Landau oscillations in (2+1)-dimensional quantum electrodynamics

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The effective action of (2+1)-dimensional quantum electrodynamics is considered in the one-loop approximation at nonzero temperature, chemical potential, and external magnetic field. It is shown that, besides the Landau oscillations in the parameter $(m^2/eH)(\mu/m-1)$, the thermodynamic potential has an oscillatory dependence on the quantity $T/\sqrt{eH}$. © *1995 American Institute of Physics.*

1. INTRODUCTION

Investigations of a number of fundamental problems in strong-interaction physics, the construction of scenarios of the development of the early universe, and astrophysical problems dominated by the presence of strong magnetic fields, high temperatures and high matter densities require the preliminary solution and analysis of the largest possible number of models of the interaction of elementary particles and radiation, and of phase transitions which allow for the presence of an external field, nonzero temperature, and nonzero matter density. In recent years some of the stages in the solution of these problems have been successfully surmounted: The Weinberg–Salam–Glashow theory of the electroweak interaction and quantum chromodynamics has been constructed, the Higgs mechanism has been discovered, making it possible to give gauge fields mass while conserving the properties of renormalizability and unitarity, and it has been found that a spontaneously broken symmetry is restored at a high temperature. The possibility of using the Furry picture to take exact account of the influence of an external field and to go beyond the scope of perturbation theory has turned out to be very important, and has opened up new possibilities for the study of physical properties of the vacuum. For example, while studying the polarization operator and mass operator in an ultrastrong field, Ritus[1] investigated the behavior of the vacuum at short distances. In Ref. 2, Fradkin developed a general technique for investigating quantum field theory at nonzero temperatures and densities by means of temperature Green's functions, derived a system of functional equations for the Green's functions both in the theory of quantized fields and in quantum statistics, considered renormalization and ways of eliminating divergences, and developed a method for the operator solution and continuous solution of the equations obtained.

Despite the successes achieved in this field of investigation, there are a number of questions that have not yet been answered because of technical difficulties or difficulties of principle. An important step on the path to realizing this program is the derivation and analysis of explicit analytical expressions for the one-loop effective potentials of the systems under consideration at a nonzero temperature. This stage, which has been passed for most models in recent years, is fraught with particular mathematical difficulties associated with the treatment of external fields and the nonzero temperature and density of the matter. Overcoming these difficulties is an important task the urgency of which arises from the present state of quantum field theory at nonzero temperatures.

Above, we have given arguments in favor of the study and analysis of the thermodynamic potential of a relativistic electron gas in a magnetic field on general theoretical and cosmological grounds. However, there is one further aspect—a study[3] of Fermi surfaces that is based on the Lax model for an ellipsoidal nonparabolic model.[4] Although this is a very approximate model, in a number of cases it permits one to make various quantitative and qualitative predictions. This model has received considerable experimental support from observations of so-called magneto-optical oscillations.

On the other hand, the study of the thermodynamic potential of relativistic particles can be important in the investigation of subtle quantum effects in solids. Recently, for example, in connection with the discovery of high-temperature superconductivity, the need to study various versions of (2+1)-dimensional (three-dimensional) quantum field theories has arisen. The point is that this phenomenon can be described using only two spatial coordinates, since the conduction electrons in materials of the type La_2CuO_4 are concentrated in planes formed by Cu and O atoms.[5] As well as in high-temperature superconductivity, three-dimensional field theories can be used to describe the fractional quantum Hall effect,[6] and also to model other physical phenomena occurring in thin films. To describe the Hall effect and high-temperature superconductivity, such parameters as the external magnetic field H, temperature T, and chemical potential μ are usually used. As a consequence of this, the study of three-dimensional field theories in the presence of these external factors is an urgent problem. The thermodynamics of the three-dimensional nonlinear σ model[7] and the three-dimensional Gross–Neveu model[8] has already appeared in the literature. In addition, three-dimensional quantum electrodynamics (QED_3) at $H\neq 0$ and $\mu\neq 0$ has been considered in Refs. 9.

Taking into account what has been said, in this paper we calculate the thermodynamic potential in the one-loop approximation in QED_3 in the presence of an external magnetic field and nonzero temperature and chemical potential. We note that the problem of the construction of a general expres-

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sion for the thermodynamic potential in spectral form in quantum field theory for a relativistic electron gas in a magnetic field has been discussed previously in Ref. 10. In the present paper, for the first time in the relativistic case, we construct for the thermodynamic potential a representation that makes it possible to obtain corrections to the oscillations in the external field to any order in the parameter eH/m^2, demonstrate explicitly the existence of a new type of oscillations in the parameter $T/\sqrt{eH}$, investigate the high-temperature ($eH/2T^2 \ll 1$) behavior of the thermodynamic potential, obtain the Stefan-Boltzmann law in QED_3 and the corrections to it that are due to the magnetic field and chemical potential, and consider the situation describing the low-temperature limit ($eH/2T^2 \gg 1$).

