Bound states of spin 1/2 particles in the field of rotating cosmic string

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(Submitted 5 August 1994)
Zh. Eksp. Teor. Fiz. **107**, 273–282 (February 1995)

The spectrum of bound states of a Dirac spin 1/2 field near a rotating gravitational cosmic string in four-dimensional spacetime has been investigated. A model of a finite-thickness string is considered. The spectrum of bound states of a spin 1/2 particle in the field of a thin string has been calculated under quite general assumptions about the internal string structure. The spectrum is demonstrated to be independent of the internal string structure.

1. INTRODUCTION

A cosmic string is a set of singularities of physical fields concentrated on a line. The existence of cosmic strings is usually linked to some of the topological properties of physical fields. It is generally believed that cosmic strings were generated in the early stages of the development of the Universe. Gravitational cosmic strings have been studied extensively in recent years.[1,2] They are generated by gravitating matter concentrated on a line. In the general case, the matter constituting string may have angular momentum, in addition to the mass. Such strings are called rotating cosmic strings.[1–4]

2. INFINITELY THIN COSMIC STRING

Let us start with an infinitely thin string. The metric of an infinitely thin rotating cosmic string can be written as follows:[1,2,4]

$$ds^2=(dt+4GJ_t d\phi)^2-dr^2-(1-4GM)^2r^2d\phi^2-(dz+4GJ_z d\phi)^2, \tag{1}$$

where M is the string mass per unit length. If the boost is in the (t,z) plane, the quantities (J_t,J_z) transform as a two-dimensional vector. Usually strings in which this vector is timelike are considered. For such a string, there is a preferred reference frame in which $J_z=0$. In this case the string metric can be written as

$$ds^2=(dt+4GJd\phi)^2-dr^2-(1-4GM)^2r^2d\phi^2-dz^2. \tag{2}$$

Now J is simply the rotational angular momentum of the string per unit of its length.

In order to investigate the bispinor field in the metric of Eq. (2), let us select a tetradic basis in this metric

$$E_a^\mu=\begin{pmatrix} 1 & & & \\ & 1 & & \\ -\dfrac{4GJ}{(1-4GM)r} & & \dfrac{1}{(1-4GM)r} & \\ & & & 1 \end{pmatrix}. \tag{3}$$

This basis is orthonormal:

$$g_{\mu\nu}E_a^\mu E_b^\nu=\eta_{ab}, \quad \eta_{ab}=\mathrm{diag}(1,-1,-1,-1).$$

Then the equation of a spin 1/2 field can be written in the form

$$(i\hat{\gamma}^a E_a^\mu D_\mu-\mu)\psi=0, \tag{4}$$

where μ is the particle mass, $\hat{\gamma}^a$ are Dirac matrices

$$\{\hat{\gamma}^a,\hat{\gamma}^b\}=2\eta^{ab}, \quad \hat{\gamma}^{0+}=\hat{\gamma}^0, \quad \hat{\gamma}^{k+}=-\hat{\gamma}^k, \tag{5}$$

and D_μ is the covariant derivative in the metric of Eq. (2):

$$D_\mu\equiv\partial_\mu+\tfrac{1}{2}\omega_{\mu;ab}\hat{\sigma}^{ab}=\partial_\mu+\tfrac{1}{4}\omega_{\mu;ab}\hat{\gamma}^a\hat{\gamma}^b. \tag{6}$$

The coefficients which determine the torsion of the basis are

$$\omega_{\mu;ab}=E_a^\nu\eta_{bc}\nabla_\mu e_\nu^c, \tag{7}$$

where e_μ^a is the basis dual to $E_a^\mu, e_\mu^a\equiv g_{\mu\nu}\eta^{ab}E_b^\nu$. The only nonzero components of the tensor $\omega_{\mu;ab}$ are

$$\omega_{2;12}=-\omega_{2;21}=1-4GM. \tag{8}$$

Because the metric of Eq. (2) and the basis (3) are invariant under a displacement along the t and z axes and under rotation about the z-axis, the solutions of Eq. (4) can be expanded in terms of eigenfunctions of the corresponding operators

$$\psi_{E,p_z,m}(x^\mu)=e^{-iEt+ip_z+im\phi}\psi_{E,p_z,m}(r). \tag{9}$$

A rotation through the angle 2π changes the sign of the wave function ψ, so m should be half-integral:

$$m-\tfrac{1}{2}\in Z. \tag{10}$$

Thus m is the projection of the total angular momentum onto the z-axis. The equation for the radial function in (9) is

$$\left[E\hat{\gamma}^0+i\hat{\gamma}^1\frac{d}{dr}-\frac{m+4GJE}{(1-4GM)r}\hat{\gamma}^2-p_z\hat{\gamma}^3+\frac{i}{2r}\hat{\gamma}^1-\mu\right]\psi=0. \tag{11}$$

