The Ginzburg–Landau expansion and the physical properties of superconductors with odd pairing

É. Z. Kuchinskiĭ, A. I. Posazhennikova, and M. V. Sadovskiĭ

Institute of Electrophysics, Ural Branch of the Russian Academy of Science, 620219 Ekaterinburg, Russia
(Submitted 15 September 1994)
Zh. Eksp. Teor. Fiz. **107**, 599–615 (February 1995)

We derive the Ginzburg–Landau expansion for superconductors whose gap function is odd in $k-k_F$. We show that for "odd" pairing there is an additional dependence of the expansion coefficients on the pairing coupling constant, which leads to a corresponding dependence of the physical characteristics of the superconductor. In contrast to the ordinary case, nonmagnetic impurities have a strong influence on the basic properties of an "odd" superconductor, and the nature of this influence differs considerably from the behavior predicted by the traditional theory of "dirty" superconductors. For one thing, this behavior manifests itself in the anomalous slope of the temperature dependence of the upper critical field H_{c2} near T_c. The field H_{c2} is studied over the entire temperature range. © *1995 American Institute of Physics.*

1. INTRODUCTION

Mila and Abrahams[1] recently suggested an interesting model of a superconducting state with an energy gap that is an odd function of the parameter $k-k_F$ (i.e., the quasiparticle energy $\xi=v_F(|\mathbf{k}|-k_F)$ measured from the Fermi level). In this case, as can easily be verified, superconductivity in the BCS approximation is also possible for an arbitrarily strong repulsion between electrons. The state occurs when there is strong repulsion, when ordinary ("even") superconductivity is suppressed and the pairing interaction is strong (the corresponding coupling constant exceeds a certain critical value).[1] Naturally, this model is attractive from the viewpoint of explaining superconductivity in highly correlated systems, although it is now clear that it can hardly be considered a realistic model for explaining high-T_c superconductivity in metal oxides, if only because of the isotropic nature of pairing inherent in the model (experiments have established that high-T_c superconductivity is anisotropic in the conducting plane and corresponds to d or anisotropic s pairs). Mila's and Abrahams's model[1] is also interesting by itself as a model of a new "exotic" superconducting state whose properties may differ considerably from those of common superconductors. In view of this it would be interesting to study a broad spectrum of such properties, which would make it possible to formulate the experimental criteria for searching for the anomalous "odd" superconductivity. A certain amount of work has already been done along this line,[2–5] although most studies were done with a view to comparing the new properties with those of high-T_c superconducting systems. The literature contains no discussion of the traditional problems of superconductivity theory as applied to the scheme of "odd" pairing of the type considered here.

The purpose of this paper is to carry out a microscopic derivation of the Ginzburg–Landau expansion in the odd pairing model and use the expansion to analyze some of the main characteristics of a superconductor near T_c. In addition we fully analyze the problem of the upper critical field H_{c2} (we will see that an "odd" superconductor is practically always a string type-II superconductor). In the process several anomalies manifest themselves, and these can be used in an experimental search for systems with odd pairing. At the same time, the results arrived at lead to additional arguments against using the model for explaining the properties of high-T_c superconductivity in metal oxides.

We recall that the model is based on demonstration of the fact[1] that the weak-binding equation in the BCS theory,

$$\Delta(\xi)=-N(0)\int_{-\infty}^{\infty}d\xi'\,V(\xi,\xi')\frac{\Delta(\xi')}{2\sqrt{\xi'^2+\Delta^2(\xi')}}\times\tanh\frac{\sqrt{\xi'^2+\Delta^2(\xi')}}{2T}\qquad(1)$$

can have a nontrivial solution $\Delta(\xi)=-\Delta(-\xi)$ (i.e., odd in $k-k_F$, with $\xi=v_F(k-k_F)$), provided that $V(\xi,\xi')$ has an attractive term even in the presence of an (infinitely) strong point repulsion (here $N(0)$ is the density of states at the Fermi level). Clearly,[1] for an odd $\Delta(\xi)$ the repulsive part of the interaction in Eq. (1) simply vanishes and the attractive term $V_2(\xi,\xi')$ can ensure pairing with nontrivial properties: the gap function $\Delta(\xi)$ vanishes at the Fermi surface, which leads to gapless superconductivity. It should be emphasized that we are speaking of an isotropic model in which the gap vanishes everywhere at the Fermi surface, which distinguishes this model from anisotropic pairing, say, of the d-type.

Thus, in what follows we assume that the interaction in Eq. (1) consists of two terms (E_F is the Fermi energy), $V(\xi,\xi')=V_1(\xi,\xi')+V_2(\xi,\xi')$,

where

$$V_1(\xi,\xi')=\begin{cases}U>0 & \text{if } |\xi|,|\xi'|<E_F,\\ 0 & \text{if } |\xi|,|\xi'|>E_F\end{cases}\qquad(2)$$

is the point repulsion of electrons, and $V_2(\xi,\xi')$ is the effective pairing interaction (attraction), which is finite for $|\xi|,|\xi'|<\omega_c$ and $|\xi-\xi'|<\omega_c$ (the latter restriction is very important), with $\omega_c \ll E_F$ acting as a characteristic frequency of bosons whose exchange gives rise to pairing. The pairing "potential" $V_2(\xi,\xi')$ can be represented by different model functions.[1] In this paper we consider the following model interaction:[3,4]

$$V_2(\xi,\xi') = \begin{cases} -V\cos\left(\dfrac{\pi}{2}\dfrac{\xi-\xi'}{\omega_c}\right) & \text{if } |\xi|,|\xi'|,|\xi-\xi'|<\omega_c, \\ 0 & \text{if } |\xi|,\xi', \text{ or } |\xi-\xi'|>\omega_c. \end{cases} \quad (3)$$

Then the integral equation for the gap is reduced to a transcendental equation and can easily be solved.[3,4]. All the main features of the model, also characteristic of other types of model pairing potential,[1] are retained.

The transition temperature of the superconductor is determined by the linearized equation

$$\Delta(\xi) = -N(0)\int_{-\infty}^{\infty} d\xi'\, V(\xi,\xi')\,\frac{\Delta(\xi')}{2\xi'}\tanh\left(\frac{\xi'}{2T_c}\right). \quad (4)$$

It can easily be verified[4] that the temperature T_c of the transition to the odd state is determined from the equation

$$1 = g\int_0^{\omega_c} d\xi \sin^2\left(\frac{\pi}{2}\frac{\xi}{\omega_c}\right)\frac{1}{\xi}\tanh\left(\frac{\xi}{2T_c}\right), \quad (5)$$

where $g = N(0)V$ is the dimensionless pairing coupling constant. In this model the pairing coupling constant has a critical value: odd pairing appears only for $g > g_c \approx 1.213$ even without point repulsion U. Actually, the ordinary "even" pairing dominates in the presence of a weak repulsive interaction, and the temperature of the respective transition is higher than the temperature of the transition to the odd-gap state. As the repulsion grows, the situation changes, and at large values of g "odd" pairing becomes preferable.[4] In what follows we assume that the system occupies a certain region on the phase diagram in the variables g and $\mu = N(0)U$, where only odd pairing is stable.