2. QED_3 IN AN EXTERNAL MAGNETIC FIELD FOR $T=0$ AND $\mu=0$

We consider first (2+1)-dimensional quantum electrodynamics in an external magnetic field for nonzero values of the temperature and chemical potential. The Lagrangian of the model has the form

$$\mathscr{L}=-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}+\bar{\psi}(i\hat{\partial}-e\hat{A}-m)\psi. \tag{1}$$

We use the formalism in which the Dirac spinors ψ are four-component spinors, and $\gamma^\mu=\mathrm{diag}(\tilde{\gamma}^\mu, -\tilde{\gamma}^\mu)$, where $\tilde{\gamma}^0=\sigma_3$ and $\tilde{\gamma}^{1,2}=i\sigma_{1,2}$. Here σ_k are the Pauli matrices, and the field tensor $F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu$.

We shall place the initial model (1) in an external uniform constant magnetic field H. It is obvious that in the zeroth approximation the effective action of this field is

$$S_0=-VH^2/2, \tag{2}$$

where $V=\int d^3x$. Next we shall calculate the one-loop quantum correction to the action (2), which has the form

$$S_1(m,H)=-i\ \ln\det\Delta+C, \tag{3}$$

where $\Delta=i\hat{\partial}-e\hat{A}-m$, A_μ is the vector potential of the external magnetic field H, and we will choose the constant C (which is independent of H) so that the following condition is fulfilled:

$$S_1(m,H)=0 \quad \text{for } H=0. \tag{4}$$

The role of the constant C is to eliminate ultraviolet divergences from the expression (3). Without loss of generality, we can assume that A_μ has the form

$$A_0=A_1=0, \quad A_2=xH.$$

We differentiate (3) with respect to the mass of the spinor field:

$$\frac{\partial S_1(m,H)}{\partial m}=i\mathrm{Tr}\Delta^{-1}=i\int d^3x\sum_{\alpha=1}^{4} G_{\alpha\alpha}(x,x'). \tag{5}$$

For simplicity, in Eq. (5) we have omitted the term $\partial C/\partial m$, which will be restored later. To obtain (5), we used the relation $\det A=\exp(\mathrm{Tr}\ln A)$, and also introduced the causal Green's function of a fermion in an external magnetic field:

$$G_{\alpha\beta}(x,x')=\Delta^{-1}_{\alpha\beta}(x,x')$$

For it we can use the following representation:[11,12]

$$G_{\alpha\beta}(x,x')=-i\sum_{\{n\}}[\theta(x^0-x^{0\prime})\psi^{(+)}_{\alpha\{n\}}(x)\bar{\psi}^{(+)}_{\beta\{n\}}(x') - \theta(x^{0\prime}-x^0)\psi^{(-)}_{\alpha\{n\}}(x)\bar{\psi}^{(-)}_{\beta\{n\}}(x')], \tag{6}$$

where $\theta(x)$ is the Heaviside step function, and $\psi^{(\pm)}_{\{n\}}(x)$ are the positive- and negative-frequency orthonormalized solutions of the Dirac equation in the presence of an external field:

$$(i\hat{\partial}-e\hat{A}-m)\psi^{(\pm)}_{\{n\}}(x)=0. \tag{7}$$

In Eqs. (6) and (7) $\{n\}$ denotes the set of both the discrete and the continuous quantum numbers over which the summation and integration are performed in Eq. (6). It follows from (7) that these solutions can be represented in the form (the symbol T denotes the operation of transposition)

$$\psi^{(\pm)T}_{1nk}(t,x,y)=\frac{1}{\sqrt{2\pi}}\exp(\mp i\varepsilon_n t+iky) \times\left(\pm\sqrt{\frac{\varepsilon_n\pm m}{2\varepsilon_n}}I_{n-1}(\xi),\sqrt{\frac{\varepsilon_n\mp m}{2\varepsilon_n}}I_n(\xi),0,0\right),$$

$$\psi^{(\pm)T}_{2nk}(t,x,y)=\frac{1}{\sqrt{2\pi}}\exp(\mp i\varepsilon_n t+iky) \times\left(0,0,\pm\sqrt{\frac{\varepsilon_n\mp m}{2\varepsilon_n}}I_{n-1}(\xi), \sqrt{\frac{\varepsilon_n\pm m}{2\varepsilon_n}}I_n(\xi)\right). \tag{8}$$

Here $\varepsilon_n=\sqrt{m^2+2eHn}$, $n=0,1,2,\ldots$; the parameter k takes all real values, and $\xi=x+k/eH$. In addition, for $n\geqslant 0$,

$$I_n(\xi)=\frac{(eH/\pi)^{1/4}}{\sqrt{2^n n!}}\exp(-\xi^2 eH/2)H_n(\xi\sqrt{eH}),$$

where $H_n(x)$ are Hermite polynomials. These functions satisfy the normalization condition

$$\int dx I_n^2(\xi)=\int\frac{dk}{eH}I_n^2(\xi)=1. \tag{9}$$

In (8) it is also assumed that $I_{-1}=0$. If now we make use of the obvious relationship

$$\theta(\pm t)e^{\mp i\varepsilon t}f(\varepsilon)=\frac{\mp 1}{2\pi i}\int_{-\infty}^{\infty}\frac{d\omega e^{-i\omega t}f(\pm\omega)}{\omega\mp\varepsilon\pm i0}$$

it is possible, taking (6)–(8) into account, to write (5) in the following form:

$$\frac{\partial S_1(m,H)}{\partial m}=\frac{im}{2\pi^2}\int d^3x\sum_{n=0}^{\infty}\int d\omega\int dk\,\frac{I^2_{n-1}(\xi)+I^2_n(\xi)}{\omega^2-\varepsilon_n^2+i0}.$$

