\sqrt{2D\sigma_a}).\qquad(30)$$

For this case we have $\langle\theta^2\rangle_\infty = 2\sqrt{2D/\sigma_a}$. From (30) it follows that the distance l_d over which the effect of absorption on the angular divergence of the radiation flux becomes significant is determined as $l_d=(\sigma_a\langle\theta^2\rangle_\infty/2)^{-1}$.

Substituting the resulting quantities $\tilde{I}^{(0)}$, $\Gamma_{\tilde{I}}$ and $\Gamma_{\tilde{Q}}$ in (22) and (23), we can obtain expressions analogous to (20) and (21) for the intensity and the second Stokes parameter in an absorbing medium. For $\tilde{Q}$, for example, in the first nonvanishing approximation in θ^2 we find

$$\tilde{Q}=\sigma\int_0^z\frac{dz'}{A_0(z-z')A_0(z')}\int_0^\infty\frac{\omega d\omega}{2\pi}J_2\left(\frac{\omega\theta}{A_0(z-z')}\right)$$
$$\times\left[\omega\frac{\partial}{\partial\omega}\frac{1}{\omega}\frac{\partial\Pi(\omega)}{\partial\omega}\right]$$
$$\times\exp\left[-\frac{\omega^2}{4}\left(A_1(z-z')+A_1(z')-\frac{\theta^2}{A_1(z)}\right)\right].\qquad(31)$$

The range of applicability of the results (28)–(31) is discussed in detail in the next section. There we also show how to generalize these results to the case of smaller values $\alpha\leq4$ of the decay index of the phase function.

4. ANGULAR SPECTRUM FOR MULTIPLE SCATTERING OF RADIATION

We begin by considering how the angular spectrum for multiple scattering of light depends on the decay index α. This analysis is of interest even though similar studies have already been carried out in Refs. 8, 14, and 38. First of all, Refs. 8, 14, and 38 did not consider how the angular spectrum is restructured in the neighborhoods of the "critical" values $\alpha=2$ and $\alpha=4$; secondly, the question of the effect absorption has on the angular divergence of the beam was not raised at all.

We carry out our treatment for fixed values of the characteristic single-scattering angle γ_0 and momentum-transfer cross section σ_{tr}. For the sake of clarity, Fig. 1 shows a diagram illustrating the change in the angular dependence of the intensity as a function of α.

In the angular spectrum for multiple light scattering we can distinguish a bulge, the range of angles near the maximum intensity for $\theta<\theta_z$ and "wings" for $\theta>\theta_z$. In the re-

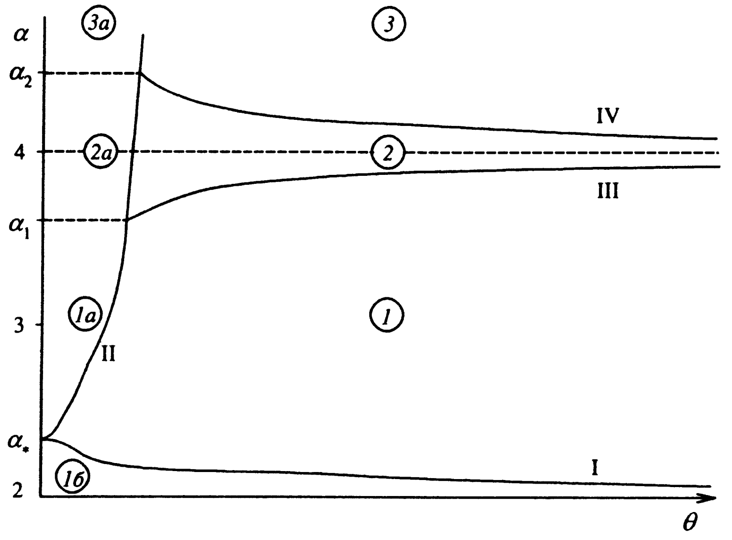

FIG. 1. Diagram illustrating the behavior of the angular spectrum $\tilde{I}(z,\theta)$ as a function of the index α. Trace I describes the behavior of $\theta=\gamma_0\exp[1/\alpha-2]$, trace II that of $\theta_z(\alpha)$, trace III that of $\theta=\gamma_0\exp[1/4-\alpha]$, trace IV that of $\theta=\gamma_0\exp[1/\alpha-4]$; $\alpha_*\approx 2+\ln(\sigma_{tr}z)/\ln\gamma_0$, $\alpha_{1,2}\approx 4\mp 2/\ln(\sigma_{tr}z/\gamma_0^2)$.

gion $\alpha>4+2/\ln(\sigma_{tr}z/\gamma_0^2)$ (regions *3* and *3*a in Fig. 1) we have $\theta_z\approx\sqrt{\sigma_{tr}z}$. Retaining the first two terms in the expansion (17) and substituting the resulting expression in Eq. (20), for $\theta<\theta_z$ we find the intensity

$$\tilde{I}_{dif}(z,\theta)\approx\frac{1}{2\pi\sigma_{tr}z}\exp\left(-\frac{\theta^2}{2\sigma_{tr}z}\right),\quad \theta\leqslant\theta_z. \tag{32}$$

In the bulge region the spectrum has a Gaussian form and results from photon diffusion in angle.[8,38] The wings of the spectrum are formed by photons which have undergone multiple deflections to small angles ($\theta<\theta_z$) and a single scattering through a relatively large angle $\theta>\theta_z$. We can establish the validity of this interpretation by estimating the optimum thickness of the layer with respect to scattering through angles $\theta>\theta_z$; it turns out to be much less than unity. The behavior of the intensity for $\theta\gg\theta_z$ is determined by the first terms not equal to ω^{2n} in the expansion (17). In the wings $\tilde{I}(z,\theta)$ is equal to

$$\tilde{I}(z,\theta)\approx(\alpha-2)(\alpha-4)\sigma_{tr}z\,\frac{\gamma_0^{\alpha-4}}{2\pi\theta^{\alpha}},\quad \theta\gg\theta_z, \tag{33}$$

In a small neighborhood of the value $\alpha=4$ (more precisely, in regions *2* and *2*a in Fig. 1) the multiple scattering process, strictly speaking, can no longer be regarded as a diffusion in angle. However, in the angular spectrum, just as for larger values of α, we can distinguish regions corresponding to the bulge and the wings. Now, retaining the third term in the expansion (17) of $\chi(\omega)$ along with the second we find that in regions *2* and *2*a of the diagram (Fig. 1) the intensity behaves as follows:

$$\tilde{I}(z,\theta)\approx\frac{\ln(2/\gamma_0)}{\pi\sigma_{tr}z\ln(\sigma_{tr}z/\gamma_0^2)}\exp\left(-\frac{\theta^2\ln(2/\gamma_0)}{2\sigma_{tr}z\ln(\sigma_{tr}z/\gamma_0^2)}\right),$$

$$\theta\leqslant\theta_z; \tag{34}$$

$$\tilde{I}(z,\theta)\approx\frac{\sigma_{tr}z}{\pi\ln(2/\gamma_0)}\,\frac{\gamma_0^{\alpha-4}}{\theta^4},\quad \theta\gg\theta_z. \tag{35}$$

As α decreases further ($\alpha<4-2/\ln(\sigma_{tr}z/\gamma_0^2)$, regions *1* and *1*a in Fig. 1) the third term begins to play the dominant role in expression (17). In this situation the characteristic multiple scattering angle θ_z varies as $\theta_z\propto(\sigma_{tr}z)^{1/\alpha-2}$, and the following expressions hold for the intensity:

$$\tilde{I}(z,\theta)$$

$$\approx\begin{cases}\dfrac{d_\alpha}{2\pi(\sigma_{tr}z)^{2/(\alpha-2)}}, & \theta\ll\theta_z, \quad (36)\\[2ex] \dfrac{4-\alpha}{2\pi(\alpha-2)}\,2^{\alpha-3}\,\dfrac{\sigma_{tr}z}{\theta^{\alpha}}, & \theta\gg\theta_z, \quad (37)\end{cases}$$

where $d_\alpha=(\alpha-2)^{(4-\alpha)/(\alpha-2)}[\Gamma(\alpha/2)/\Gamma(3-\alpha/2)]^{2/(\alpha-2)}$. In the case $\alpha=3$, which corresponds to the Henyey–Grinstein phase function,[1,36–38] the quantity $\tilde{I}(z,\theta)$ is equal to

$$\tilde{I}_{\alpha=3}(z,\theta)=\frac{\sigma_{tr}z}{2\pi\sqrt{[\theta^2+(\sigma_{tr}z)^2]^3}}. \tag{38}$$

It is easy to see that expression (38) goes over to Eqs. (36) and (37) with $\alpha=3$ in the corresponding limiting cases.