We take the Dirac matrices in the form

$$\hat{\gamma}^0=\begin{pmatrix}\hat{\sigma}_0 & \\ & -\hat{\sigma}_0\end{pmatrix}, \quad \hat{\gamma}^k=\begin{pmatrix} & -\hat{\sigma}_k \\ \hat{\sigma}_k & \end{pmatrix}. \tag{12}$$

As a result, we have the wave functions of the free state

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$$\begin{pmatrix} \sqrt{|E_\perp+\mu|}J_{\epsilon(m_e-1/2)}(kr) \\ -i\epsilon\dfrac{p_z}{E+E_\perp}\sqrt{|E_\perp-\mu|}J_{\epsilon(m_e+1/2)}(kr) \\ \dfrac{p_z}{E+E_\perp}\sqrt{|E_\perp+\mu|}J_{\epsilon(m_e-1/2)}(kr) \\ i\epsilon\sqrt{|E_\perp-\mu|}J_{\epsilon(m_e+1/2)}(kr) \end{pmatrix},$$

$$\begin{pmatrix} -i\epsilon\dfrac{p_z}{E+E_\perp}\sqrt{|E_\perp-\mu|}J_{\epsilon(m_e-1/2)}(kr) \\ \sqrt{|E_\perp+\mu|}J_{\epsilon(m_e+1/2)}(kr) \\ -i\epsilon\sqrt{|E_\perp-\mu|}J_{\epsilon(m_e-1/2)}(kr) \\ -\dfrac{p_z}{E+E_\perp}\sqrt{|E_\perp+\mu|}J_{\epsilon(m_e+1/2)}(kr) \end{pmatrix}, \tag{13}$$

where

$$E_\perp^2=E^2-p_z^2,\quad k^2=E_\perp^2-\mu^2,\quad |E_\perp|>\mu, \tag{14}$$

$$m_e=\frac{m+4GJE}{1-4GM}, \tag{15}$$

and ϵ may assume two values, $+1$ or -1. Equation (13) includes two column vectors corresponding to two string states. If m_e is an integer, a similar solution is possible and can be expressed in terms of Neumann functions Y_μ, but it does not satisfy the quadratic integrability condition.

The wave modes (13) found by solving Eq. (11) should satisfy the quadratic integrability condition. When $|m_e|\geqslant 1/2$ holds, this condition unambiguously determines the sign of ϵ

$$\epsilon=\operatorname{sign}(m_e).$$

But if we have $|m_e|<1/2$, all modes in (13) are quadratically integrable, regardless of the sign of ϵ, with the implication that the basis consisting of the of functions (13) is linearly dependent. However, this is not the only difficulty. Let us rewrite the Dirac equation (4)

$$\frac{\partial}{\partial t}\psi=-i\hat{H}\psi, \tag{16}$$

by explicitly separating the time derivative and the Hamiltonian. In the space of the derived solutions, the Hamiltonian is not symmetric

$$(v,\hat{H}u)\neq(\hat{H}v,u). \tag{17}$$

The following device is used[6–9] in order to bypass this difficulty. First we can limit the set of admissible functions by the condition $\psi(0)=0$ and make the Hamiltonian symmetric (Hermitian). Then all solutions with $|m_e|<1/2$ are excluded and the Hamiltonian becomes nonself-adjoint, i.e. the domain of $\hat{H}^+$ is wider than that of $\hat{H}$. The next step is to construct a self-adjoint extension of the Hamiltonian.