The gap function in the odd-pairing model considered here has the form[4]

$$\Delta(\xi) = \begin{cases} \Delta_0(T)\sin\left(\dfrac{\pi}{2}\dfrac{\xi}{\omega_c}\right) & \text{if } |\xi|<\omega_c, \\ 0 & \text{if } |\xi|>\omega_c, \end{cases} \quad (6)$$

and the temperature dependence of $\Delta_0\,(T)$ is determined by the following equation:

$$1 = g\int_0^{\omega_c} d\xi' \sin^2\left(\frac{\pi}{2}\frac{\xi'}{\omega_c}\right) \times \frac{\tanh\left(\sqrt{\xi^2+\Delta_0^2(T)\sin^2\left(\frac{\pi}{2}\frac{\xi}{\omega_c}\right)}/2T\right)}{\sqrt{\xi^2+\Delta_0^2(T)\sin^2\left(\frac{\pi}{2}\frac{\xi}{\omega_c}\right)}}. \quad (7)$$

The temperature dependence of $\Delta_0\,(T)$ resembles the one in the BCS theory but the two are not identical.[4]

Normal (nonmagnetic) impurities strongly suppress odd pairing.[3,4] In this case the transition temperature is determined by the equation

$$1 = g\int_0^{\omega_c}\frac{d\xi'}{\xi'}\sin^2\left(\frac{\pi}{2}\frac{\xi}{\omega_c}\right)\int_{-\infty}^{\infty}\frac{d\omega}{\pi}\tanh\left(\frac{\omega+\xi'}{2T_c}\right)\frac{\gamma}{\omega^2+\gamma^2}, \quad (8)$$

where γ is the rate of electron scattering by impurities randomly distributed in space. Superconductivity disappears at $\gamma \sim T_{c0}$, where T_{c0} is the transition temperature without scattering and is determined by Eq. (4). The critical scattering rate γ_c corresponding to destruction of superconductivity is determined from the following equation:[3,4]

$$1 = \frac{2}{\pi} g\int_0^{\omega_c}\frac{d\xi}{\xi}\sin^2\left(\frac{\pi}{2}\frac{\xi}{\omega_c}\right)\arctan\left(\frac{\xi}{\gamma_c}\right). \quad (9)$$

For $g \simeq g_c$ this implies the dependence $\gamma_c \sim (g-g_c) \to 0$, which reflects the narrowing of the superconductivity region on the phase diagram in variables T_c and γ. In this sense destruction of superconductivity proceeds even faster than if we were to introduce magnetic impurities into a common superconductor.

In comparison to an approach of the BCS type, the Ginzburg–Landau theory makes it possible, at least in principle, to describe a broader set of physical properties at the expense of narrowing the temperature range to $T \sim T_c$. This makes the microscopic derivation of the Ginzburg–Landau expansion coefficients very important, because knowing them immediately leads to several useful conclusions.

2. THE GINZBURG–LANDAU EXPANSION

2.1. The case of a pure superconductor

As is the common practice, we select the gap function (6) as the order parameter in powers of which the Ginzburg–Landau expansion is done. We also assume that generally the amplitude $\Delta_0(T)$ is a slowly varying function of the spatial coordinates. Correspondingly, in the momentum space there emerges a Fourier component of the order parameter,

$$\Delta(\xi,T,\mathbf{q}) = \Delta_{\mathbf{q}}(T)\sin\left(\frac{\pi}{2}\frac{\xi}{\omega_c}\right), \quad (10)$$

and the Ginzburg–Landau expansion for the difference of free energies of a superconducting and normal states for small values of q has the form

$$F_s - F_n = A|\Delta_{\mathbf{q}}|^2 + q^2C|\Delta_{\mathbf{q}}|^2 + \tfrac{1}{2}B|\Delta_{\mathbf{q}}|^4. \quad (11)$$

Our problem consists is to find the microscopic expressions for the coefficients A, B, and C.

The Ginzburg–Landau expansion can easily be found by examining the diagrams shown in Fig. 1. It is reduced to the ordinary loop expansion for the free energy of an electron in the fluctuation field of the order parameter of the form (10). The only point requiring explanation is the need to subtract the second diagram in Fig. 1: this procedure, as one can easily verify, ensures that the coefficient A in Eq. (11) vanishes at the transition point $T = T_c$. All calculations are done in the standard manner, with allowance for the fact that the

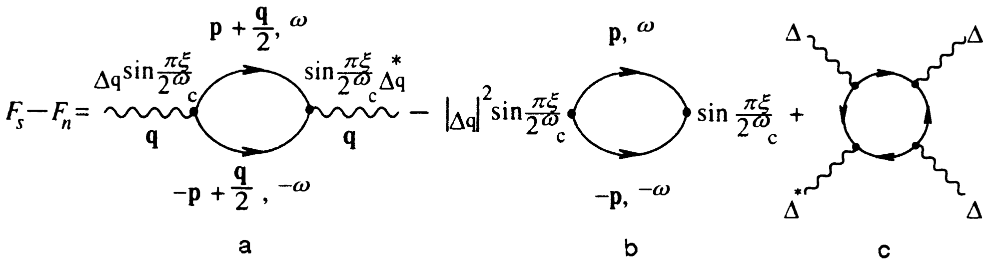

FIG. 1. The diagrammatic representation of the Ginzburg–Landau expansion. The wavy lines correspond to the fluctuations of the order parameter (10), and the solid lines to the Matsubara Green's functions of the electron. The diagram (b) is calculated at $T=T_c$.

temperature T is close to the transition temperature T_c; some details are given in Appendix 1. As a result the Ginzburg–Landau expansion coefficients can be written as follows:

$$A=A_0K_A, \quad B=B_0K_B, \quad C=C_0K_C, \tag{12}$$

where by A_0, B_0, and C_0 we denote the standard expressions for the expansion coefficients in the ordinary theory of "even" superconductors,[6]

$$A_0=N(0)\frac{T-T_c}{T_c}, \tag{13}$$

$$B_0=N(0)\frac{7\zeta(3)}{8\pi^2T_c^2}, \tag{14}$$

$$C_0=N(0)\frac{7\zeta(3)}{48\pi^2}\frac{v_F^2}{T_c^2}\approx N(0)\xi_0^2, \tag{15}$$

and all the features of the odd-pairing model are contained in the dimensionless combinations K_A, K_B, and K_C:

$$K_A=\int_0^{\omega_c/T_c}dx\,\frac{\sin^2\left(\frac{\pi}{2}\frac{T_c}{\omega_c}x\right)}{2\cosh^2(x/2)}, \tag{16}$$

$$K_B=\frac{4\pi^2}{7\zeta(3)}\int_0^{\omega_c/T_c}dx\,\frac{\sin^2\left(\frac{\pi}{2}\frac{T_c}{2\omega_c}x\right)}{x^2}\times\left[\frac{\tanh(x/2)}{x}-\frac{1}{2\cosh^2(x/2)}\right], \tag{17}$$

$$K_C=\frac{1}{2}-\frac{4}{7\zeta(3)}\sum_{n=0}^{\infty}\frac{\exp\left[-\pi^2\frac{T_c}{\omega_c}(2n+1)\right]}{(2n+1)^3}\times\left[1+\pi^2\frac{T_c}{\omega_c}(2n+1)+\frac{\pi^4}{2}\frac{T_c^2}{\omega_c^2}(2n+1)^2\right]. \tag{18}$$