As α approaches $\alpha=2$, Eq. (37) is correct beginning at $\theta>[\sigma_{tr}z/(\alpha-2)]^{1/(\alpha-2)}>\theta_z$, i.e., the regions of the bulge and wings in the spectrum are found to be distinct.

Near $\alpha=\alpha_*\approx 2+\ln(\sigma_{tr}z)/\ln\gamma_0$ the width θ_z of the bulge decreases, becoming on the order of the single-scattering angle γ_0. Under these conditions for $\theta<\theta_*=\gamma_0\exp[1/(\alpha-2)]$ (region *1*b in Fig. 1), for the intensity $\tilde{I}(z,\theta)$ we find from (20)

$$\tilde{I}(z,\theta)\approx\frac{\nu_1\exp(\nu_2)}{2\pi\theta^{2-\nu_1}}+\frac{1}{4\pi\theta_*^2}$$

$$\times\left(\exp\left\{-\frac{\sigma_{tr}z}{\alpha-2}\left[\left(\frac{1}{\theta_*}\right)^{\alpha-2}-1\right]\right\}-\theta_*^{\nu_1}\exp(\nu_2)\right), \tag{39}$$

where

$$\nu_1=\sigma_{tr}z\left(\frac{2}{\gamma_0}\right)^{\alpha-2}, \quad \nu_2=\nu_1\left[\ln\frac{2}{\gamma_0}-\frac{1-(\gamma_0/2)^{\alpha-2}}{\alpha-2}\right].$$

In the small neighborhood $\alpha<2+[\ln(2/\gamma_0)]^{-1}$ the second term in (39) can be omitted and we find for the intensity

$$\tilde{I}(z,\theta)\approx\frac{\sigma_{tr}z}{2\pi\theta^{2-\sigma_{tr}z}}. \quad (40)$$

For the case in which $\alpha=2$ holds exactly Eq. (40) was derived in Refs. 6, 8, and 12–14.

For $\alpha<2-[\ln(2/\gamma_0)]^{-1}$ the ratio σ_{tr}/σ which characterizes the single-scattering angular anisotropy no longer contains the small parameter γ_0, i.e., we have $\sigma_{tr}/\sigma\approx2-\alpha$. Hence for $\alpha<2$ the small-angle multiple scattering regime exists only in the small neighborhood $2-\alpha\ll1$. For smaller values of the exponent α the light flux isotropizes after a small number of collisions, and the small-angle approximation is inadequate to describe multiple scattering.

Up until now we have ignored the effect of absorption on the shape of the angular spectrum of the radiation. For the case of weak absorption in the medium ($l_a>l_{tr}$) this effect starts to be important for $z\gg l_{tr}$, when the angular distribution has isotropized considerably. Absorption in this case results only in a small anisotropy of the angular spectrum.[36,37] On the other hand, for strong absorption ($l_a<l_{tr}$) the angular spectrum always remains extended in the direction of small scattering angles.[9–11,36,37] The asymptotic ("deep") regime of propagation, in which the shape of the angular spectrum and the attenuation rate of the total flux remain unchanged, sets in for strong absorption at $z<l_{tr}$. Hence the small-angle propagation regime remains correct for arbitrary z.

In the asymptotic regime the radiation intensity $\tilde{I}(z,\theta)$ can be represented as a product of two functions, one of which depends only on the depth z and the other only on the scattering angle θ (Ref. 36). The z dependence is exponential, and the attenuation rate is determined by the lowest eigenvalue corresponding to Eq. (24) for the "stationary" problem. The angular dependence of the scattered light intensity is described by the corresponding eigenfunction. The absorption changes the shape of the angular spectrum noticeably, but it continues to depend on the decay index α of the phase function.

The spectrum can be calculated most easily in the region of the wings. For this it suffices to find the form of the expansion of $\tilde{I}(z,\omega)$ for small ω. Substituting the expansion of $\tilde{I}(z,\omega)$ in a power series in ω in Eq. (24) and equating the coefficients of powers of ω, we find

$$\tilde{I}(z,\omega)=\left(1-\frac{1}{4}\langle\theta^2\rangle_\infty\omega^2+\frac{\Gamma(2-\alpha/2)}{\alpha^2(\alpha-2)\Gamma(\alpha/2)}\times\frac{4-\alpha}{1-(2/\gamma_0)^{\alpha-4}}\frac{\sigma_{tr}}{\sigma_a}\omega^\alpha+\dots\right)\tilde{E}(z), \quad (41)$$

where $\tilde{E}(z)=\tilde{I}(z,\omega=0)=\int d\boldsymbol{\theta}\tilde{I}(z,\theta)$ is the radiation flux at depth z.

The series (41) corresponds to the following asymptotic form of $\tilde{I}(z,\theta)$:

$$\tilde{I}(z,\theta)=\frac{2^{\alpha-3}(4-\alpha)}{1-(2/\gamma_0)^{\alpha-4}}\frac{\sigma_{tr}}{\pi\sigma_a}\frac{\tilde{E}(z)}{\theta^{\alpha+2}}. \quad (42)$$

Thus, owing to the increase in the probability of absorption of strongly deflected photons, the intensity rate of decay $\tilde{I}\propto\theta^{-\alpha}$ at large z becomes $\tilde{I}\propto\theta^{-\alpha-2}$. This change in the asymptotic decay law for the intensity was demonstrated in Ref. 39 for the case of an exactly soluble scattering model in a two-dimensional medium.

As for the region where the angle θ is relatively small (i.e., the bulge in the spectrum), here the dependence of the intensity on α becomes less marked as z increases.

For $\alpha>4+2[\ln(\theta_z^2/\gamma_0^2)]^{-1}$ the bulge in the spectrum, as in the case when absorption is present, forms as a result of photon angular diffusion. In this case in the expansion for $\chi(\omega)$ we can retain just the quadratic term. The solution of Eq. (24) with this function $\chi(\omega)$ is determined by the expression (28), which leads to the following simple formula for the intensity:

$$\tilde{I}(z,\theta)=\frac{1}{\pi A_0(z)\langle\theta^2\rangle_z}\exp\left\{-\frac{\theta^2}{\langle\theta^2\rangle_z}\right\}, \quad (43)$$

where

$$\langle\theta^2\rangle_z=A_1(z) \quad (44)$$

is the mean square multiple-scattering angle at depth z. From (43) for $z\ll l_d=\sqrt{l_a l_{tr}}$ we obtain the previous result (32), while for $z\gg l_d$ the radiation distribution in the asymptotic regime is

$$\tilde{I}(z,\theta)=\frac{2}{\pi\langle\theta^2\rangle_\infty}\exp\left(-\frac{z}{l_d}-\frac{\theta^2}{\langle\theta^2\rangle_\infty}\right), \quad (45)$$

where $\langle\theta_2\rangle_\infty=2\sqrt{l_a/l_{tr}}$ is the asymptotic value of the mean square multiple-scattering angle. For strong absorption, when Eq. (45) holds, we have $\langle\theta^2\rangle_\infty<1$.