However, de Sousa Gerbert and Jakiw[3], who studied a similar problem in three-dimensional spacetime, demonstrated that this technique cannot be applied to rotating cosmic strings because the quantity which serves as a time-invariant inner product of two functions is negative for the domain $r<4GJ/(1-4GM)$, in which the causality principle is violated. Still, the same authors proved that it is sufficient to impose some linear constraints on the Hamiltonian domain to make it self-adjoint. As concerns the problem discussed, these constraints are as follows: the admissible solutions of Eq. (4) are

$$\cos\gamma_\uparrow\begin{pmatrix} \sqrt{|E_\perp+\mu|}J_{m_e-1/2}(kr) \\ -i\dfrac{p_z}{E+E_\perp}\sqrt{|E_\perp-\mu|}J_{m_e+1/2}(kr) \\ \dfrac{p_z}{E+E_\perp}\sqrt{|E_\perp+\mu|}J_{m_e-1/2}(kr) \\ i\sqrt{|E_\perp-\mu|}J_{m_e+1/2}(kr) \end{pmatrix} +\sin\gamma_\uparrow\begin{pmatrix} \sqrt{|E_\perp+\mu|}J_{-(m_e-1/2)}(kr) \\ i\epsilon\dfrac{p_z}{E+E_\perp}\sqrt{|E_\perp-\mu|}J_{-(m_e+1/2)}(kr) \\ \dfrac{p_z}{E+E_\perp}\sqrt{|E_\perp+\mu|}J_{-(m_e-1/2)}(kr) \\ -i\epsilon\sqrt{|E_\perp-\mu|}J_{-(m_e+1/2)}(kr) \end{pmatrix},$$

$$\cos\gamma_\downarrow\begin{pmatrix} -i\dfrac{p_z}{E+E_\perp}\sqrt{|E_\perp-\mu|}J_{m_e-1/2}(kr) \\ \sqrt{|E_\perp+\mu|}J_{m_e+1/2}(kr) \\ -i\sqrt{|E_\perp-\mu|}J_{m_e-1/2}(kr) \\ -\dfrac{p_z}{E+E_\perp}\sqrt{|E_\perp+\mu|}J_{m_e+1/2}(kr) \end{pmatrix} +\sin\gamma_\downarrow\begin{pmatrix} i\dfrac{p_z}{E+E_\perp}\sqrt{|E_\perp-\mu|}J_{-(m_e-1/2)}(kr) \\ \sqrt{|E_\perp+\mu|}J_{-(m_e+1/2)}(kr) \\ i\sqrt{|E_\perp-\mu|}J_{-(m_e-1/2)}(kr) \\ -\dfrac{p_z}{E+E_\perp}\sqrt{|E_\perp+\mu|}J_{-(m_e+1/2)}(kr) \end{pmatrix}. \tag{18}$$

Here $\gamma_\uparrow$ and $\gamma_\downarrow$ are constants which characterize the internal properties of the string. These constants in fact determine the boundary conditions on the wave functions as $r\to 0$.

Solving Eq. (11) for $|E|<\mu$, we can also obtain the bound-state wave functions

$$\begin{pmatrix} \sqrt{\mu+E_\perp}K_{m_e-1/2}(kr) \\ -i\dfrac{p_z}{E+E_\perp}\sqrt{\mu-E_\perp}K_{m_e+1/2}(kr) \\ \dfrac{p_z}{E+E_\perp}\sqrt{\mu+E_\perp}K_{m_e-1/2}(kr) \\ i\sqrt{\mu-E_\perp}K_{m_e+1/2}(kr) \end{pmatrix},$$

$$\begin{pmatrix} -i\dfrac{p_z}{E+E_\perp}\sqrt{\mu-E_\perp}K_{m_e-1/2}(kr) \\ -\sqrt{\mu+E_\perp}K_{m_e+1/2}(kr) \\ -i\sqrt{\mu-E_\perp}K_{m_e-1/2}(kr) \\ \dfrac{p_z}{E+E_\perp}\sqrt{\mu+E_\perp}K_{m_e+1/2}(kr) \end{pmatrix}, \tag{19}$$

where

$$E_\perp^2=E^2-p_z^2,\quad k^2=\mu^2-E_\perp^2,\quad |E_\perp|<\mu, \tag{20}$$

and m_e is determined by Eq. (15), as previously. The boundary conditions at $r\to 0$ should be now applied to the solutions of Eq. (11). An additional difficulty in this case is that both m_e and the order of Bessel functions are functions of energy. These difficulties can be overcome by making additional assumptions about the energy dependence of the boundary conditions.[3] One way or another, these conditions prohibit existence of bound states, except at some discrete energy values.