These dimensionless quantities are functions of T_c/ω_c, which in turn depends in a known manner[3,4] on the dimensionless pairing coupling constant g. The corresponding functions K_A, K_B, and K_C on g are shown in Fig. 2. Thus, in contrast to the ordinary case, where the Ginzburg–Landau expansion coefficients depend on the pairing coupling constant only through the respective dependence of the transition temperature T_c, there emerges an additional nonmonotonic dependence on the coupling constant.

As is known, the Ginzburg–Landau equations determine two characteristic lengths, the coherence length and the magnetic-field penetration depth.[6]

The coherence length at a given temperature, $\xi(T)$, serves as the characteristic scale of variations in the order parameter Δ, i.e., is actually the "size" of the Cooper pair:

$$\xi^2(T)=-\frac{C}{A}. \tag{19}$$

In ordinary superconductors

$$\xi^2_{\rm BCS}(T)=-\frac{C_0}{A_0}, \tag{20}$$

$$\xi_{\rm BCS}(T)\approx 0.74\frac{\xi_0}{\sqrt{1-T/T_c}}, \tag{21}$$

where $\xi_0=0.18v_F/T_c$. In our case we have

$$\frac{\xi^2(T)}{\xi^2_{\rm BCS}(T)}=\frac{K_C}{K_A}. \tag{22}$$

The dependence of this ratio on g is shown in Fig. 3.

For the magnetic-field penetration depth in an ordinary superconductor we have

$$\lambda_{\rm BCS}(T)=\frac{1}{2}\frac{\lambda_0}{\sqrt{1-T/T_c}}, \tag{23}$$

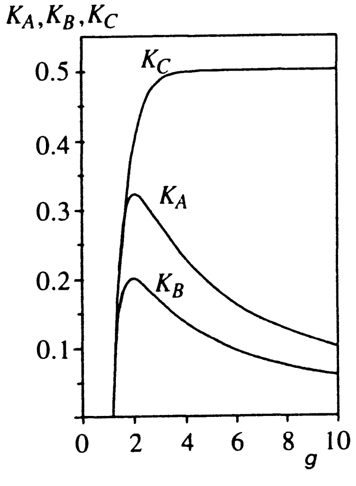

FIG. 2. The coefficients K_A, K_B, and K_C as functions of the coupling constant g.

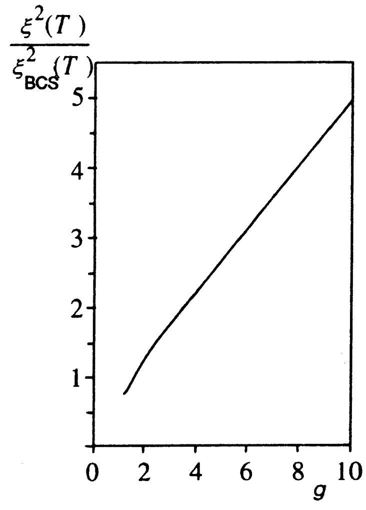

FIG. 3. The dimensionless coherence length as a function of the coupling constant g.

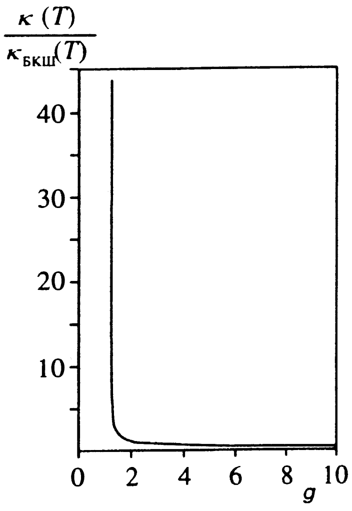

FIG. 5. The relative Ginzburg–Landau parameter as a function of the coupling constant g.

where $\lambda_0^2 = mc^2/4\pi ne^2$ determines the penetration depth at $T=0$. Generally we have the following expression for the penetration depth in terms of the Ginzburg–Landau expansion coefficients:

$$\lambda^2(T) = -\frac{c^2}{32\pi e^2}\frac{B}{AC}. \tag{24}$$

Then in our model

$$\frac{\lambda(T)}{\lambda_{\mathrm{BCS}}(T)} = \left(\frac{K_B}{K_A K_C}\right)^{1/2}. \tag{25}$$

Figure 4 shows the dependence of this ratio on the pairing coupling constant. The fact that $\lambda \to \infty$ as $g \to g_c$ is physically understandable and related to the disappearance of odd pairing.

Let us examine the Ginzburg–Landau parameter

$$\kappa = \frac{\lambda(T)}{\xi(T)} = \frac{c}{4eC}\sqrt{B/2\pi}. \tag{26}$$

As is known, the type of superconductor depends on the value of the parameter κ: superconductors with $\kappa < 1/\sqrt{2}$ belong to type I and those with $\kappa > 1/\sqrt{2}$ to type II. Accordingly, in the odd-pairing model considered here we have

$$\frac{\kappa}{\kappa_{\mathrm{BCS}}} = \frac{\sqrt{K_B}}{K_C}, \tag{27}$$

where

$$\kappa_{\mathrm{BCS}} = \frac{3c}{\sqrt{7\zeta(3)}e}\frac{T_c}{v_F^2\sqrt{N(0)}} \tag{28}$$

is the Ginzburg–Landau parameter for the ordinary case. Figure 5 shows $\kappa/\kappa_{\mathrm{BCS}}$ as a function of g. The diagram clearly demonstrates that for all reasonable values of g (near g_c) an odd superconductor is sure to be a strong type-II superconductor. In this connection we recall that the asymptotic behavior of large $g \gg g_c$ is unphysical, because actually all our reasoning is based on weak-binding equations of the BCS type (1), while the transition to the tight-binding range requires[4] more careful consideration in the spirit of Nozieres and Schmitt-Rink.[7] No such consideration has been done for odd pairing.