Integrating here over the variable k [see (9)], and going over to Euclidean space ($\omega\to i\omega$), we obtain

$$\frac{\partial S_1(m,H)}{\partial m}=\frac{VeHm}{2\pi^2}\int d\omega\sum_{n=0}^{\infty}\frac{\alpha_n}{\omega^2+\varepsilon_n^2}, \tag{10}$$

where V is given in (2), and $\alpha_n=2-\delta_{n0}$. In Sec. 3 this formula will serve as the basis for introducing the temperature and chemical potential. But here we shall continue the calculation of S_1. First we use in (10) the familiar equality

$$(\omega^2+\varepsilon_n^2)^{-1}=\int_0^{\infty}d\alpha\,\exp[-\alpha(\omega^2+\varepsilon_n^2)],$$

and, in the resulting expression, integrate over ω and sum over n. As a result, restoring the previously omitted term $\partial C/\partial m$, we have

$$\frac{\partial S_1(m,H)}{\partial m}=\frac{VeHm}{2\pi^{3/2}}\int_0^{\infty}\frac{d\alpha}{\sqrt{\alpha}}e^{-\alpha m^2}\coth(eH\alpha)+\frac{\partial C}{\partial m}. \tag{11}$$

Besides the condition (4) at $H=0$, of course, the derivative $\partial S_1/\partial m$ should also vanish. By virtue of this, it follows from (11) that

$$\frac{\partial C}{\partial m}=-\frac{Vm}{2\pi^{3/2}}\int_0^{\infty}\frac{d\alpha}{\alpha^{3/2}}\exp(-\alpha m^2). \tag{12}$$

Now, in order to obtain the expression for the one-loop correction to the effective action, it is necessary to integrate Eq. (11) over the fermion mass in the range (m,∞). Taking into account (12), and also the rather obvious physical requirement that $S_1(m,H)\to 0$ as $m\to\infty$, we have the final expression

$$S_1(m,H)=-\frac{V}{4\pi^{3/2}}\int_0^{\infty}\frac{d\alpha}{\alpha^{5/2}}\exp(-\alpha m^2)\times[eH\alpha\coth(eH\alpha)-1]. \tag{13}$$

We note that this can also be obtained by Schwinger's proper-time method.[13] It is clear that for S_1 from (13) the condition (4) is fulfilled.

3. ALLOWANCE FOR THE TEMPERATURE AND CHEMICAL POTENTIAL

In order to calculate $S_{1T\mu}(H)$, the one-loop correction to the effective action at nonzero temperature T and chemical potential μ, we shall use the method of summation over Matsubara frequencies. We shall assume that for $T=0$ and $\mu=0$ a physical quantity can be represented in the form of an integral over the Euclidean energy variable. Then to obtain the corresponding quantity for T, $\mu\neq 0$ it is sufficient to make the following transformation:

$$\int\frac{d\omega}{2\pi}(\cdots)\to\frac{1}{\beta}\sum_k(\cdots),\quad \omega\to(2k+1)\frac{\pi}{\beta}-i\mu, \tag{14}$$

where $\beta=1/T$ and $k=0,\pm1,\pm2,\ldots$. Now, making use of (10), by means of (14) we can write an equation for the one-loop correction to the effective action for T, $\mu\neq 0$:

$$\frac{\partial S_{1T\mu}(H)}{\partial m}=\frac{VeHm}{\pi\beta}\sum_{k=-\infty}^{\infty}\sum_{n=0}^{\infty}\frac{\alpha_n}{[(2k+1)\pi T-i\mu]^2+\varepsilon_n^2}. \tag{15}$$

Here, as in (10), for simplicity we have omitted the term $\partial C/\partial m$, which has the form (12). After straightforward algebraic transformations in (15) we can sum over k. For this it is necessary to invoke the relation[14]

$$\sum_{k=0}^{\infty}[(2k+1)^2+a^2]^{-1}=\frac{\pi}{4a}\tanh\left(\frac{\pi a}{2}\right).$$

As a result, we obtain

$$\frac{\partial S_{1T\mu}(H)}{\partial m}=\frac{VeHm}{2\pi}\sum_{n=0}^{\infty}\frac{\alpha_n}{\varepsilon_n}-F(T,H,\mu), \tag{16}$$

where

$$F(T,H,\mu)=\frac{VeHm}{2\pi}\sum_{n=0}^{\infty}\frac{\alpha_n}{\varepsilon_n}\{[1+\exp(-\beta(\varepsilon_n+\mu))]^{-1}+[1+\exp(-\beta(\varepsilon_n-\mu))]^{-1}\}.$$