For $\alpha<4+2(\ln(\theta_z^2/\gamma_0^2))^{-1}$ the above approximation for $\chi(\omega)$ fails, and these results become inapplicable. In this case we can use the self-consistent diffusion approximation[40] to calculate $\tilde{I}(z,\theta)$ in the region of the bulge. The essential idea of this approximation is as follows. We assume that a layer of the medium of a given thickness z is characterized by a certain value of the diffusion coefficient D_z. Then the intensity $\tilde{I}(z,\theta)$ of the radiation passing through the layer can be described by the Gaussian distribution (43). In determining the diffusion coefficient D_z we assume that the main contribution to D_z comes from photons deflected through angles less than the typical multiple-scattering angle in the layer, i.e., $\sqrt{\langle\theta^2\rangle_z}$, and we set

$$D_z=\frac{\pi}{2}\sigma\int_0^{\eta\sqrt{\langle\theta^2\rangle_z}}\gamma^3d\gamma\chi(\gamma). \quad (46)$$

In the case $\alpha>4$ the integral in (46) for $\sqrt{\langle\theta^2\rangle_z}\gg\gamma_0$ is independent of the upper limit and the definition (45) is the same as the usual definition of the diffusion coefficient [cf. Eq. (27)]. For $\alpha\leqslant4$ Eqs. (44) and (46) constitute a self-consistent prescription for calculating the intensity $\tilde{I}$ of the radiation. The usual diffusion approximation for $\alpha\leqslant4$ is inapplicable, since the integral (27) diverges at the upper limit.

The numerical constant η which enters into (46) is determined by comparing the value of $\bar{I}$ found in the self-consistent approximation and the result of solving the transport equation without treating absorption [cf. Eqs. (34), (36), and (39)]. For a phase function of the form (16) we find $\eta=[\Gamma(3-\alpha/2)]^{1/(4-\alpha)}$, $2\leqslant\alpha\leqslant4$.

Note that this method for calculating the angular distribution was used for the first time in the work of Williams[41] (see also Refs. 5 and 42) to describe multiple electron scattering (where $\alpha=4$ holds).

Close to the value $\alpha=4$ (where $\alpha=4$ $(|\alpha-4|<[\ln(\langle\theta^2\rangle_z/\gamma_0^2)]^{-1})$ holds) the value of D_z is related to the mean square multiple-scattering angle $\langle\theta^2\rangle_z$ through the following system of equations:

$$D_z=\frac{1}{4}\,\sigma\gamma_0^2\ln\frac{\langle\theta^2\rangle_z}{\gamma_0^2},\qquad(47)$$

$$\langle\theta^2\rangle_z=\sqrt{\frac{8D_z}{\sigma_a}}\tanh(z\sqrt{2D_z\sigma_a}).\qquad(48)$$

According to (47) and (48), at moderate depths $z\ll l_d$ we have $D_z\approx1/4\sigma\gamma_0^2\ln\sigma z$, and we come back to the result (34). In the asymptotic regime $(z>l_d)$ we have $D_\infty=1/4\sigma\gamma_0^2\ln(\langle\theta^2\rangle_\infty/\gamma_0^2)$ and

$$\langle\theta^2\rangle_\infty=\sqrt{\frac{2\sigma\gamma_0^2}{\sigma_a}\ln\sqrt{\frac{2\sigma}{\sigma_a\gamma_0^2}}}.$$

In the region $2<\alpha<4$ the diffusion coefficient satisfies

$$D_z=\frac{\alpha-2}{4(4-\alpha)}\,\sigma\gamma_0^{\alpha-2}\Gamma\left(3-\frac{\alpha}{2}\right)\langle\theta^2\rangle_z^{2-\alpha/2},\qquad(49)$$

and $\langle\theta_2\rangle_z$ continues to be given by expression (48). Substituting (49) in (48) we find a simple equation for the quantity $\Lambda=D_z/D_{z\to\infty}$:

$$[\Lambda(\xi)]^{\alpha/2(4-\alpha)}=\tanh[\xi\sqrt{\Lambda(\xi)}],\qquad(50)$$

where

$$\xi=\frac{z}{l_d}=\frac{1}{2}\,\sigma_a z\langle\theta^2\rangle_\infty,\quad \langle\theta^2\rangle_\infty=\left[2^{\alpha-2}\,\frac{\sigma_{tr}}{\sigma_a}\,\Gamma\left(3-\frac{\alpha}{2}\right)\right]^{2/\alpha}.\qquad(51)$$

Equation (50) for $\xi\ll1$ implies $\Lambda(\xi)=\xi^{(4-\alpha)/(\alpha-2)}$, and we obtain (36). For $\xi\gg1$ the function $\Lambda(\xi)$ is equal to unity and for $\bar{I}(z,\theta)$ expression (45) continued to hold for $\theta\leqslant\sqrt{\langle\theta^2\rangle_\infty}$; here the quantity $\langle\theta^2\rangle_\infty$ is now defined using Eq. (51).

Close to $\alpha=2$ (i.e., for $\alpha=2$ $(|\alpha-2|<[\ln(2/\gamma_0)]^{-1})$ the quantity D_z is expressed in terms of $\langle\theta^2\rangle_z$ using

$$D_z=\frac{1}{8}\,\sigma\langle\theta^2\rangle_z\left(\ln\frac{2}{\gamma_0}\right)^{-1},\qquad(52)$$

which together with (48) leads to the equation

$$\sqrt{\Lambda(\xi)}=\tanh[\xi\sqrt{\Lambda(\xi)}],\qquad(53)$$

where $\langle\theta^2\rangle_\infty=\sigma/\sigma_a\ln(2/\gamma_0)$. Equation (53) has a nontrivial solution only for $\xi>1$. This results from the following consideration.

For values of α lying close to $\alpha=2$ it makes sense to talk about a bulge in the angular distribution only for a depth $z>l_d$. At smaller depths $z<l_d$, as shown above [cf. Eq. (39)] the multiple-scattering angular spectrum has no bulge [i.e., the quantity $\bar{I}(z,\theta\to0)$ in question is undefined]. Note that this change in the shape of the angular spectrum as the depth increases is due to the effect of absorption in the medium.

For small-angle multiple scattering the relative size of the polarization correction $\delta\bar{I}_{pol}$ to the intensity at arbitrary values of α is small compared with unity. Specifically, at small depths ($z\ll l_d$ the second term in (20) yields

$$\frac{\delta\bar{I}_{pol}}{\bar{I}}\sim\begin{cases}\sigma_{tr}z\theta^2, & \alpha<\alpha_*,\\ \theta_z^{2(\alpha-1)}, & \alpha_*<\alpha\leqslant3,\\ \theta_z^4, & 3\leqslant\alpha.\end{cases}\qquad(54)$$

Accordingly, in the asymptotic regime $(z\gg l_d)$ we find for arbitrary α directly from Eqs. (13) and (14)

$$\frac{\delta\bar{I}_{pol}}{\bar{I}}\sim\langle\theta^2\rangle_\infty^2.\qquad(55)$$

In Eqs. (54) and (55) the values of $\langle\theta^2\rangle_z$ and $\langle\theta^2\rangle_\infty$ for each α are determined by the relations given above.

One further remark is needed regarding $\delta\bar{I}_{pol}$. From the energy conservation law (12) it follows that the right-hand side of Eq. (13) should vanish at $\omega=0$. On the other hand, Eq. (21) implies that the second Stokes parameter varies as $Q\propto\omega^{\alpha-2}$ close to the value $\omega=0$. Consequently, the right-hand side of Eq. (13) for small $\omega\to0$ is proportional to $\omega^{2\alpha-6}$ and for $\alpha\leqslant3$ it does not vanish at the point $\omega=0$. This contradiction is not real; it is associated with the inapplicability of Eqs. (7) for $\theta>1$ and that of Eqs. (13) and (14) for very small $\omega\sim1/\theta\ll1$.