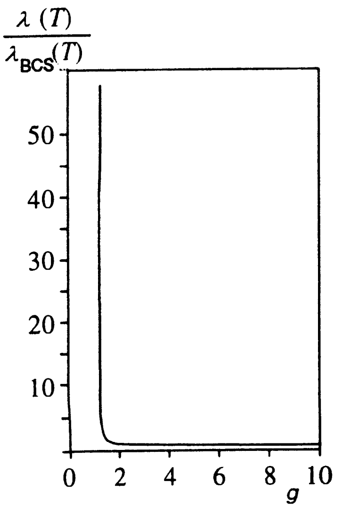

FIG. 4. The dimensionless magnetic-field penetration depth as a function of the coupling constant g.

2.2. Effect of "normal" impurities

We examine a superconductor containing "normal" (nonmagnetic) impurities. When deriving the diagrammatic Ginzburg–Landau expansion, we must allow for the corresponding scattering processes, i.e., diagrams of the type shown in Figs. 6(a) and (b). Clearly, the contribution of diagrams of the type in Fig. 6(b) is virtually nil because the functions at the vertices [the factors $\sin(\xi)$] are odd in ξ. Thus, for an odd superconductor the loop expansion has the form of Figs. 6(c) and (d) (to within second-order terms), where the electron lines stand for total averaged Green's

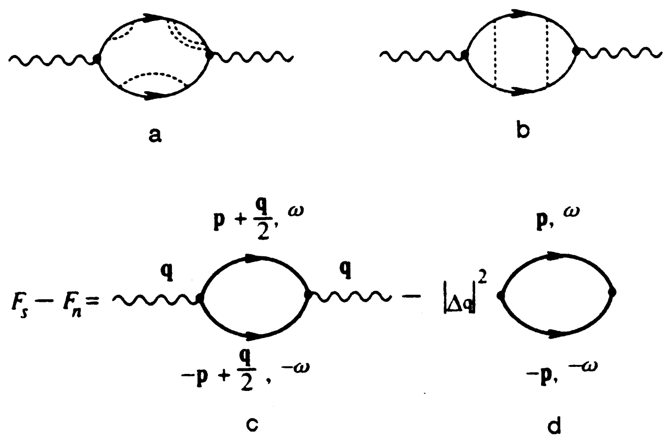

FIG. 6. Diagrammatic expansion of the free energy in the presence of impurity scattering. The dashed lines correspond to scattering by impurities.

functions with allowance for scattering on impurities. "Diffusion" renormalization caused by diagrams of the type in Fig. 6(b), which is a characteristic feature in the ordinary theory of "dirty" superconductors,[8] does not occur. In this sense the structure of all the expressions is closer to that of the theory of ordinary "pure" superconductors. Note, however, that in the sense of the ordinary criterion $\xi_0 \ll l$, where l is the mean free path, the "dirty" limit is entirely inaccessible because, as noted above, such a superconducting state is destroyed by impurity scattering even at $\gamma \sim T_c$ (Refs. 3 and 4). Below we discuss the essential changes in the coefficients A and C brought on by impurity scattering. The details of the calculations can be found in Appendix 1.

The Ginzburg–Landau expansion coefficients are again represented in the form (12), and impurity scattering leads to

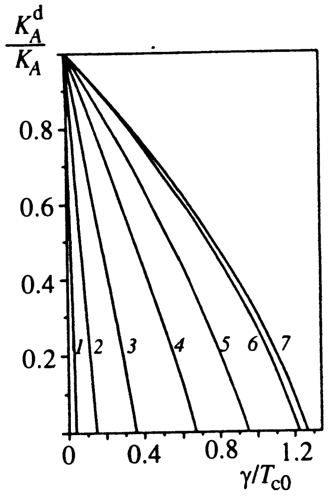

FIG. 7. The normalized coefficient K_A^d as a function of the impurity scattering rate for different values of the coupling constant g: curve *1*, $g=1.22$; curve *2*, $g=1.24$; curve *3*, $g=1.3$; curve *4*, $g=1.5$; curve *5*, $g=2.0$; curve *6*, $g=5.0$; and curve *7*, $g=10.0$.

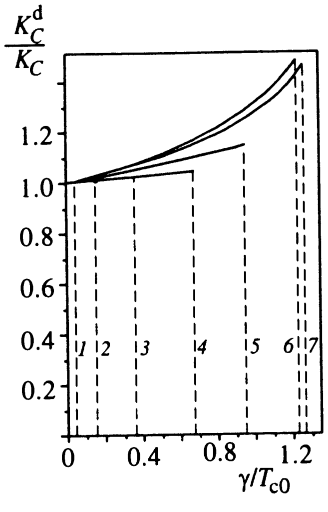

FIG. 8. The normalized coefficient K_C^d as a function of the impurity scattering rate for different values of the coupling constant g: curve *1*, $g=1.22$; curve *2*, $g=1.24$; curve *3*, $g=1.3$; curve *4*, $g=1.5$; curve *5*, $g=2.0$; curve *6*, $g=5.0$; and curve *7*, $g=10.0$.

renormalization of the dimensionless functions K_A and K_C, which now acquire the form

$$K_A^d = \frac{T_{c0}}{T_c}\int_0^{\omega_c/T_c}\frac{dx}{x}\sin^2\left(\frac{\pi}{2}\frac{T_{c0}}{\omega_c}x\right) \times \int_{-\infty}^{\infty}\frac{dy}{2\pi}\frac{y+x}{\cosh^2\left(\frac{y+x}{2}\frac{T_{c0}}{T_c}\right)}\frac{\gamma/T_{c0}}{y^2+\gamma^2T_{c0}^2}, \tag{29}$$

$$K_C^d = \frac{4\pi^3}{7\zeta(3)}\frac{T_c}{T_{c0}}\sum_{n=0}^{\infty}\left\{\frac{1}{\left[(2n+1)\pi\frac{T_c}{T_{c0}}+\frac{\gamma}{T_{c0}}\right]^3} - \frac{\exp\left(-\pi\frac{T_{c0}}{\omega_c}\left[(2n+1)\pi\frac{T_c}{T_{c0}}+\frac{\gamma}{T_{c0}}\right]\right)}{\left[(2n+1)\pi\frac{T_c}{T_{c0}}+\frac{\gamma}{T_{c0}}\right]^3} \times\left[1+\pi\frac{T_{c0}}{\omega_c}\left[(2n+1)\pi\frac{T_c}{T_{c0}}+\frac{\gamma}{T_{c0}}\right]+\frac{\pi^2}{2}\frac{T_{c0}^2}{2\omega_c^2} \times\left[(2n+1)\pi\frac{T_c}{T_{c0}}+\frac{\gamma}{T_{c0}}\right]^2\right]\right\}. \tag{30}$$

Here T_{c0} is the temperature of the transition without impurities, T_c is the true temperature of the transition in the impurity system determined by Eq. (8), and γ is the rate of electron scattering on impurities. In the limit of $\gamma \to 0$ Eqs. (29) and (30) naturally transform into Eqs. (16) and (18).