Note that the first term in the right-hand side of (16), together with $\partial C/\partial m$, coincides with $\partial S_1/\partial m$. This is easy to show if we integrate over ω in (10). Consequently, after integration of both sides of Eq. (16) over the fermion mass in the range (m,∞), we obtain

$$S_{1T\mu}(H)=S_1(m,H)+\tilde{S}_1(T,\mu,H), \tag{17}$$

where $S_1(m,H)$ does not depend on T or μ and is represented in (13). Henceforth, we shall be interested only in the second term in (17):

$$\tilde{S}_1(T,\mu,H)=\frac{VeH}{2\pi\beta}\left[\Phi(0)+2\sum_{k=1}^{\infty}\Phi(k)\right], \tag{18}$$

where

$$\Phi(k)=\ln[1+\exp(-\beta(\varepsilon_x-\mu))]+\ln[1+\exp(-\beta(\varepsilon_k+\mu))]. \tag{19}$$

Using the Poisson summation formula[15]

$$\Phi(0)+2\sum_{k=1}^{\infty}\Phi(k)=2\int_0^{\infty}\Phi(x)dx+4\,\mathrm{Re}\sum_{k=1}^{\infty}\int_0^{\infty}\Phi(x)e^{2\pi ikx}dx, \tag{20}$$

we transform (18) to the form

$$\tilde{S}_1(T,\mu,H)=\tilde{S}_1(T,\mu)+W(T,\mu,H). \tag{21}$$

The first term here (apart from the sign) is the thermodynamic potential of the (2+1)-dimensional electron-positron ideal gas for $H=0$:

$$\tilde{S}_1(T,\mu)=\frac{VeH}{\pi\beta}\int_0^\infty \Phi(x)dx=\frac{V}{2\pi\beta}\int_0^\infty dx\ \ln\{[1+\exp(-\beta(\sqrt{m^2+x}+\mu))][1+\exp(-\beta(\sqrt{m^2+x}-\mu))]\}.$$

In correspondence with (19) and (20), the function W in (21) is a sum of two terms:

$$W(T,\mu,H)=W_+ + W_-,$$

where $W_-=W_+(\mu\to-\mu)$, and

$$W_+=\frac{2VeH}{\pi\beta}\sum_{k=1}^\infty\int_0^\infty dx\ \cos(2\pi kx)\ln[1+\exp(\beta\mu-\beta\sqrt{m^2+2eHx})]. \quad (22)$$

We first perform a Laplace transformation of the logarithmic function in (22) (Ref. 16):

$$\ln(1+e^z)=\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} ds\ \frac{\pi s}{s^2\sin(\pi s)}e^{sz},$$

where $0<\sigma<1$ and $z=\beta\mu-\beta\sqrt{m^2+2eHx}$, and then, in order to get rid of the square root in the exponent, make use of the integral representation[17]

$$\exp(-\beta s\sqrt{m^2+2eHx})=\sqrt{\frac{s\beta m}{2\pi}}\int_0^\infty \frac{dt}{\sqrt{t}}\exp\left[-\frac{s\beta m}{2}\left(t+\frac{m^2+2eHx}{m^2t}\right)\right].$$

After this, in (22) we can integrate over x, using the formula[14]

$$\int_0^\infty e^{-px}\cos(bx)dx=p/(b^2+p^2).$$

As a result we obtain

$$W_+=\frac{V(eH)^2}{i\beta\pi\sqrt{2\pi}}\int_{\sigma-i\infty}^{\sigma+i\infty} ds\ \frac{(s\beta m)^{3/2}\exp(\beta s\mu)}{s\sin(\pi s)}\times\sum_{k=1}^\infty\int_0^\infty dt\sqrt{t}\ \frac{\exp\left[-\frac{s\beta m}{2}\left(t+\frac{1}{t}\right)\right]}{(s\beta eH)^2+(2\pi ktm)^2}. \quad (23)$$

Up to now, we have not assumed any restrictions on T, μ, and H. We now assume that the temperature is low, i.e., that $\beta m\gg1$ and $\sigma\beta m\gg1$, where $\sigma=\mathrm{Re}\ s$. Consequently, to estimate the integral over the variable t in (23) we can apply the method of steepest descent, since the exponent contains the large parameter $s\beta m$. The saddle point in this case can be taken in the form $t_0=1$, and the function W_+ becomes equal to

$$W_+=\sum_{k=1}^\infty\frac{mV}{\beta^2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} ds\ \frac{\exp[\beta s(\mu-m)]}{\sin(\pi s)\left[s^2+\left(\frac{2\pi km}{\beta eH}\right)^2\right]}. \quad (24)$$

If $\mu<m$ holds, in the calculation of the integral over s in (24) the integration contour must be closed in the right half-plane of the complex variable s. In this case the integral will be equal to the sum of the residues of the integrand at the points $s_n=n$, with $n=1,2,\ldots$. It is obvious that for $\mu<m$ the function W_+ is exponentially small at low temperatures. For $\mu>m$ the integration contour in (24) must be closed in the left half-plane. In this case, for each fixed value of k the integrand in (24) has simple poles at the real points $s_n=-n$ ($n=0,1,2,\ldots$), and also at two points on the imaginary axis, $s_\pm=\pm i2\pi km/\beta eH$. The contributions from the residues at the real points s_n form the monotonic part of the thermodynamic potential (this is discussed in more detail in Sec. 5). If, on the other hand, we sum the residues at the points $s_\pm$, we can obtain the oscillatory part of the thermodynamic potential:

$$W_+^{\mathrm{osc}}=-\theta(\mu-m)\frac{VeH}{\pi\beta}\sum_{k=1}^\infty\frac{\cos\left(2\pi\frac{m^2}{eH}\left(\frac{\mu}{m}-1\right)k\right)}{k\sinh\left(\frac{2\pi^2mk}{\beta eH}\right)}. \quad (25)$$