5. DEGREE OF POLARIZATION OF MULTIPLY SCATTERED RADIATION

5.1. Weakly absorbing media

To analyze the degree of polarization we break up the range of variation of α into several subregions $\alpha>6$; $|\alpha-6|<[\ln(\sigma z)]^{-1}$; $4<\alpha<6$; $|\alpha-4|<[\ln(\sigma z)]^{-1}$; $2<\alpha<4$; $|\alpha-2|\ll1$. In each of these the formulas for the polarization properties of light are quite different.

In the case of rapidly decaying phase functions ($\alpha>6$) it follows from (20) and (21) together with the expansion (17) that the degree of polarization near the bulge in the angular spectrum ($\theta<\theta_z,\theta_z\sim\sqrt{\sigma_{tr}z}$) is equal to

$$P=-\frac{|c_\alpha|}{|a_\alpha|}\frac{\theta^2}{D_z}=-\frac{1}{8}\frac{\langle\gamma^4\rangle}{\langle\gamma^2\rangle}\frac{\theta^2}{\sigma_{tr}z}.\qquad(56)$$

For small values of α, ($4<\alpha<6$) the degree of polarization in the region of the bulge falls off with depth more slowly than is implied by (56),

$$P=-p(\alpha,z)\theta^2,\qquad(57)$$

where

$$p(\alpha,z)=2^{-\alpha}\Gamma\left(3-\frac{\alpha}{2}\right)(\alpha-2)(\alpha-4)\left[\frac{2\gamma_0^2}{\sigma_{tr}z}\right]^{(\alpha-4)/2}\qquad(58)$$

In the transition regions $|\alpha-4|$, $|\alpha-6|<[\ln(\sigma z)]^{-1}$ the quantity P for $\theta<\theta_z$ [where $\theta_z \sim \sqrt{\sigma z \gamma_0^2 \ln(\sigma z)}$ holds for $\alpha\approx 4$ (Ref. 31) and $\theta_z \sim \sqrt{\sigma_{tr} z}$ holds for $\alpha\approx 6$] contains the logarithmic factor

$$P=-\frac{\theta^2}{4\ln(\sigma z)}, \quad |\alpha-4|<[\ln(\sigma z)]^{-1}; \tag{59}$$

$$P=-\frac{\ln(\sigma z)}{\sigma z}\frac{\theta^2}{2}, \quad |\alpha-6|<[\ln(\sigma z)]^{-1}. \tag{60}$$

In the region $\alpha_*<\alpha<4$, to lowest nonvanishing order in $\gamma_0/\theta\ll 1$, we find that in the region of the bulge of the angular spectrum $(\theta<\theta_z\sim(\sigma_{tr}z)^{1/(\alpha-2)})$ the degree of polarization is independent of depth:

$$P=-\frac{4-\alpha}{8}\theta^2, \tag{61}$$

In the wings of the spectrum $(\theta>\theta_z)$ the degree of polarization for all $\alpha>\alpha_*$ is equal to

$$P=-\frac{\theta^2}{2}. \tag{62}$$

For $\alpha<\alpha_*$ in the range of angles $\theta<\theta_*$ we find for the degree of polarization

$$P=-\frac{\theta^2}{2}$$

$$\times\frac{1+\frac{\sigma_{tr}z}{8\nu_1}\frac{\theta^{2-\nu_1}}{\theta_*^{\alpha}}\left(\exp\left\{-\frac{\sigma_{tr}z}{\alpha-2}\left[\left(\frac{1}{\theta_*}\right)^{\alpha-2}-1\right]-\nu_2\right\}-\theta_*^{\nu_1}\right)}{1+\frac{\theta^{2-\nu_1}}{2\nu_1\theta_*^2}\left(\exp\left\{-\frac{\sigma_{tr}z}{\alpha-2}\left[\left(\frac{2}{\theta_*}\right)^{\alpha-2}-1\right]-\nu_2\right\}-\theta_*^{\nu_1}\right)}.$$

As α decreases further, $\alpha<2+[\ln(2/\gamma_0)]^{-1}$, the angular spectrum in the whole range of angles $\theta>\gamma_0$ falls off with increasing θ algebraically [cf. Eq. (40)] and the degree of polarization is determined by expression (62).

Note that for the Henyey–Grinstein phase function $(\alpha=3)$ the degree of polarization can be described by a single general formula over the whole range of variation of angles (Fig. 2)

$$P=-\frac{\sigma_{tr}^4 z^4}{2\theta^2}\left[\sqrt{1+\left(\frac{\theta}{\sigma_{tr}z}\right)^2}-1\right]^2\left[1+\left(\frac{\theta}{\sigma_{tr}z}\right)^2\right]. \tag{63}$$

Expression (63) for $\theta\ll\sigma_{tr}z$ and $\theta\gg\sigma_{tr}z$ goes over to (61) and (62) respectively.

As can be seen from the above results, for relatively large angles θ, where the intensity falls off as $I\propto\theta^{-\alpha}$, the quantity P is the same as the degree of polarization for singly scattered radiation.[1,31] This behavior holds in general. It is related to the following fact. The number of collisions N in a layer of thickness z, as a result of which a photon is deflected through an angle $\theta>[\sigma_{tr}z/(\alpha-2)]^{1/(\alpha-2)}\geqslant\theta_z$, is equal to the product of the total number of collisions $N\sim\sigma z$ by the probability of deflection through an angle $\theta>[\sigma_{tr}z/(\alpha-2)]^{1/(\alpha-2)}$ in a single scattering event. A simple estimate shows that the quantity N is of order unity. Consequently, the radiation intensity in this range of angles develops as a result of many scattering events through angles $\theta\ll[\sigma_{tr}z/(\alpha-2)]^{1/(\alpha-2)}$ and a single scattering through a relatively large angle $\theta>[\sigma_{tr}z/(\alpha-2)]^{1/(\alpha-2)}$. It is just this latter process which has the greatest effect on the polarization properties of the radiation and determines the magnitude of P.

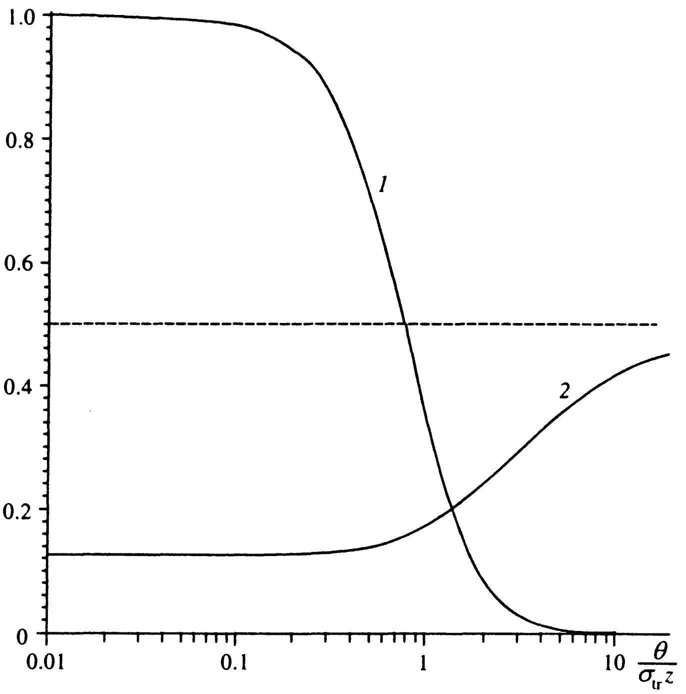

FIG. 2. Characteristics of multiple scattering of light for the Henyey-Grinstein phase function $(\alpha=3)$. *1*) Angular spectrum (arbitrary index); *2*) the function $p(\alpha=3,z)=|P|/\theta^2$; the broken trace represents the value $p(\alpha,z)=1/2$ corresponding to single scattering of light.