Figures 7 and 8 show the diagrams, obtained by numerical methods, of the ratios K_A^d/K_A and K_C^d/K_C as functions of

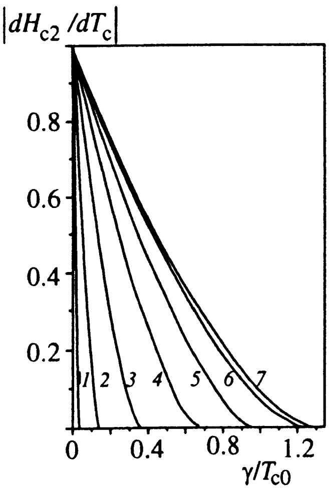

FIG. 9. The slope of the upper critical field H_{c2} vs impurity scattering rate curve for different values of the coupling constant g: curve *1*, $g=1.22$; curve *2*, $g=1.24$; curve *3*, $g=1.3$; curve *4*, $g=1.5$; curve *5*, $g=2.0$; curve *6*, $g=5.0$; and curve *7*, $g=10.0$. The field's derivative is normalized to its value in the absence of scattering.

the scattering rate γ. The most important dependence emerges for the coefficient K_A^d, which rapidly decreases as γ grows and vanishes as $\gamma \to \gamma_c$.

The behavior of the Ginzburg–Landau expansion coefficients A and C determines, as is known, the temperature dependence of the upper critical magnetic field near T_c (see Ref. 6):

$$H_{c2}(T)=\frac{\phi_0}{2\pi\xi^2(T)}=-\frac{\phi_0}{2\pi}\frac{A}{C}, \tag{31}$$

where $\phi_0=c\pi/e$ is the quantum of magnetic flux. From this we can easily find the slope of the temperature dependence of H_{c2} near T_c, i.e., the temperature derivative of the magnetic field:

$$\left|\frac{dH_{c2}}{dT}\right|_{T_c}=\frac{24\pi\phi_0}{7\zeta(3)v_F^2}T_c\frac{K_A^d}{K_C^d}. \tag{32}$$

Figure 9 shows $|dH_{c2}/dT|_{T_c}$ as a function of disorder (the scattering rate γ). We see that the "slope" of the field decreases rapidly as disorder grows and vanishes as $\gamma \to \gamma_c$. This behavior is the opposite of that in the ordinary theory, where in the "pure" limit the slope of H_{c2} does not change if impurities are added, while in the "dirty" limit it grows with γ (see Ref. 8). This anomalous behavior can serve as an experimental criterion in searching for superconductors with "odd" pairing. We note in this connection that high-Tc superconductivity oxides do not exhibit such behavior and, apparently, the odd-pairing model is unable to describe the observed anomalies in H_{c2} (Ref. 9). This fact can be considered an additional argument against applying models of this type to high-Tc superconductivity systems.[3,4]

3. THE UPPER CRITICAL FIELD

Concerning the above-mentioned anomalies in the behavior of H_{c2} in the odd-pairing model it would be interesting to do a complete study of the temperature dependence of the upper critical field in such systems with allowance for impurity scattering. This requires stepping outside the Ginzburg–Landau theory and using a microscopic approach, i.e., from the standpoint of the picture of Cooper instability in an external magnetic field. The problem of Cooper instability, which leads to odd pairing in the absence of a field, was considered in Ref. 4. Here we use a similar approach. We start our analysis immediately with a system with impurities.

The diagrams that determine scattering and interaction in the Cooper channel are shown in Fig. 10. The vertex part of the impurity scattering in the ladder approximation is shown in Fig. 10(a) and the vertex part allowing for pairing interaction (represented by a dot on the diagram) in Fig. 10(b). For the latter we have the following integral equation:

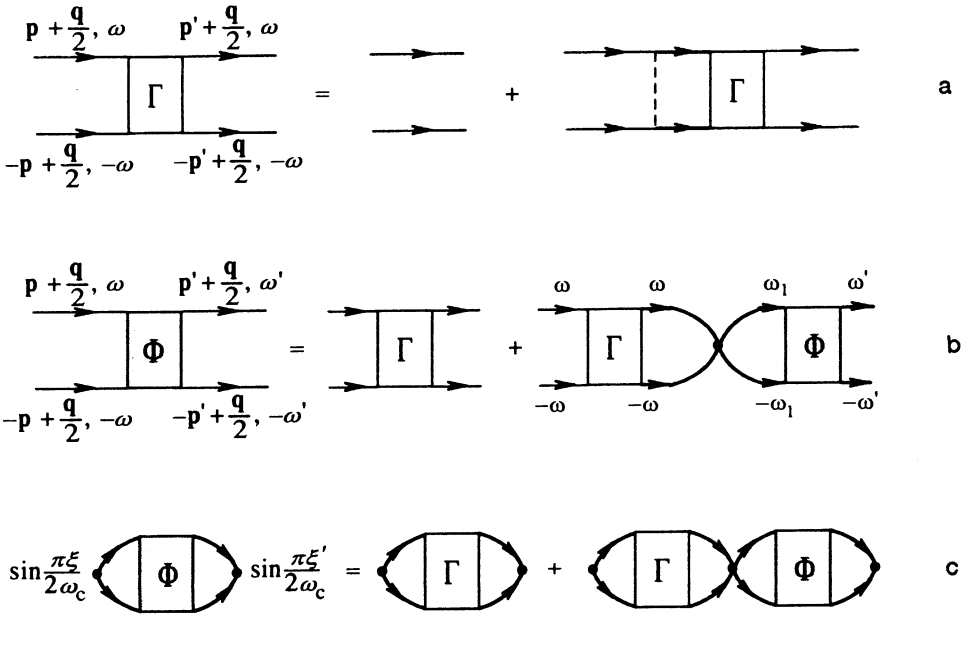

FIG. 10. The diagrams that determine Cooper instability in a system with impurities: (a) the vertex part of the impurity scattering in the Cooper channel; (b) the vertex part of the pairing interaction; and (c) the response function, whose divergence determines odd pairing.