3. NON-ZERO-THICKNESS STRINGS

It has been shown above that we are forced to introduce some additional constants into the theory when studying the behavior of particles near an infinitely thin cosmic string. These constants are functions of the internal string structure. They cannot be determined *a priori* and have to be found by considering realistic models of finite-thickness cosmic strings. Similar models of nonrotating strings have been studied elsewhere.[12,13]

Any attempts to construct a rotating cosmic string generated by real gravitating matter run into difficulties because causality is violated near the string.[14,15] These difficulties can be overcome in the following way. It is known that parallel cosmic strings do not interact. The spacetime metric generated by several strings is given by the equation[2,4]

$$ds^2=\eta_{AB}\omega^A\omega^B-\delta_{ab}\omega^a\omega^b,\quad A=0,3,\quad a=1,2,$$

$$\omega^a=\prod_{i=1}^{N}|\mathbf{r}-\mathbf{r}_i|^{-4GM_i}dx^a,$$

$$\omega^A=dx^A+4G\sum_{i=1}^{N}J_i^A\,\frac{(x-x_i)dy-(y-y_i)dx}{(\mathbf{r}-\mathbf{r}_i)^2}, \tag{21}$$

where M_i and J_i^A are the linear mass and angular momentum densities of the strings. In the exterior this metric is identical to that of a single rotating string (1) with total angular momentum and spin $M=\Sigma_{i=1}^N M_i$ and $J^A=\Sigma_{i=1}^N J_i^A$.

We can position an infinite number of parallel strings with zero thickness and infinitesimal parameters M_i and J_i^A to obtain a completely static configuration which is equivalent to a finite-thickness cosmic string. Assuming that the string is axially symmetric and $J_i^3=0$ holds (the string components are fixed with respect to one another), we can write the string metric as

$$ds^2=(dt+S(r)d\phi)^2-dr^2-a(r)^2d\phi^2-dz^2. \tag{22}$$

Here $S(r)$ represents the angular momentum distribution and $a(r)$ the string mass distribution. In the limit $r\to 0$ we have $S(r)\to 0$ and $a(r)\sim r$, which corresponds to the metric of free space. At infinity the metric corresponds to that of an infinitely thin rotating string (2). Assume that beyond some radius R these two metrics coincide. Then for $r>R$ we have

$$S(r)=4GJ,\quad a(r)=(1-4GM)r+\text{const}.$$

Thus R can be treated as the radius of the string core. We assume that the string thickness is small in comparison with the typical radius determined by the angular momentum ($R\ll|4GJ|$), as well as with the particle mass and energy ($\mu R\ll 1, ER\ll 1$). Moreover, the natural conditions in this problem are that the density of the string mass distribution is nonnegative throughout space and the sign of the string angular momentum density is identical to that of the total string momentum. In this case $da(r)/dr$ should drop monotonically from 1 at $r=0$ to $1-4GM$ at $r=R$, and $S(r)$ should rise monotonically in this region from 0 to $4GJ$.

Let us select an orthonormal tetradic basis in the metric of (22) similar to that in Eq. (3), i.e.,

$$E_a^\mu=\begin{pmatrix}1&&&\\&1&&\\-\frac{S}{a}&&\frac{1}{a}&\\&&&1\end{pmatrix}. \tag{23}$$

The equation for the bispinor field in this case is similar to (4), but the coefficients which determine the basis torsion are different:

$$\omega_{0;12}=-\omega_{0;21}=-\frac{S'}{2a},\quad \omega_{1;02}=-\omega_{1;20}=-\frac{S'}{2a}$$

$$\omega_{2;01}=-\omega_{2;10}=-\frac{S'}{2},\quad \omega_{2;12}=-\omega_{2;21}=a'-\frac{SS'}{2a}. \tag{24}$$

The other components of $\omega_{\mu;ab}$ are equal to zero. The equation for the radial function (11) should be rewritten in the form

$$\left[E\hat\gamma^0+i\hat\gamma^1\frac{d}{dr}-\frac{m+SE}{a}\hat\gamma^2-p_z\hat\gamma^3+\frac{iS'}{4a}\hat\gamma^0\hat\gamma^1\hat\gamma^2+\frac{ia'}{2a}\hat\gamma^1-\mu\right]\psi=0. \tag{25}$$