Expression (62) determines the main contribution to the value of P for relatively large angles θ. On the other hand, the deviation δP in the degree of polarization from the value determined by Eq. (62) $(P=-\theta^2/2+\delta P)$ increases as a function of θ. It follows from (20) and (21) that for $\theta>\theta_z$ the quantity δP behaves for as $\theta^{4-\alpha}$ when θ increases. This is clearly discernible also in the case of the Henyey–Grinstein phase function [Eq. (63)], for which $\delta P\propto\theta$ holds.

5.2. Strongly absorbing media

Absorption not only changes the angular distribution of multiply scattered radiation, it can also have a major effect on the dependence of the degree of polarization on the depth z. Let us analyze the behavior of P in strongly absorbing $(l_a<l_{tr})$ media.

For a phase function $\chi(\gamma)$ that falls off with increasing γ more rapidly than γ^{-4} we find from expressions (31) and (43)

$$P=-\frac{\sigma\theta^2}{32}A_0(z)A_1(z)\int_0^z\frac{dz'}{A_0^3(z-z')A_0(z')}$$

$$\times \int_0^\infty \omega^3 d\omega \left[\omega \frac{\partial}{\partial \omega} \frac{1}{\omega} \frac{\partial \chi(\omega)}{\partial \omega} \right]$$

$$\times \exp\left[-\frac{\omega^2}{4} [A_1(z-z') + A_1(z')] \right]. \tag{64}$$

Like (43), expression (64) is applicable in describing the propagation of radiation in the region of small deflection angles $\theta \leqslant \theta_z$ (i.e., in the region of the bulge in the angular spectrum. For phase functions with a decay index $\alpha > 6$ expression (64) leads to the following result:

$$P = -\frac{\theta^2}{8} \frac{\sigma}{\sigma_d} \frac{\langle \gamma^4 \rangle}{\langle \theta^2 \rangle_\infty^2} f(\sigma_d z), \tag{65}$$

where $\langle \gamma^4 \rangle = 8\gamma_0^4/(\alpha-4)(\alpha-6)$, $\sigma_d = l_d^{-1} = 1/2\sigma_a \langle \theta^2 \rangle_\infty$, and the function $f(\xi)$ is equal to

$$f(\xi) = 2\int_0^\xi d\xi' \left(\frac{\cosh \xi'}{\sinh \xi} \right)^2 = \frac{1}{\sinh^2 \xi} \left(\xi + \frac{1}{2} \sinh(2\xi) \right). \tag{66}$$

At relatively small depths $z \ll l_d$, when the effect of absorption on the angular distribution of the radiation is negligible, expression (65) goes over to (56). As z increases the degree of polarization drops off monotonically, and in the asymptotic regime $z \gg l_d$ it approaches its limiting value

$$P_\infty = -\frac{\theta^2}{8} \frac{\sigma}{\sigma_d} \frac{\langle \gamma^4 \rangle}{\langle \theta^2 \rangle_\infty^2} = -\frac{\theta^2}{8} \sqrt{\frac{\sigma_a \langle \gamma^2 \rangle}{2\sigma}} \frac{\langle \gamma^4 \rangle}{\langle \gamma^2 \rangle^2}. \tag{67}$$

In the vicinity of $\alpha = 6$ ($|\alpha-6| < [\ln(\langle \theta^2 \rangle_z/\gamma_0^2)]^{-1}$) the degree of polarization is determined by

$$P = -\frac{\theta^2}{4} \frac{\sigma}{\sigma_d} \frac{\gamma_0^4}{\langle \theta^2 \rangle_\infty^2} f(\sigma_d z | \alpha = 6), \tag{68}$$

where the function f differs from (66) by the logarithmic factor

$$f(\xi | \alpha = 6) = 2\int_0^\xi d\xi' \left(\frac{\cosh \xi'}{\sinh \xi} \right)^2 \times \ln \frac{\langle \theta^2 \rangle_\infty \sinh(\xi)}{\gamma_0^2 \cosh(\xi - \xi') \cosh \xi'}. \tag{69}$$

For $z \ll l_d$ relations (68) and (69) go over to (60). In the asymptotic regime we find from (68) and (69)

$$P_\infty = -\frac{\theta^2}{8} \frac{\sigma}{\sigma_d} \frac{\gamma_0^4}{\langle \theta^2 \rangle_\infty^2} \ln\left\{ \frac{\langle \theta^2 \rangle_\infty^2}{\gamma_0^4} \right\} = -\frac{\theta^2}{8} \sqrt{\frac{\sigma_a \gamma_0^2}{2\sigma}} \ln \frac{2\sigma}{\sigma_a \gamma_0^2}. \tag{70}$$

In (70) the analog of $\langle \gamma^4 \rangle$ is the quantity $\gamma_0^4 \ln(\langle \theta^2 \rangle_\infty^2/\gamma_0^4)$ and we have $\langle \gamma^2 \rangle = \gamma_0^2$.

In the range $4 < \alpha < 6$ Eq. (64) yields

$$P(z,\theta) = -\frac{\theta^2}{8} \left(\frac{\alpha}{2} - 1 \right) \Gamma \times \left(3 - \frac{\alpha}{2} \right) \frac{\sigma}{\sigma_d} \frac{\gamma_0^{\alpha-2}}{\langle \theta^2 \rangle_\infty^{(\alpha-2)/2}} f(\sigma_d z | \alpha < 6), \tag{71}$$

where

$$f(\xi | \alpha < 6) = 2\int_0^\xi d\xi' [\cosh(\xi - \xi')]^{(\alpha-6)/2} \left(\frac{\cosh \xi'}{\sinh \xi} \right)^{(\alpha-2)/2} = \begin{cases} 2\xi^{2-\alpha/2}, & \xi \ll 1, \\ 2 \dfrac{2^{2-\alpha/2} - 1}{2 - \alpha/2}, & \xi \gg 1. \end{cases} \tag{72}$$

The result (57), (58) follows from (71), (72) for $z \ll l_d$, and in the "deep" regime we have

$$P_\infty = -\frac{\theta^2}{2^{\alpha/2}} (2^{\alpha/2-2} - 1) \left(\frac{\alpha}{2} - 1 \right) \times \Gamma\left(3 - \frac{\alpha}{2} \right) \left[\frac{\sigma_a}{2\sigma} \gamma_0^2 \left(\frac{\alpha}{2} - 2 \right) \right]^{\alpha/4 - 1}. \tag{73}$$

According to (73), the degree of polarization P_∞ increases as the index α decreases.

For $\alpha < 4 + 2[\ln(\langle \theta^2 \rangle_z/\gamma_0^2)]^{-1}$ we should use the functions A_0 and A_1 calculated in the self-consistent diffusion approximation in relation (64). The values of these functions $A_i(z')$, $A_i(z-z')$, $i = 0, 1$ in a layer of thickness z are calculated assuming (see Sec. 4 above) that the layer is characterized by its own value of the diffusion coefficient D_z. The above remarks imply that the degree of polarization near $\alpha = 4$ ($|\alpha - 4| < [\ln(\langle \theta^2 \rangle_z/\gamma_0^2)]^{-1}$) is given by

$$P(z,\theta) = -\frac{\theta^2}{32} \frac{\sigma\gamma_0^2}{D_z} f\left(\sqrt{\frac{D_z}{D_\infty}} \sigma_d z \middle| \alpha = 4 \right),$$

$$f(\xi | \alpha = 4) = 2(\xi \coth \xi - \ln \cosh \xi), \tag{74}$$

where D_z is determined by the system of equations (47), (48).