For the function W_-, which is obtained from W_+ by the replacement $\mu\to-\mu$, we can again use the method of steepest descent to obtain an approximate expression for $T\to0$. Taking into account what has been said above, and the obvious relation $x/\sinh x\to1$ as $x\to0$, we have for W_+^{osc} from (25) as $\beta\to\infty$:

$$W_+^{\mathrm{osc}}(\mu,H)=-\theta(\mu-m)\frac{V(eH)^2}{2\pi^3m}\sum_{k=1}^\infty\frac{1}{k^2}\times\cos\left[2\pi\frac{m^2}{eH}\left(\frac{\mu}{m}-1\right)k\right]=-\theta(\mu-m)\frac{V(eH)^2}{2\pi m}B_2\left(\frac{m^2}{eH}\left(\frac{\mu}{m}-1\right)\right), \quad (26)$$

where $B_2(x)$ is a periodic function with period unity, which, for $x\in[0,1]$, is the ordinary second-order Bernoulli polynomial[17]

$$B_2(x)=x^2-x+\frac{1}{6},\quad x\in[0,1].$$

From this it follows that, for sufficiently low temperatures, the effective action of QED_3 (i.e., apart from the sign, the thermodynamic potential of the electron gas in an external magnetic field) in the one-loop approximation has an oscillatory dependence on the parameter $(m^2/eH)(\mu/m-1)$.

This is the usual way of obtaining the oscillations. However, the question of the accuracy of the approximate calculations and the question of whether physically significant terms are lost in such calculations remain open. In the nonrelativistic case this question was posed in the original paper of Landau,[18] while in the relativistic case this question is no less important, as indicated by the recent publication Ref. 19.

4. EXACT EXPRESSION FOR THE OSCILLATORY PART OF THE THERMODYNAMIC POTENTIAL OF A RELATIVISTIC ELECTRON GAS IN A MAGNETIC FIELD

To solve this problem we propose the following procedure for exact calculation of the thermodynamic potential W_+ of the electron component of the system. In Eq. (23), after expanding in a series the denominator of the fraction in the integrand over the variable t, and performing the integration over t, we obtain

$$W_+=\frac{V(eH)^2}{i\pi^{3/2}\beta}\int_{\sigma-i\infty}^{\sigma+i\infty}ds\,\frac{(sm\beta)^{3/2}e^{s\mu\beta}}{s\,\sin(\pi s)}\times\sum_{k=1}^{\infty}(s\beta eH)^{-2}\sum_{l=0}^{\infty}\times(-1)^l\left(\frac{2\pi mk}{s\beta eH}\right)^{2l}K_{2l+3/2}(sm\beta). \tag{27}$$

Next, for the modified Bessel function $K_{2l+3/2}(sm\beta)$, we make use of the representation[17]

$$K_{2l+3/2}(sm\beta)=\sqrt{\frac{\pi}{2sm\beta}}\,e^{-sm\beta}\times\sum_{r=0}^{2l+1}\frac{\Gamma[2(l+1)+r]}{\Gamma[2(l+1)-r]}\frac{(2sm\beta)^{-r}}{r!}. \tag{28}$$

From (27), using (28), we obtain the following expression for the thermodynamic potential of an ideal electron gas (without interaction) in a magnetic field:

$$W_+=\frac{Vm}{i\pi\beta^2}\int_{\sigma-i\infty}^{\sigma+i\infty}ds\,\frac{e^{s\beta(\mu-m)}}{s^2\sin(\pi s)}\sum_{k=1}^{\infty}\sum_{l=0}^{\infty}(-1)^l\left(\frac{2\pi mk}{s\beta eH}\right)^{2l}\left\{1+\frac{(4l+3)^2-1^2}{8sm\beta}+\frac{[(4l+3)^2-1^2][(4l+3)^2-3^2]}{2!(8sm\beta)^2}+\cdots+\frac{[(4l+3)^2-1^2][(4l+3)^2-3^2]\cdots[(4l+3)^2-(4l+1)^2]}{(2l+1)!(8sm\beta)^{2l+1}}\right\}. \tag{29}$$