Now let us investigate bound states of particles in this metric. The general problem for an arbitrary metric (22) is rather difficult. Before investigating its general form, let us first consider a simplified string model. Let

$$a(r)=\begin{cases}r, & r<R,\\ (1-4GM)r+4GMR, & r>R,\end{cases}$$

$$S(r)=\begin{cases}0, & r<R,\\ 4GJ, & r>R.\end{cases} \tag{26}$$

This model corresponds to the configuration in which the total mass and angular momentum are concentrated on the cylinder surface $r=R$. Without rotation this metric would correspond to the "flower-pot" model.[12,13] For $r<R$ space is absolutely flat. In this region the radial part of a bound-state wave function is expressed in terms of the Bessel function of imaginary argument I_ν,

$$\begin{pmatrix} \sqrt{\mu+E_\perp} I_{|m-1/2|}(kr) \\ i\,\dfrac{p_z}{E+E_\perp}\sqrt{\mu-E_\perp} I_{|m+1/2|}(kr) \\ \dfrac{p_z}{E+E_\perp}\sqrt{\mu+E_\perp} I_{|m-1/2|}(kr) \\ -i\sqrt{\mu-E_\perp} I_{|m+1/2|}(kr) \end{pmatrix},$$

$$\begin{pmatrix} -i\,\dfrac{p_z}{E+E_\perp}\sqrt{\mu-E_\perp} I_{|m-1/2|}(kr) \\ \sqrt{\mu+E_\perp} I_{|m+1/2|}(kr) \\ -i\sqrt{\mu-E_\perp} I_{|m-1/2|}(kr) \\ -\dfrac{p_z}{E+E_\perp}\sqrt{\mu+E_\perp} I_{|m+1/2|}(kr) \end{pmatrix}, \tag{27}$$

where $k^2=\mu^2-E_\perp^2$. It follows from Eq. (25) that when passing through the point $r=R$ the wave function changes as follows:

$$\psi(R+0)=\exp\left[\frac{S(R+0)-S(R-0)}{4R}\,\hat{\gamma}^0\hat{\gamma}^2\right]\psi(R-0)$$

$$=\exp\left[\frac{4GJ}{4R}\,\hat{\gamma}^0\hat{\gamma}^2\right]\psi(R-0). \tag{28}$$

Assume $J>0$. Since the string radius is assumed to be small, $R\ll 4GJ$, the factor in the exponent of (28) is much greater than unity. Therefore at $r=R+0$ we have

$$\psi_4\approx i\psi_1, \quad \psi_3\approx -i\psi_2, \quad \psi=\begin{pmatrix}\psi_1\\ \psi_2\\ \psi_3\\ \psi_4\end{pmatrix}, \tag{29}$$

and Eq. (29) is accurate, neglecting $\exp(-R/GJ)$. Considering that $R\ll 4GJ$ holds, this is quite good accuracy. Outside the string core ($r>R$), the space metric coincides with the metric of (2). Therefore the wave function of (29) should be smoothly joined to the wave functions of bound states in Eq. (19). Taking into account that $R\ll 4GJ$ and using the asymptotic forms of the modified Bessel functions of the second kind, we obtain

$$m_e\approx 0. \tag{30}$$

By virtue of Eq. (15), this equation uniquely determines the energy

$$E\approx -\frac{m}{4GJ}. \tag{31}$$

Moreover, only one of the two modes defined by (19) can exist. At $p_z=0$ this is the mode with the spin projection $\sigma_3=-1$. If the string angular momentum has the opposite sign ($J<0$), the result is similar, namely, the energy is determined by the same equation (31), and only one of the two modes of (19) is realized. At $p_z=0$ this is the mode with the spin projection $\sigma_3=+1$. In addition to (31) of course, the energy must lie between $-\mu$ and $+\mu$.

Now let us consider the general string model. The radial function $\psi(r)$ is described by Eq. (25). We rewrite it in the form

$$\frac{d}{dr}\,\psi=\left[iE\hat{\gamma}^0\hat{\gamma}^1+\frac{i}{a}\,\hat{\gamma}^1\hat{\gamma}^3(m+SE)+ip_z\hat{\gamma}^1\hat{\gamma}^3 + \frac{S'}{4a}\,\hat{\gamma}^0\hat{\gamma}^2-\frac{a'}{2a}+i\mu\hat{\gamma}^1\right]\psi. \tag{32}$$

In the limit $r\to 0$, $\psi(r)$ scales as a power. By integrating Eq. (32) over the interval from 0 to R, we can find $\psi(r)$ at $r=R$. It must be a linear combination of the bound-state wave functions in (19). If the integration interval is extended to infinity, the wave function should vanish exponentially.