At smaller depths $z \ll l_d$, expression (74) goes over to (59), while for $z \gg l_d$ it approaches a limit

$$P_\infty = -\frac{\theta^2}{2} \frac{\ln 2}{\ln(2\sigma/\sigma_a \gamma_0^2)}. \tag{75}$$

As α decreases further ($2 < \alpha < 4$) the value of the degree of polarization continues to increase:

$$P(z,\theta) = -\frac{\theta^2}{8} \left(2 - \frac{\alpha}{2} \right) \left(\frac{D_\infty}{D_z} \right)^{\alpha/4} f\left(\sqrt{\frac{D_z}{D_\infty}} \sigma_d z \middle| \alpha < 4 \right), \tag{76}$$

where the ratio D_z/D_∞ is found from Eq. (50).

For $z \ll l_d$ we find Eq. (61) from (76). As regards the limiting value P_∞ of the degree of polarization, it is found to be smaller than the quantity (61),

$$P_\infty = -\frac{\theta^2}{4} (2^{2-\alpha/2} - 1). \tag{77}$$

The α dependence of P_∞ is monotonic: in the range of values $2 < \alpha < 4$, P_∞ varies from $P_\infty(\alpha=2) = -\theta^2/4$ to $P_\infty(\alpha=4) = -\theta^2/2[\ln 2/\ln(2\sigma/\sigma_a \gamma_0^2)]$.

The above results (67), (70), (73), (75), and (77) imply that the asymptotic value P_∞ of the degree of polarization can be written in the form

$$P_\infty = \theta^2 \Psi_a\left(\frac{\sigma_a \gamma_0^2}{\sigma} \right), \tag{78}$$

where $\Psi_a(x)$ is some function. Thus, P_∞ depends on the optical properties of the medium only through the combination $\sigma_a\gamma_0^2/\sigma$. Physically, the quantity $\sigma_a\gamma_0^2/\sigma$ is proportional to the difference in the probabilities of absorption of unscattered and singly scattered photons over the mean free path l. Since $\gamma_0^2\ll 1$ holds and in realistic situations we have $\sigma_a/\sigma<1$, the parameter $\sigma_a\gamma_0^2/\sigma$ is generally small.

The value of the degree of polarization in the wings of the distribution ($\theta>\theta_z$) does not depend on the optical properties of the medium and agrees with the result that follows from the single-scattering law. For the case $z\ll l_d$ this was shown in the previous section. It is also quite simple to prove this assertion for large depths.

Substituting the expansion (41) into Eq. (14) and following the same procedure as in the derivation of Eq. (41) itself, we find for $\tilde{Q}(z,\omega)$ an expansion in small ω. After inverting the Bessel transformation for the second Stokes parameter we find

$$\tilde{Q}(z,\theta)=-\frac{4-\alpha}{1-(2/\gamma_0)^{\alpha-4}}\cdot\frac{\Gamma(2-\alpha/2)}{\alpha\Gamma(\alpha/2)}\frac{\sigma_{tr}}{\sigma_a}\int_0^\infty\frac{\omega^{\alpha-1}d\omega}{2\pi}\times J_2(\omega\theta)\left(1-\frac{1}{4}\langle\theta^2\rangle_\infty\omega^2+\dots\right)\tilde{E}(z)$$
$$=-\frac{2^{\alpha-4}(4-\alpha)}{1-(2/\gamma_0)^{\alpha-4}}\frac{\sigma_{tr}}{\pi\sigma_a}\frac{\tilde{E}(z)}{\theta^\alpha}+\dots\ . \qquad (79)$$

As in (42), Eq. (79) holds for $\theta\gg\sqrt{\langle\theta^2\rangle_\infty}$. The ratio of (79) to (42) yields $P=-\theta^2/2$.

6. MULTIPLE SCATTERING FROM LARGE PARTICLES: CORRECTIONS TO THE BORN APPROXIMATION

Now we consider propagation of light in a medium consisting of transparent spheres of large radius ($k_0a|n-1|\gg 1$, $k_0a\ \mathrm{Im}\ n\ll 1$), when the Born approximation is inapplicable. This situation occurs quite frequently in various experiments on multiple scattering of light.[3,20,23–25]

In the range of angles γ greater than the diffraction angle ($\gamma\gg\lambda/a$) the phase function for single scattering averaged over the range of angles $\Delta\gamma>\lambda/a$ can be represented as an expansion in the number of collisions of rays with a surface scatterer:[31]

$$\chi(\gamma)=\chi^{(1)}(\gamma)+\chi^{(2)}(\gamma)+\chi^{(3)}(\gamma)+\dots\ . \qquad (80)$$

The expansion (80) is obtained neglecting interference between waves which undergo different numbers of reflections.

The quantities $\chi^{(1)}$, $\chi^{(2)}$, and $\chi^{(3)}$ result respectively from rays deflected as a result of specular reflection from the surface, passing through the scatterer, and also from having experienced one additional internal reflection from the surface of the sphere. At scattering angles $\gamma\ll\sqrt{|n-1|}$ the second term in (80) dominates. In the range of angles $\lambda/a\ll\gamma\ll\sqrt{|n-1|}$ it is determined by expression (16) with $\alpha=4$, $\chi_0=1/2\pi$, and $\gamma_0=2|n-1|$, while for $\gamma\geqslant 2\sqrt{2|n-1|}$ it vanishes identically.[31,43] It is just this term which makes the principal contribution to the total scattering cross section $\sigma=2\pi a^2$. The first and third terms in (80) are important for $\gamma>\sqrt{|n-1|}$. In this range of angles they are all equal to lowest order in $|n-1|$. Their total contribution to (80) is given by the asymptotic formula (16) at large angles ($\gamma\gg\gamma_0$) with $\alpha=4$, $\chi_0=1/4\pi$, and $\gamma_0=2|n-1|$. The relative magnitude of the contribution of $\chi^{(1)}$ and $\chi^{(3)}$ to the total cross section is small, $\sigma^{(1),(3)}/\sigma\sim|n-1|\ll 1$.

Using the expansion (17) for $\alpha=4$ we find for the quantity $1-\chi(\omega)$ in the range $\omega<1/\gamma_0\sim|n-1|^{-1}$ of interest to us

$$1-\chi(\omega)=\begin{cases}\frac{1}{2}|n-1|^2\omega^2\ln\frac{1}{\omega^2|n-1|^2}, & |n-1|^{-1/2}\ll\omega\ll|n-1|^{-1},\\ \frac{1}{4}|n-1|^2\omega^2\ln\frac{1}{\omega^2|n-1|^3}, & \omega\ll|n-1|^{-1/2}.\end{cases} \qquad (81)$$

From expression (81) it follows that under these conditions the intensity $\tilde{I}(z,\theta)$ for multiple scattering of radiation near the bulge ($\theta<\theta_z$) is determined by a relation analogous to Eq. (34),

$$\tilde{I}(z,\theta)=\frac{1}{2\pi\sigma|n-1|^2zL_n}\exp\left(-\frac{\theta^2}{2\sigma|n-1|^2zL_n}\right), \qquad (82)$$

where $L_n=\ln(\sigma z)$ holds for $z<l[|n-1|\ln(1/|n-1|)]^{-1}$ and $L_n=\ln\sqrt{\sigma z/|n-1|}$ holds for $z>l[|n-1|\ln(1/|n-1|)]^{-1}$. The intensity of the wings ($\theta>\theta_z$) of the angular spectrum is equal to the product of (80) and σz.