The representation (29) that we have obtained for the thermodynamic potential makes it possible to carry out a complete analysis of this potential. It immediately becomes obvious that the method of steepest descent, used by various authors, corresponds to just the first term in the curly brackets in the expression (29). For greater clarity in the investigation of (29), we represent this expression in the form

$$\begin{aligned}W_+=\frac{Vm}{i\pi\beta^2}\int_{\sigma-i\infty}^{\sigma+i\infty}ds\,\frac{e^{s\beta(\mu-m)}}{s^2\sin(\pi s)}\sum_{k=1}^{\infty}\sum_{l=0}^{\infty}\times(-1)^l\left(\frac{2\pi mk}{2\beta eH}\right)^{2l}\Bigg\{1+\frac{3^2-1^2}{8sm\beta}+l\Bigg[\frac{24}{8sm\beta}\\+\frac{24(3^2-1^2)}{2!(8sm\beta)^2}+\frac{24(3^2-1^2)(3^2-5^2)}{3!(8sm\beta)^3}+\cdots\\+\frac{24(3^2-1^2)(3^2-5^2)\cdots(3^2-(4l+1)^2)}{(2l+1)!(8sm\beta)^{2l+1}}\Bigg]\\+l^2\Bigg[\frac{4^2}{8sm\beta}+\frac{4^2(3^2-1^2)+24^2}{2!(8sm\beta)^2}\\+\frac{4^2(3^2-1^2)(3^2-5^2)+24^2[(3^2-1^2)+(3^2-5^2)]}{3!(8sm\beta)^3}\\+\cdots\Bigg]+\cdots+l^{2(2l+1)}\frac{(4^2)^{2l+1}}{(2l+1)!(8sm\beta)^{2l+1}}\Bigg\}.\end{aligned} \tag{30}$$

It is easy to see that in (30) the following types of sums over the index l appear:

$$\sum_{l=0}^{\infty}(-1)^l l^n(x^2)^l=\left(x^2\frac{d}{dx^2}\right)^n\frac{1}{1+x^2}, \tag{31}$$

$$\sum_{l=0}^{\infty}(-1)^l l^n\frac{(x^2)^l}{(2l)!}=\left(x^2\frac{d}{dx^2}\right)^n\cos x, \tag{32}$$

$$\sum_{l=0}^{\infty}(-1)^l l^n\frac{(x^2)^l}{(2l+1)!}=\left(x^2\frac{d}{dx^2}\right)^n\frac{\sin x}{x}. \tag{33}$$

Here, $n=0,1,2,\ldots$, and $x=2\pi mk/s\beta eH$. It is entirely evident that the expression for W_+ can be represented in the form

$$W_+=\sum_{n=0}^{\infty}W_+^{(n)},$$

where the value of the index n corresponds to the power of the index l in (31)–(33), and it is easy to obtain the following improvement of the answer (25), (26) containing Landau oscillations that we found above by the method of steepest descent:

$$W_+^{(0)}=\frac{Vm}{i\pi\beta^2}\int_{\sigma-i\infty}^{\sigma+i\infty}ds\,\frac{e^{s\beta(\mu-m)}}{s^2\sin(\pi s)}\sum_{k=1}^{\infty}\times\frac{1+1/sm\beta}{1+(2\pi mk/s\beta eH)^2}$$

$$=-\theta(\mu-m)\frac{VeH}{\pi\beta}\sum_{k=1}^{\infty}\times\left\{\frac{1}{k}\cos\left[\frac{2\pi m^2}{eH}\left(\frac{\mu}{m}-1\right)k\right]+\frac{1}{2\pi k^2}\frac{eH}{m^2}\sin\left[\frac{2\pi m^2}{eH}\left(\frac{\mu}{m}-1\right)k\right]\right\}\frac{1}{\sinh(2\pi^2 mk/\beta eH)}. \quad (34)$$

In the zero-temperature limit ($T\to0$) we obtain

$$W_+^{(0)}|_{T=0}=-\theta(\mu-m)\frac{V(eH)^2}{2\pi m}\left\{B_2\left(\frac{m^2}{eH}\left(\frac{\mu}{m}-1\right)\right)+\frac{1}{3}\frac{eH}{m^2}B_3\left(\frac{m^2}{eH}\left(\frac{\mu}{m}-1\right)\right)\right\}.$$

This improved form (34) contains a field correction in the parameter eH/m^2. Terms from (30) of an analogous type, containing sums of the form (31), are obtained by differentiation and contain higher powers of the parameter eH/m^2 and B_i Bernoulli polynomials of higher degrees.

We shall now analyze the consequences of structures of the type (32). First we shall consider the case that is obtained for $n=0$, i.e.,

$$\sum_{k=1}^{\infty}\cos\left(\frac{2\pi mk}{s\beta eH}\right). \quad (35)$$

To calculate this we shall use a procedure that effectively corresponds to Borel summation (or to one of the types of regularization). Instead of (35) we sum the expression

$$\lim_{\alpha\to0}\left\{\sum_{k=1}^{\infty}e^{-k\alpha}\cos(k\xi)\right\}=\lim_{\alpha\to0}\frac{1}{2}\left(\frac{\sinh\alpha}{\coth\alpha-\cos\xi}-1\right)=-\frac{1}{2}.$$

Terms of this type make no contribution to the final result.

The third type of term, of the form (33), after summation over the index k, leads us to Bernoulli polynomials that are periodically continued along the entire axis, i.e.,

$$\sum_{k=1}^{\infty}\frac{\sin(kx_1)}{k}=-\pi B_1\left(\frac{x_1}{2\pi}\right),\quad n=0,\quad x_1\in(0,2\pi),$$

where $x_1=\pi/(2s\beta\sqrt{eH})^2$. Thus, we have oscillations in the new parameter $T/\sqrt{eH}$. This answer is strictly valid in the low-temperature region, i.e., for $m\beta\gg1$, in which the use of the procedure we have proposed is correct.