$$\Phi_{\mathbf{pp}'}(\mathbf{q}\omega\omega')=\Gamma_{\mathbf{pp}'}(\mathbf{q}\omega)\delta_{\omega\omega'}$$

$$-T\sum_{\omega_1}\sum_{\mathbf{p}_1\mathbf{p}_2}\Gamma_{\mathbf{pp}_1}(\mathbf{q}\omega)V(\mathbf{p}_1\mathbf{p}_2)\Phi_{\mathbf{p}_2\mathbf{p}'}(\mathbf{q}\omega_1\omega'), \tag{33}$$

where the impurity vertex $\Gamma_{\mathbf{pp}'}(\mathbf{q}\omega)$ is determined, as noted earlier, by the ladder approximation [Fig. 10(a)], and $V(\mathbf{pp}')$ is the pairing interaction. Following Ref. 4, we introduce vertices summed over frequencies,

$$\Gamma_{\mathbf{pp}'}(\mathbf{q})=-T\sum_{\omega}\Gamma_{\mathbf{pp}'}(\mathbf{q}\omega), \tag{34}$$

$$\Phi_{\mathbf{pp}'}(\mathbf{q})=-T\sum_{\omega\omega'}\Phi_{\mathbf{pp}'}(\mathbf{q}\omega\omega'), \tag{35}$$

which satisfy an equation that follows from Eq. (33):

$$\Phi_{\mathbf{pp}'}(\mathbf{q})=\Gamma_{\mathbf{pp}'}(\mathbf{q})-T\sum_{\mathbf{p}_1\mathbf{p}_2}\Gamma_{\mathbf{pp}_1}(\mathbf{q})V(\mathbf{p}_1\mathbf{p}_2)\Phi_{\mathbf{p}_1\mathbf{p}'}(\mathbf{q}). \tag{36}$$

It is convenient to look for the odd Cooper instability using the equation shown in Fig. 10(c), which is actually the function of the response to the odd-order-parameter fluctuations we are interested in:

$$\Pi(\mathbf{q})=\Pi_0(\mathbf{q})-\Pi_0(\mathbf{q})V\Pi(\mathbf{q}), \tag{37}$$

where the loop diagrams represent

$$\Pi_0(\mathbf{q})=\sum_{|\xi_{\mathbf{p}}|,|\xi_{\mathbf{p}'}|<\omega_c}\sin\left(\frac{\pi}{2}\frac{\xi_{\mathbf{p}}}{\omega_c}\right)\Gamma_{\mathbf{pp}'}(\mathbf{q})\sin\left(\frac{\pi}{2}\frac{\xi_{\mathbf{p}'}}{\omega_c}\right), \tag{38}$$

$$\Pi(\mathbf{q})=\sum_{|\xi_{\mathbf{p}}|,|\xi_{\mathbf{p}'}|<\omega_c}\sin\left(\frac{\pi}{2}\frac{\xi_{\mathbf{p}}}{\omega_c}\right)\Phi_{\mathbf{pp}'}(\mathbf{q})\sin\left(\frac{\pi}{2}\frac{\xi_{\mathbf{p}'}}{\omega_c}\right), \tag{39}$$

and we have explicitly allowed for the pairing interaction in our model Eq. (3). Solving Eq. (37) yields

$$\Pi(\mathbf{q})=\frac{\Pi_0(\mathbf{q})}{1+V\Pi_0(\mathbf{q})}. \tag{40}$$

It is the divergence of this expression (the zero of the denominator means the divergence of the response) that determines the point of odd Cooper instability.

While calculating $\Pi_0(\mathbf{q})$ it is easy to see that because the vertices are odd functions of ξ only the first diagram in the ladder of Fig. 10(a) contributes to this quantity and that the diffusion contribution vanishes. We have

$$\Pi_0(\mathbf{q})=-T\sum_{\omega}\sum_{|\xi|<\omega_c}\sin^2\left(\frac{\pi}{2}\frac{\xi}{\omega_c}\right)\times G_\omega(\mathbf{p}+\mathbf{q}/2)G_{-\omega}(\mathbf{p}-\mathbf{q}/2)$$
$$=-TN(0)\sum_{\omega}\int_{-\omega_c}^{\omega_c}d\xi\sin^2\left(\frac{\pi}{2}\frac{\xi}{\omega_c}\right)\times\frac{1}{2}\int_{-1}^{1}dt\,\frac{1}{\xi-v_{\mathrm{F}}qt/2+i(\omega+\gamma)}\times\frac{1}{\xi+v_{\mathrm{F}}qt/2-i(\omega+\gamma)}$$
$$=-TN(0)\sum_{\omega}\int_{-\omega_c}^{\omega_c}d\xi\sin^2\left(\frac{\pi}{2}\frac{\xi}{\omega_c}\right)\frac{1}{4\theta\xi}\times\ln\left(\frac{(\theta+\xi)^2+\Omega^2}{(\theta+\xi)^2-\Omega^2}\right), \tag{41}$$

where $\Omega=\omega+\gamma$ and $\theta=v_{\mathrm{F}}q/2$. Then the divergence of (40) is determined from the equation

$$1=gT\sum_{\omega}\int_{-\omega_c}^{\omega_c}d\xi\sin^2\left(\frac{\pi}{2}\frac{\xi}{\omega_c}\right)\frac{1}{4\theta\xi}\ln\left(\frac{(\theta+\xi)^2+\Omega^2}{(\theta+\xi)^2-\Omega^2}\right), \tag{42}$$

which at $q=0$ is reduced to

$$1=gT\sum_{\omega}\int_{-\omega_c}^{\omega_c}d\xi\,\frac{\sin^2[(\pi/2)(\xi/\omega_c)]}{\xi^2+(\omega+\gamma)^2}, \tag{43}$$

from which Eq. (8) can be obtained by standard methods [cf. Eq. (47)].

For a system in an external magnetic field H the Cooper-pair momentum $\mathbf{q}$ is replaced in the ordinary manner by $\mathbf{q}-(2e/c)\mathbf{A}$, where $\mathbf{A}$ is the vector potential. Then, as before, the Cooper instability is determined by Eq. (42), only θ is defined as $\theta=v_{\mathrm{F}}q_0/2$, where q_0 is the minimum eigenvalue of the operator $(\mathbf{q}-(2e/c)\mathbf{A})^2$, equal, as is known,[6] to $\sqrt{2\pi H/\phi_0}$, and where ϕ_0 is the quantum of magnetic flux introduced earlier, equal to twice the electron charge. The equation emerging in the process determines the upper critical field over the entire temperature range.