We may omit the term $a'/2a$ from Eq. (32) because it yields a common coefficient (which is approximately a power of r) to the wave function. We will also temporarily neglect terms with E, p_z, and μ because their absolute values are bounded over the entire region. Then we arrive at

$$\frac{d}{dr}\,\psi=\left[\frac{i}{a}\,\hat{\gamma}^1\hat{\gamma}^2(m+SE)+\frac{S'}{4a}\,\hat{\gamma}^0\hat{\gamma}^2\right]\psi. \tag{33}$$

This equation can be split into two independent ones for the two pairs of components. For one pair of components it can be rewritten (after substituting ψ_4 for $i\psi_4$) as

$$\frac{d}{dr}\begin{pmatrix}\psi_1\\ \psi_4\end{pmatrix}=\left[\frac{m+SE}{a}\begin{pmatrix}-1 & \\ & 1\end{pmatrix}+\frac{S'}{4a}\begin{pmatrix} & 1\\ 1 & \end{pmatrix}\right]\begin{pmatrix}\psi_1\\ \psi_4\end{pmatrix}, \tag{34}$$

and in a similar form for the other pair of components. We can reduce the case $J<0$ to that with $J>0$ by swapping the components ψ_1 and ψ_4, so we assume $J>0$ in what follows. Similarly, substituting ψ_4 with $-\psi_4$, we can reduce the problem to the case in which $m>0$. Now we will separately consider two options. First assume that $m_e\equiv(m+S(R)E)/(1-4GM)>0$. Then using Eq. (34) we can prove that in all cases $0\leqslant\psi_1/\psi_4\leqslant 1$. Then, in the domain $r>R$ the wave function diverges and does not represent a bound state. The case $m+S(R)E<0$ is more complicated. Let us calculate the value of R_1 at which $m+S(R_1)E=0$. Then for $r<R_1$ we have $m+SE>0$ and in all cases $0\leqslant\psi_1/\psi_4\leqslant 1$. Besides, unless the entire string angular momentum is concentrated near $r=0$, $\psi_1(R_1)/\psi_4(R_1)$ is of an order of unity, i.e. it does not tend to zero at small string radius $R\to 0$. In this case the condition $m+SE<0$ is satisfied in the domain $r>R_1$, and the ratio ψ_1/ψ_4 increases monotonically, whereas the opposite is required. Thus we can see that for $m_e\equiv(m+S(R)E)/(1-4GM)\neq 0$ the wave function does not represent a bound state. A bound state is thus possible only in the transitional region $m_e\approx 0$.

Finally, we will consider the terms with E, p_z, and μ in Eq. (32). Their total contribution is bounded by a constant, which we denote as C. Then we can prove that for $m_e>0$ and for $m_e<0$ and $r<R_1$ the relative error due to these terms is less than $2Cr\ll 1$. If $m_e<0$ and $R_1<r<R$, this error can be estimated [using the condition $\psi_1(R_1)/\psi_4(R_1)\sim 1$ and a similar one for the other pair of components] to be of the same order of magnitude. Hence the contribution these terms make to the wave function for $r<R$ is negligible. The spectrum of bound states is determined by smoothly joining the exterior wave functions $\psi(R)$ to the bound states given by

Eq. (19) in the interior of the $r=(R)$ surface. The resulting spectrum is therefore identical to that derived from the simplified equation (26).

We can also estimate the accuracy of Eq. (31) for the energy spectrum. In the simplified case (26) the error estimate is obtained by joining smoothly the functions from Eq. (29) to those in Eq. (19):

$$\Delta E \sim \frac{\ln \dfrac{\mu+E}{\mu-E}}{16GJ \ln \dfrac{1}{kR}}. \tag{35}$$

Thus the accuracy of Eq. (31) is rather poor (the error expression involves logarithms of the energy), especially at small binding energies $|E| \approx \mu$.

The author is grateful to D. V. Gal'tsov for helpful discussions and to M. I. Zel'nikov for help in preparing the manuscript.

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Translation was provided by the Russian Editorial office.