Note that, as can be seen from (82), the spread in the multiple scattering angle at depths $z>l[|n-1|\ln(1/|n-1|)]^{-1}$ is the sum of two terms,

$$\langle\theta^2\rangle_z\approx\sigma|n-1|^2z\ln\frac{1}{|n-1|}+\sigma|n-1|^2z\ln(\sigma z). \qquad (83)$$

The first term in (83) results from small-angle diffusion of rays that have crossed the boundary of different scatterers twice. The second term in (83) describes multiple scattering of rays that have undergone external or internal specular reflection from the boundaries of the scatterers and deflection in each collision through a relatively large ($\gamma>\sqrt{|n-1|}$) angle.

Corrections to the Born approximation are especially important for the magnitude of the degree of polarization of the scattered waves. From the small-angle expansion of the exact scattering matrix[44] we find that in the case of very large irregularities ($k_0a|n-1|\geqslant 1$) the diagonal elements d_{11} and d_{22} continue to be determined by expressions (8), while for the function Π which appears in the nondiagonal elements of (9) and (10) we now must use the expression

$$\Pi(\gamma)=\frac{1}{2\sigma\gamma^2}(|A_\parallel(\gamma)|^2-|A_\perp(\gamma)|^2), \tag{84}$$

where $A_\parallel$ and $A_\perp$ are the scattering amplitudes of the waves polarized respectively parallel and perpendicular to the scattering plane.[1,31,44]

In analogy with Eq. (80) we write the quantity $\Pi(\gamma)$ as a sum

$$\Pi(\gamma)=\Pi^{(1)}(\gamma)+\Pi^{(2)}(\gamma)+\Pi^{(3)}(\gamma)+\dots, \tag{85}$$

where each term describes polarization of rays undergoing a different number of interactions with the boundary of a scatterer.

Using the results of the asymptotic representation of the exact Mie solution for large scatterers,[31] we find

$$\Pi(\gamma)=\begin{cases}\Pi^{(2)}(\gamma)\approx\frac{1}{4}\chi^{(2)}(\gamma), & \gamma\ll\sqrt{|n-1|},\\ \Pi^{(1)}(\gamma)+\Pi^{(3)}(\gamma)\approx-\frac{1}{2}[\chi^{(1)}(\gamma)+\chi^{(3)}(\gamma)], & \gamma\gg\sqrt{|n-1|}.\end{cases} \tag{86}$$

From (86) we find without difficulty the degree of polarization of singly scattered radiation

$$P=\frac{\Pi(\gamma)\gamma^2}{\chi(\gamma)}=\begin{cases}\gamma^2/4, & \gamma\ll\sqrt{|n-1|},\\ -\gamma^2/2, & \gamma\gg\sqrt{|n-1|}.\end{cases} \tag{87}$$

According to (87), at large angles the Born approximation still holds, while at relatively small angles the degree of polarization in general has the opposite sign.

The function $\Pi(\omega)$ corresponding to expression (86) is equal to

$$\Pi(\omega)=\begin{cases}\frac{1}{8}\left(1-\left|n-1\right|^2\omega^2\ln\frac{1}{\omega^2|n-1|^2}\right), & |n-1|^{-1/2}\ll\omega\ll|n-1|^{-1},\\ \frac{1}{8}\left(1+\left|n-1\right|^2\omega^2\ln\frac{1}{\omega^2}\right), & \omega\ll|n-1|^{-1/2}.\end{cases} \tag{88}$$

At depths $z<l[|n-1|\ln(1/|n-1|)]^{-1}$ the angle $\theta\sim\sqrt{|n-1|}$ enters the region of the wings of the angular spectrum (i.e., $\sqrt{|n-1|}>\theta_z$). In this case, using expression (86) and the results of the previous section, we find for the degree of polarization of multiply scattered waves in the region of the bulge in the spectrum ($\theta\ll\theta_z$)

$$P=\frac{\theta^2}{8\ln(\sigma z)}. \tag{89}$$

For $z>l[|n-1|\ln(1/|n-1|)]^{-1}$ the situation is different. The opposite inequality $\sqrt{|n-1|}<\theta_z$ holds, and hence collisions with deflections through angles $\gamma>\sqrt{|n-1|}$ play the principal role. The main contribution to (20) and (21) comes from values $\omega\ll(\sqrt{|n-1|})^{-1}$. The degree of polarization has a value close to that found above in the Born approximation,

$$P=-\frac{\theta^2}{4\ln(\sigma z/|n-1|)}. \tag{90}$$

At large depths, where absorption effects become important, when $l\ll l_a\ll l_{tr}$ holds, the width of the angular spectrum is always greater than $\sqrt{|n-1|}$. Using this inequality we find for the degree of polarization

$$P(z,\theta)=-\frac{\theta^2}{32}\frac{\sigma|n-1|^2}{D_z}f\left(\sqrt{\frac{D_z}{D_\infty}}\sigma_d z\middle|\alpha=4\right), \tag{91}$$

where the diffusion coefficient is $D_z=1/4(\sigma|n-1|^2)\ln(\langle\theta^2\rangle_z/|n-1|^3)$, and the function $f(\xi|\alpha=4)$ is defined by Eq. (74). For small values of z expression (91) goes over to (90), and for $z\to\infty$ it yields

$$P_\infty=-\frac{\theta^2}{2}\frac{\ln 2}{\ln(\sigma/\sigma_a|n-1|^4)}. \tag{92}$$

As for the degree of polarization on the wings of the angular spectrum ($\theta\gg\theta_z$), in all cases considered above it coincides with the degree of polarization of singly scattered radiation and is determined by expression (87).

7. CONCLUSION

Summarizing the results of the above calculations, we paint a qualitative picture of the dependence of the degree of polarization of multiply scattered radiation on the medium parameters.

The behavior of the degree of polarization P near the bulge in the angular distribution of the radiation can conveniently be classified using the diagram given in Fig. 3. The diagram indicates the regions in which the function

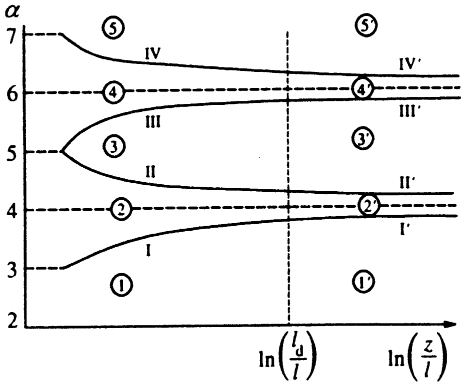

FIG. 3. Diagram for the quantity $p(\alpha,z)=|P|/\theta^2$ as a function of z and α. Traces I–IV describe the following equations: for $z<l_d$, I) $\alpha=4-1/\ln(z/l)$, II) $\alpha=4+1/\ln(z/l)$, III) $\alpha=6-1/\ln(z/l)$, IV) $\alpha=6+1/\ln(z/l)$; for $z>l_d$, I') $\alpha=4-1/\ln(l_d/l)$, II') $\alpha=4+1/\ln(l_d/l)$, III') $\alpha=6-1/\ln(l_d/l)$, IV') $\alpha=6+1/\ln(l_d/l)$.

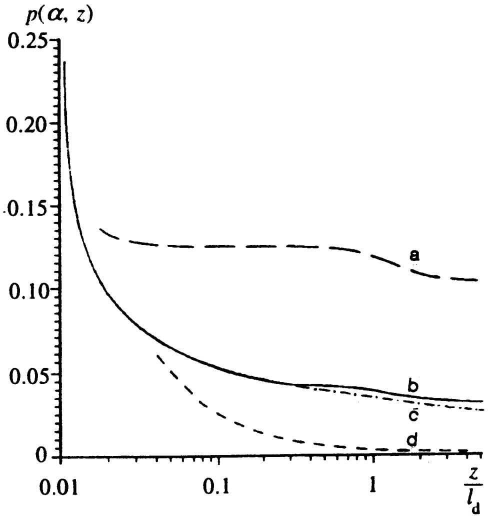

FIG. 4. Plot of the function $p(\alpha,z)=|P|/\theta^2$ as a function of the depth z for different values of α ($\sigma_a\gamma_0^2/\sigma=3\cdot10^{-5}$): a) $\alpha=3$; b) $\alpha=11/3$; c) $\alpha=4$; d) $\alpha=7$.