The most convenient way to solve such an equation numerically is to convert the sum over discrete frequencies to an integral. Details are given in Appendix 2. As a result, instead of (42) there emerges the following equation for H_{c2}:

$$1=2gT\int_{-\omega_c}^{\omega_c}d\xi\,\sin^2\left(\frac{\pi}{2}\frac{\xi}{\omega_c}\right)\times\int_0^\infty dx\,\sin\left(\frac{v_{\mathrm{F}}x\sqrt{\frac{\pi H}{2\phi_0}}}{2\pi T}\right)\times\frac{\exp\left[-\left(\frac{\gamma}{2\pi T}+\frac{1}{2}\right)x\right]\sin\left(\frac{\xi x}{2\pi T}\right)}{v_{\mathrm{F}}\sqrt{\frac{\pi H}{2\phi_0}}\,\xi x(1-\exp(-x))}. \tag{44}$$

Figure 11 shows the results of solving this equation numerically, which demonstrate the temperature dependence of H_{c2} for different degrees of disorder in the system. The qualitative picture of the temperature dependence of H_{c2} differs little from the ordinary, but there is a clear-cut rapid decrease in the slope of the H_{c2} vs T curve near T_c as the scattering rate grows. This decrease, as can easily be verified,

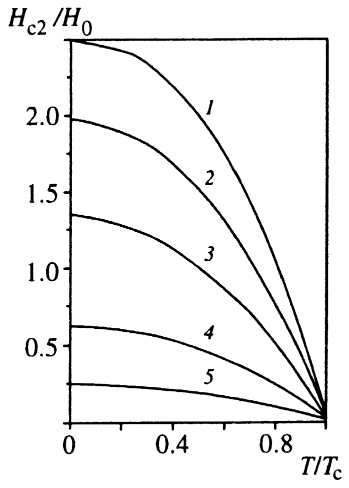

FIG. 11. The temperature behavior of the upper critical field for systems with different degrees of disorder γ/T_{c0}. The pairing coupling constant $g=2$, the magnetic field is in units of $H_0 = (2/\pi)\phi_0 T_{c0}/v_F^2$ and the temperature is normalized to T_c, which is disorder-dependent. Curve 1, $\gamma/T_{c0}=0.2$; curve 2, $\gamma/T_{c0}=0.25$; curve 3, $\gamma/T_{c0}=0.5$; curve 4, $\gamma/T_{c0}=0.75$; and curve 5, $\gamma/T_{c0}=0.87$.

is described by the equations of the Ginzburg–Landau theory obtained earlier. Note that the qualitative shape of the temperature dependence of H_{c2} is very similar over a broad range of values of the pairing coupling constant g; hence Fig. 11 presents the data for only one value of g. We note once more that as the disorder increases the established anomalous behavior of H_{c2} clearly distinguishes between odd pairing and ordinary even pairing and can be used to establish experimentally the existence of such "exotic" superconductors.

4. CONCLUSION

We have done a microscopic analysis of the Ginzburg–Landau expansion coefficients in the model of pairing that is odd in $k-k_F$. We have shown that these coefficients acquire a strong dependence on the pairing coupling constant, a dependence that cannot be reduced to that of T_c, characteristic of the ordinary "even" case. We have also analyzed the corresponding behavior for the characteristic parameters of the Ginzburg–Landau theory.

In examining the effect of scattering by normal impurities, the main theoretical conclusion to be drawn is the absence of diffusion renormalization, which leads to dimensional dependencies of the coefficients characteristic of the theory of "pure" superconductors. At the same time, the sharp suppression of T_c by impurity scattering and the strong dependence of the Ginzburg–Landau expansion coefficients on disorder cause the behavior of the upper critical field H_{c2} as a function of the degree of order to deviate dramatically from that in both the "pure" and "dirty" limits of the ordinary theory of even pairing. This difference is revealed in the sharp decrease in the slope of the H_{c2} vs T curve near T_c as impurity scattering grows, which is of interest as an experimental criterion in the search for systems with odd pairing.

We have resolved the problem of the behavior of H_{c2} over the entire temperature range by analyzing odd Cooper instability in an external magnetic field. Again, a characteristic feature of this analysis is the absence of diffusion renormalization of the equations, which is directly related to the fact that the order parameter (the gap) is odd in $k-k_F$. The results are in full agreement with the conclusions drawn on the basis of the Ginzburg–Landau approximation.

This work was supported by the Scientific Council on High-Tc Superconductivity and carried out as part of Project No. 93-001 of the National Program for Superconductivity Studies. The work was also partially supported by Grant No. RGL000 from the Soros Fund and Grant No. 93-02-2066 from the Russian Fund for Fundamental Studies.

APPENDIX 1

Below we give the details of calculations needed in deriving the above expressions for the Ginzburg–Landau expansion coefficients. To within terms quadratic in q, the expression corresponding to the diagram in Fig. 1(a) is

$$\text{Fig. 1(a)}=-|\Delta_{\mathbf{q}}|^2\frac{T}{(2\pi)^3}\sum_\omega\int d\mathbf{p}\sin^2\left(\frac{\pi\xi}{2\omega_c}\right)G_\omega(\mathbf{p}+\mathbf{q}/2)G_{-\omega}(-\mathbf{p}+\mathbf{q}/2)\approx-|\Delta_{\mathbf{q}}|^2N(0)\times\left\{\int_0^{\omega_c}d\xi\sin^2\left(\frac{\pi\xi}{2\omega_c}\right)\frac{1}{\xi}\tanh\frac{\xi}{2T}+\frac{\pi v_F^2q^2}{48}T\sum_\omega\frac{1}{|\omega|^3}\left[2-2\exp\left(-\pi\frac{|\omega|}{\omega_c}\right)\times\left(1+\pi\frac{|\omega|}{\omega_c}+\pi^2\frac{|\omega|^2}{2\omega_c^2}\right)\right]\right\}. \quad (45)$$

Here $G_\omega(\mathbf{p})=(i\omega-\xi_{\mathbf{p}})^{-1}$ is the ordinary Matsubara Green's function of the electron, and $\omega=(2n+1)\pi T$. What is left is to expand the integrand in powers of $T-T_c$ and put $T=T_c$ in the term containing the small quantity q^2. Using Eq. (5) for T_c, we can easily see that the contribution of the diagram in Fig. 1(b) is

$$\text{Fig. 1(b)}=-|\Delta_{\mathbf{q}}|^2\left\{\frac{T}{(2\pi)^3}\sum_\omega\int d\mathbf{p}G_\omega(\mathbf{p})G_{-\omega}(-\mathbf{p})\sin^2\left(\frac{\pi\xi}{2\omega_c}\right)\right\}=-\frac{N(0)}{g}|\Delta_{\mathbf{q}}|^2, \quad (46)$$

which cancels out with the zero-order term in $T-T_c$ and q in the expression for the diagram in Fig. 1(a). As a result we arrive at Eqs. (12), (16), and (18) defining the coefficients A and C.