5. MONOTONIC PART OF THE THERMODYNAMIC POTENTIAL OF AN ELECTRON GAS IN A MAGNETIC FIELD

Above, we have investigated the oscillatory part of the thermodynamic potential of an ideal electron gas in a uniform and constant magnetic field. We now discuss the monotonic part of the thermodynamic potential, for which it is convenient to use the following ideas.

In the high-temperature region ($m\beta\ll1$ and $\sqrt{eH}/T\sqrt{2}\ll1$) it is convenient to transform the expression (23) to the form (see, e.g., Refs. 20 and 21):

$$W_+=\frac{eH}{2\pi}\sqrt{\frac{2}{\pi\beta}}\sum_{l=1}^{\infty}\frac{(-1)^l}{l^{1/2}}\left\{\sum_{n=0}^{\infty}(2eHn+m^2)^{1/4}K_{1/2}(l_\beta\sqrt{2eHn+m^2})-\frac{\sqrt{m}}{2}K_{1/2}(l\beta m)\right\}e^{l\mu\beta}.$$

after which, using a contour-integral representation,[22] we find

$$W_+=-\frac{(2eH)^{5/4}}{8\pi}\sqrt{\frac{2}{\pi\beta}}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}ds(2^{-s+1/2}-1)\zeta\times\left(s+\frac{1}{2}\right)\left(\frac{2}{eH\beta^2}\right)^{s/2}\Gamma\left(\frac{s+1/2}{2}\right)\Gamma\left(\frac{s-1/2}{2}\right)$$
$$\times\left\{\sum_{n=0}^{\infty}\left(n+\frac{m^2}{2eH}\right)^{-(1-2s)/4}\times{}_2F_1\left(\frac{s+1/2}{2},\frac{s-1/2}{2};\frac{1}{2};\frac{\mu^2}{2eHn+m^2}\right)\right\}$$
$$-\frac{2eH}{8\pi}\sqrt{\frac{2m}{\pi\beta}}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}ds(2^{1/2-s}-1)\zeta\left(s+\frac{1}{2}\right)\times\left(\frac{2}{m\beta}\right)^{s}\Gamma\left(\frac{s+1/2}{2}\right)\Gamma\left(\frac{s-1/2}{2}\right)\times{}_2F_1\left(\frac{s+1/2}{2},\frac{s-1/2}{2};\frac{1}{2};\frac{\mu^2}{m^2}\right),$$

where ${}_2F_1(x,y;\alpha;\beta)$ is a hypergeometric function and $\zeta(x)$ is the Riemann zeta-function.[17] A detailed description of the choice of integration contour is given in, e.g., Refs. 22 and 23. As a result of straightforward transformations we obtain, for $eH/2T^2\ll1$,

$$W_+=-\frac{3\zeta(3)T^3}{4\pi}-\frac{T\mu^2\ln 2}{2\pi}+\frac{(2eH)^{3/2}}{4\pi}\zeta\left(-\frac{1}{2},\frac{m^2}{2eH}\right)-\frac{meH}{4\pi}+\frac{m^2T\ln 2}{2\pi}$$
$$+\frac{T^3}{\pi}\sum_{k=2}^{\infty}\frac{\sqrt{\pi}\zeta(3-2k)}{\Gamma(k+1/2)k!}(2^{2k-2}-1)\left(\frac{\mu}{2T}\right)^{2k}$$
$$+\frac{TeH}{\pi}\sum_{k=1}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^j\Gamma(1/2-j)}{j!k!\Gamma(k+1/2)}\zeta(-2(k+j)+1)(4^{k+j}-1)\left(\frac{\mu}{2T}\right)^{2k}\left\{\left(\frac{eH}{2T^2}\right)^j\zeta\left(-j,\frac{m^2}{2eH}\right)-\frac{1}{2}\left(\frac{m^2}{4T^2}\right)^j\right\}. \quad (36)$$

As can be seen from (36), we have the Stefan–Boltzmann law in (2+1)-dimensional theory (the first term), and also corrections to it due to both the nonzero density of matter and the influence of the magnetic field. Taking into account the zero-temperature contribution, i.e., the terms S_0 (2) and S_1 (13), we find that the general expression for the effective potential at the level of the one-loop approximation is described by the following formulas. In the high-temperature region ($eH/2T^2\ll 1$),

$$V_{\mathrm{eff}}=S_0+S_1(m,eH)+W_+=-\frac{3\zeta(3)T^3}{4\pi}-\frac{\mu^2T\ln 2}{2\pi}+\frac{m^2T\ln 2}{2\pi}+\frac{m^2\mu^2}{16T}-\frac{eHT}{2\pi}\left\{\frac{eH}{2T^2}\zeta\left(-1,\frac{m^2}{2eH}\right)-\frac{m^2}{8T^2}\right\}+o\left(\left(\frac{eH}{T}\right)^3\right).$$