$p(\alpha,z)=|P|/\theta^2$ displays different behavior, reflecting the way P changes as a function of the depth z and the decay index α of the phase function. Regions *1* and *1'* in the diagram correspond to the range $2<\alpha<4$, with (region *1'*) and without (region *1*) absorption effects. The degree of polarization in regions *1* and *1'* is determined by Eqs. (61) and (76). Other values of α correspond to regions *2*, *2'* and *3*, *3'*, etc. The degree of polarization P in regions *2* and *2'* is determined by expressions (59) and (74) respectively; that in regions *3* and *3'* by Eqs. (57), (58), and (71); that in regions *4* and *4'* by Eqs. (60) and (68); and that in regions *5* and *5'* by Eqs. (56) and (65).

Using the diagram in Fig. 3 we can readily see how different values of α change the dependence of P on the depth z. For example, the behavior of the degree of polarization in a turbulent ($\alpha=11/3$) absorbing medium has the following properties. At depths $z\leqslant(20-30)l$ the quantity $p(\alpha,z)$ falls off as $p(\alpha=11/3,z)\propto[\ln \sigma z]^{-1}$. As z increases $[z\geqslant(20-30)l]$ the function $p(\alpha=11/3,z)$ goes over to a plateau $p\approx0.04$. For $z\geqslant l_a^{6/11}l^{5/11}$ absorption effects begin to pull apart and the value of $p(\alpha=11/3,z)$ decreases. In the deep regime p approaches its minimum value $p_\infty\approx0.03$.

Figure 4 shows $p(\alpha,z)$ as a function of depth for different values of α. As can be seen from the figure, the degree of polarization falls off monotonically as z increases. As α decreases the absolute value $|P|$ grows (for a fixed value of the parameter $\sigma_a\gamma_0^2/\sigma$), and the depth dependence $P(z)$ becomes more gradual.

As the size of the irregularities in the medium increases and the deviations from the Born approximation grow, the behavior of the degree of polarization can change drastically. As shown above for the case of a medium consisting of refracting particles of a single radius, the degree of polarization P for multiple scattering of light as a function of depth becomes more complicated. Nevertheless, P retains features inherent in single scattering. For relatively small values of z the degree of polarization is positive; then it changes sign and falls off approximately as in the case of Born scattering.

Thus, the analysis of the degree of polarization of scattered radiation can be an additional means (along with analysis of the angular spectrum) for studying structural variation in multiple-scattering random media.

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APPENDIX A

To prove the identity (12) we make a change of variables $\boldsymbol{\theta}=\boldsymbol{\theta}'+\boldsymbol{\gamma}$ in the corresponding term in Eq. (7), taking into account the obvious identity

$$\theta^2(1-\cos 2\psi)\equiv\frac{2}{\theta'^2}\,\theta_\alpha\theta'_\beta(\theta_\alpha\theta'_\beta-\theta_\beta\theta'_\alpha), \tag{A1}$$

where α, $\beta=x,y$. As a result we find for the left-hand side of (12)

$$\int d\boldsymbol{\theta}'Q(z,\theta')\int d\boldsymbol{\gamma}\chi(\gamma)\left\{\frac{2}{\theta'^2}(\theta'_\alpha+\gamma_\alpha)\theta'_\beta[(\theta'_\alpha+\gamma_\alpha)\theta'_\beta-(\theta'_\beta+\gamma_\beta)\theta'_\alpha]-\gamma^2\right\}. \tag{A2}$$

Then using the obvious relations

$$\int d\boldsymbol{\gamma}\chi(\gamma)\gamma_\alpha=0,$$

$$\int d\boldsymbol{\gamma}\chi(\gamma)\gamma_\alpha\gamma_\beta=\frac{\delta_{\alpha\beta}}{2}\int d\boldsymbol{\gamma}\chi(\gamma)\gamma^2,$$

we find that (A2) vanishes identically.

When the conditions for the Born approximation (2) do not hold the proof of the identity (12) is unchanged. Now, however, instead of $\chi(\gamma)$ we must use $\Pi(\gamma)$.

APPENDIX B

Let us consider the evaluation of the right-hand side of Eq. (14). The corresponding term in Eq. (13) can be found analytically.

From Eq. (7) it follows that the desired quantity $\tilde{\Phi}(z,\theta)\equiv d_{21}\tilde{Q}$ can be written

$$\tilde{\Phi}(z,\theta)=\tilde{\Phi}_1(z,\theta)+\tilde{\Phi}_2(z,\theta), \tag{B1}$$

where

$$\tilde{\Phi}_1(z,\theta)=-\sigma\int_0^\infty \omega d\omega \Pi(\omega)\int_0^\infty \theta'^3 d\theta'[J_0(\omega\theta)J_0(\omega\theta') - J_2(\omega\theta)J_2(\omega\theta')]\tilde{I}(z,\theta'), \tag{B2}$$

$$\tilde{\Phi}_2(z,\theta)=-\sigma\int_0^\infty \frac{\omega d\omega}{2\pi}\Delta_\omega \Pi(\omega)J_0(\omega\theta)\tilde{I}(z,\omega). \tag{B3}$$

Using the recurrence relations for the Bessel function,[35] we transform (B2) into

$$\tilde{\Phi}_1(z,\theta)=\sigma\left[\int_0^\infty \frac{\omega d\omega}{2\pi}J_0(\omega\theta)\Pi(\omega)\Delta_\omega\tilde{I}(z,\omega) + \int_0^\infty \frac{\omega d\omega}{2\pi}J_2(\omega\theta)\Pi(\omega)\left\{\frac{\partial^2}{\partial\omega^2} - \frac{1}{\omega}\frac{\partial}{\partial\omega}\right\}\tilde{I}(z,\omega)\right], \tag{B4}$$

where

$$\tilde{I}(z,\omega)=2\pi\int_0^\infty \theta d\theta J_0(\omega\theta)\tilde{I}(z,\theta).$$

Substituting (B3) and (B4) into (B1) for

$$\tilde{\Phi}(z,\omega)=2\pi\int_0^\infty \theta d\theta J_2(\omega\theta)\tilde{\Phi}(z,\theta)$$

we find

$$\tilde{\Phi}(z,\theta)=-\sigma\left[\frac{2}{\omega^2}\int_0^\omega \frac{\omega' d\omega'}{2\pi}\{\tilde{I}(z,\omega')\Delta_{\omega'}\Pi(\omega') - \Pi(\omega')\Delta_{\omega'}\tilde{I}(z,\omega')\}+\Pi(\omega)\Delta_\omega\tilde{I}(z,\omega) - \tilde{I}(z,\omega)\Delta_\omega\Pi(\omega)-\Pi(\omega)\left\{\frac{\partial^2}{\partial\omega^2} - \frac{1}{\omega}\frac{\partial}{\partial\omega}\right\}\tilde{I}(z,\omega)\right]. \tag{B5}$$

To derive (B5) we use the identity[45]

$$\int_0^\infty J_n(bt)J_{n+1}(at)dt=\begin{cases}0, & a<b\\ \dfrac{b^n}{a^{n+1}}, & a>b\end{cases}. \tag{B6}$$

Using elementary transformations in (B5) we find the right-hand side of Eq. (14).

[1)]In contrast to the scalar transport theory, where it is always enough to retain only the first three terms in the expansion (17) in the analysis of the asymptotic form of the intensity at angles $\theta > \gamma_0$ (Ref. 8).

[2)]Equations (13) and (14) are unsuited for calculating the Stokes parameters of back-scattered light. However, the intensity of these photons is low.

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