The calculations are done in a similar manner for an impurity superconductor, the only difference being that the electron Green's function now has the form

$G_\omega(\mathbf{p})=[i\omega-\xi_\mathbf{p}+i\gamma\,\mathrm{sign}\omega)]^{-1}$. Accordingly, e.g., the expression for the diagram in Fig. 6(d) contains the sum

$$T_c\sum_\omega \frac{1}{(\omega+\gamma)^2+\xi^2}=\int_{-\infty}^{\infty}\frac{d\omega}{\pi}\frac{1}{2\xi}\tanh\left(\frac{\omega+\xi}{2T_c}\right)\frac{\gamma}{\omega^2+\gamma^2} \tag{47}$$

and is reduced, if we allow for Eq. (8) for T_c, to

$$\text{Fig. 6(d)}=-N(0)|\Delta_\mathbf{q}|^2\int_0^{\omega_c}\frac{d\xi}{\xi}\sin^2\left(\frac{\pi\xi}{2\omega_c}\right)$$

$$\times\int_{-\infty}^{\infty}\frac{d\omega}{\pi}\tanh\left(\frac{\omega+\xi}{2T_c}\right)\frac{\gamma}{\omega^2+\gamma^2}=-\frac{N(0)}{\gamma}|\Delta_\mathbf{q}|^2. \tag{48}$$

This contribution cancels out with the zero-order terms in $T-T_c$ and q^2 in the expression for the diagram in Fig. 6(c). As a result the Ginzburg–Landau expansion coefficients contain the true temperature T_c of the transition in the presence of impurities. Similarly, one must allow for the scattering rate γ in the part of the diagram in Fig. 6(c) reflecting contributions of orders $T-T_c$ and q^2. All this leads to expressions for the Ginzburg–Landau expansion coefficients A and C written above for the impurity case.

The simplest way to find the coefficient B is to use the expression for the difference in free energies of the superconducting and normal phases at an arbitrary temperature in the homogeneous case:

$$F_s-F_n=N(0)\int_{-\infty}^{\infty}d\xi\left\{\frac{|\Delta(\xi)|^2}{2\epsilon}\tanh\left(\frac{\epsilon}{2T}\right)\right.$$

$$\left.-2T\ln\left(\frac{\cosh(\epsilon/2T)}{\cosh(\xi/2T)}\right)\right\}, \tag{49}$$

where $\epsilon=\sqrt{\xi^2+|\Delta(\xi)|^2}$, and $\Delta(\xi)$ is defined in Eqs. (6) and (7). Expanding this difference in a power series in Δ_0 and $T-T_c$ and employing Eq. (5) for T_c, we get

$$F_s-F_n=N(0)\frac{\Delta_0^2}{g}+N(0)$$

$$\times\int_0^{\omega_c}d\xi\left\{-\frac{\Delta_0^2}{\xi}\sin^2\left(\frac{\pi\xi}{2\omega_c}\right)\tanh\left(\frac{\xi}{2T_c}\right)\right.$$

$$+(T-T_c)\Delta_0^2\frac{\sin^2\left(\frac{\pi\xi}{2\omega_c}\right)}{2T_c^2\cosh^2\left(\frac{\xi}{2T_c}\right)}$$

$$+\frac{1}{4}\frac{\Delta_0^4}{\xi^2}\sin^4\left(\frac{\pi\xi}{2\omega_c}\right)\left(\frac{1}{\xi}\tanh\left(\frac{\xi}{2T_c}\right)\right.$$

$$\left.\left.-\frac{1}{2T_c}\frac{1}{\cosh^2\left(\frac{\xi}{2T_c}\right)}\right)\right\}. \tag{50}$$

Using once more Eq. (5) for T_c, we isolate the coefficient A found earlier in the term with Δ_0^2. Then the factor of $\Delta_0^4/2$ is the coefficient B,

$$B=\frac{N(0)}{2}\int_0^{\omega_c}\frac{d\xi}{\xi^3}\sin^4\left(\frac{\pi\xi}{2\omega_c}\right)\tanh\left(\frac{\xi}{2T_cx}\right)$$

$$-\frac{N(0)}{4T_c}\int_0^{\omega_c}\frac{d\xi}{\xi^2}\frac{\sin^4\left(\frac{\pi\xi}{2\omega_c}\right)}{\cosh^2\left(\frac{\xi}{2T_c}\right)}, \tag{51}$$

from which Eq. (17) follows immediately.

APPENDIX 2

Examination of the block $\Pi_0(\mathbf{q})$ introduced earlier reveals the need to calculate a sum over frequencies of the form

$$S=\sum_\omega\frac{1}{\xi-\theta t+i(\gamma+\omega)}\frac{1}{\xi+\theta t-i(\gamma+\omega)}. \tag{52}$$

One can obtain a convenient representation of this sum in integral form by employing the well-known expression

$$\sum_{n=0}^{\infty}\frac{1}{n+a}\frac{1}{n+b}=\frac{1}{b-a}[\psi(b)-\psi(a)] \tag{53}$$

and the integral representation of the logarithmic derivative of the gamma function Γ,

$$\psi(z)=\int_0^\infty\left(\frac{e^-}{x}-\frac{e^{-zx}}{1-e^{-x}}\right),\quad \mathrm{Re}\,z>0, \tag{54}$$

so that

$$\psi(b)-\psi(a)=\int_0^\infty\frac{e^{-ax}-e^{-bx}}{1-e^{-x}},$$

$$\mathrm{Re}\;a>0,\quad \mathrm{Re}\;b>0. \tag{55}$$

We then get

$$S=\frac{1}{\pi T\xi}\int_0^\infty dx\,\sin\left(\frac{\xi x}{2\pi T}\right)$$

$$\times\frac{\exp\left(-\left(\frac{\gamma}{2\pi T}+\frac{1}{2}\right)x\right)\exp\left(-i\frac{\theta tx}{2\pi T}\right)}{1-\exp(-x)}. \tag{56}$$

Integration with respect t is elementary, with the result that Eq. (42) is reduced to Eq. (44).

[1] F. Mila and E. Abrahams, Phys. Rev. Lett. **67**, 2379 (1991).
[2] Q. P. Li and R. Joynt, Mod. Phys. Lett. **6**, 1145 (1992).
[3] É. Z. Kuchinskiĭ and M. V. Sadovskiĭ, Pis'ma Zh. Eksp. Teor. Fiz. **57**, 494 (1993) [JETP Lett. **57**, 515 (1993)].
[4] É. Z. Kuchinskiĭ, M. V. Sadovskiĭ, and M. A. Érkabaev, Zh. Eksp. Teor. Fiz. **104**, 3550 (1993) [JETP **77**, 692 (1993)].
[5] M. Dobroliubov, E. Langmann, and P. C. E. Stamp, Europhys. Lett. **26**, 141 (1994).

[6] P. G. de Gennes, *Superconductivity of Metals and Alloys*, W. A. Benjamin, Reading, Mass. (1966).
[7] P. Nozieres and S. Schmitt-Rink, J. Low Temp. Phys. **59**, 195 (1985).
[8] L. P. Gor'kov, Zh. Eksp. Teor. Fiz. **37**, 1407 (1959) [Sov. Phys. JETP **10**, 998 (1960)].
[9] M. V. Sadovskiĭ, in *Studies of High-Temperature Superconductors*, Vol. 11, edited by A. V. Narlikar, Nova Science Publishers, New York (1993), pp. 131–224.

Translated by Eugene Yankovsky