In the low-temperature region ($eH/2T^2\gg 1$) we can obtain the expression

$$V_{\mathrm{eff}}=S_0+S_1(m,eH)-\frac{eHT}{4\pi}\exp\left[\frac{m}{T}\left(\frac{\mu}{m}-1\right)\right].$$

6. CONCLUSION

Thus, we have analyzed completely the thermodynamic potential of a relativistic ideal electron gas in a magnetic field in (2+1)-dimensional QED, and have shown that, besides the usual terms oscillatory in the parameter $(m^2/eH)(\mu/m-1)$, discussed by Landau[18] and Peierls,[24] there exist terms that have oscillations in the parameter $T/\sqrt{eH}$. It should be noted that oscillations in the parameter $T/\sqrt{eH}$ were noted previously in our paper[25] on the thermodynamic potential of the gluon gas in an SU(2) gauge theory in the presence of an Abelian chromomagnetic field. This fact (the oscillations) was attributed by us to the presence of a tachyon mode in the gluon spectrum. However, the above analysis indicates that this type of oscillation is not determined by the statistics of the particles (the statistics is important for the Landau oscillations) but is a consequence of the discretization of the spectrum in the given field. We note one further fact that follows from our analysis. A number of authors[19,26] suppose that the presence of the Riemann zeta-function $\zeta(-1/2, m^2/eH)$ in the expression (36) for the thermodynamic potential is evidence for oscillations of this quantity in the parameter eH/m^2. At the same time, from an estimate of the magnitude of the field it becomes clear that $eH/m^2\sim 1$ even for fields on the order of the characteristic Schwinger field, i.e., $H=4.41\times 10^{13}$ G, and this field can increase to $H\sim 10^{18}$ G is only in cosmic conditions, e.g., on the surface of pulsars.[27,28] Thus, in laboratory conditions the only types of oscillations of the thermodynamic potential that can actually be observed and distinguished are those in the two parameters $(m^2/eH)(\mu/m-1)$ and $T/\sqrt{eH}$.

We recall also that in the literature[29–31] allowance has been made for the Landau-level splitting due to the electron spin. However, in these papers this effect was reduced to the semiphenomenological introduction of a phase difference between the oscillations induced by electrons with different spin orientations relative to the field. As a result, the semiclassical result of Refs. 32 in this approach should be multiplied by the quantity $\cos(\pi\Delta\varepsilon T/H)$, where $\Delta\varepsilon=ge\hbar H/2mc$ is the energy difference between levels with different spin orientations ($g=2.0023$ is the g-factor). From this it can be seen that the parameter introduced in those papers, despite its dependence on the temperature, differs from the parameter that arises in our work.

It is easy to see that our procedure for analysis of the present model problem does not depend essentially on the dimensionality of space, and the results can be carried over straightforwardly to the theory in any number of dimensions. This may be important in connection with the fact that in (3+1)-dimensional QED in the investigation of metals and, especially, semimetals at liquid-helium temperatures one uses the entire series corresponding to the Landau oscillations—a series that has been obtained by the method of steepest descent and, as we have seen, does not take into account all the characteristic features of the interaction of the electrons with the magnetic field.

The difference between the relativistic problem of the determination of the thermodynamic potential of an electron gas in a constant uniform magnetic field H, which we have solved with the electron spectrum $\varepsilon_{ns}=\sqrt{m^2+2eH(n+s)}$ ($s=\pm 1, n=1,2,...$) from the nonrelativistic problem, the spectrum of which has a quadratic dependence on the momenta,[15,16,18,24,33] is extremely interesting from the point of view that, apart from changes of notation for the variables, we have effectively solved the Lax model.[4,33] Thus, it is possible that the additional oscillations that appear in this case at the semiclassical level can be explained in terms of ellipsoidal constant-energy surfaces but not in terms of parabolic bands. The corresponding semiclassical explanation of the phenomenon of Landau oscillations is proposed in, e.g., the paper by Lifshits and Kosevich.[32] As is well known, in the first approximation the semiclassical description works with smooth functions. From a geometrical point of view, we have a simple topological surface—a sphere (or, apart from scale changes, an ellipsoid). The inclusion of higher orders leads us to the need to consider more-complicated topological structures—the torus, the pretzel, etc. In this case it is clear that the number of possible periods in the physical interpretation of the result obtained on such a surface is increased.[34] In quantum mechanics, a very simple but nontrivial example is the anharmonic oscillator,[35] for which the deviation from a harmonic law leads to nontrivial consequences. It is evident that we are also dealing with an analogous situation in the present exactly solvable problem, in which the electron spectrum differs from a quadratic spectrum. Taking into account what has been said, it would be interesting to find further exactly solvable problems with analogous nontrivial results. It is possible that this will throw light on the manifestation of nontrivial topological effects in physically observable situations.

Note also that we have considered the ideal electron gas. It is possible that allowance for the interaction will distort the picture that we have described and make the observation of oscillations of the conjectured type substantially more difficult. This depends, on the one hand, on the intensity of the magnetic field in the sample, and, on the other, on the degree of uniformity of the sample, i.e., the question of the limits of applicability of the model described to a real physical situation becomes important.

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