Adiabatic wave–particle interaction in a weakly inhomogeneous plasma

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Nonlinear periodic solutions of the Vlasov–Maxwell system of equations are found, describing the collisionless damping of a Langmuir wave in a plasma with small longitudinal density gradients, assuming rapid phase mixing of the resonant particles. By means of the adiabatic approximation the spatial evolution of the wave and the deformation of the distribution function near the Cherenkov resonance are treated. The nonlinear dispersion relation for the wave is found, taking into account the contribution of the phase-space structures (electron voids and beams of trapped electrons) that develop in the resonant interaction process. The propagation and transport of the energy of a wave perturbation are discussed, taking into account the interaction with the resonant particles.

1. INTRODUCTION

The exchange of momentum and energy between waves and resonant particles is frequently the dominant mechanism of wave dynamics in the plasma. The collection of physical phenomena for which the wave–particle interaction plays the dominant role is exceptionally diverse. It ranges from beam–plasma interactions in devices for controlled thermonuclear fusion and plasma electronics, all the way to artificial geophysical phenomena and active space experiments. In recent years studies in this area have also been stimulated by the search for new techniques for accelerating charged particles[1] and the progress in nonlinear (chaotic) dynamics.[2]

Two interaction regimes can be distinguished, depending on the intensity of the plasma waves. In the linear regime, when the wave amplitude E_0 is small enough that $\gamma/\omega_b \gg 1$ holds, where γ is the Landau damping rate and $\omega_b = \sqrt{ekE_0/m}$ is the characteristic (*bounce*) frequency for electrons in the potential wells of a wave with wave number k, the wave damps exponentially. As is well known, collisionless damping is described by perturbation theory, where the Vlasov–Poisson system is first linearized.[3] Van Kampen[4] (see also Refs. 5 and 6) has given an elegant interpretation of Landau damping in terms of the spreading of a superposition of time-independent eigenmodes of the plasma (Van Kampen modes) making up the initial wave perturbation.

In the nonlinear regime ($\gamma/\omega_b \ll 1$) the resonant particles undergo rapid phase mixing. The waves do not damp significantly, but close to resonance ($V \approx u = \omega/k$) the modulated plateau which is characteristic of waves of finite amplitude develops on the distribution function. Since the wave amplitude has barely changed, the initial-value problem to first order reduces to the self-consistent solution of the kinetic equation.[7–9] Besides confirming the results of theory, experimental studies of wave dynamics in the nonlinear regime[10,11] have revealed a nonlinear frequency shift and satellite instability–phenomena which are also due to the resonant particles.[12,13]

The details of the wave–particle interaction in a weakly homogeneous plasma are related to the spatial dependence of the phase velocity, as a result of which the ratio $\gamma(u)/\omega_b$ can vary over a wide range for essentially any wave amplitude. Consequently, the wave damping can easily pass from one regime to the other. This important property is closely related to the continuous renewal of the resonant region, which competes with the phase mixing of the resonant particle.[14,15] In addition, in the limit $\gamma/\omega_b \ll 1$ beams of accelerated trapped particles can develop.[14,16–18] The corresponding wave-damping mechanism is very different from Landau damping and cannot be described in linear theory.

In the present work we consider the spatial evolution of a Langmuir wave propagating in a collisionless plasma with small longitudinal density gradients. We will be primarily interested in nonlinear phenomena which are specific to inhomogeneous plasmas and are not describable by perturbation theory, even though the wave amplitude is small. These include the irreversible deformation of the resonant particle distribution, the formation of voids and beams of trapped electrons in the resonant region of phase space, and also the corresponding wave-damping mechanisms associated with negative and positive density gradients. The ultimate goal of this work is the derivation and analysis of a closed set of equations that self-consistently describe the spatial evolution of a wave including these effects.

The method employed below, which is based on the adiabatic approximation,[19–25] operates on the electron distribution function over the whole range of velocities, including the nonresonant region (thermal electrons).[26] In contrast to previous work on closely related problems,[14–18] we can therefore perform a rigorous calculation of all the moments of the distribution function of interest and identify the contributions of the resonant and nonresonant components of the plasma. This provides a transparent physical picture of the propagation and transport of wave energy, and also the processes by which momentum and energy are exchanged between the wave and the resonant particles. In solving this rigorous self-consistent problem we also touch on questions regarding the role of anharmonicity in the plasma oscillations which are normally left out of consideration.

The process of phase mixing of the resonant electrons in a weakly inhomogeneous plasma implies that a time-independent periodic wave is closely related to the well-

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known Bernstein–Greene–Kruskal (BGK) waves.[27] Consequently, the problem posed in this work can also be regarded as dealing with the dynamics of BGK waves in a weakly inhomogeneous plasma. Note, however, that even the class of periodic BGK waves is quite broad, since the form of the distribution function in the coordinate frame comoving with the wave and the self-consistent field are far from uniquely determined (attempts to narrow the category of BGK solutions were undertaken, e.g., in Refs. 28 and 29). It is significant that in the solution of the problem of the wave evolution (the initial-value problem) this uncertainty is removed in a natural way, and those BGK solutions which actually arise are selected from among the entire class. In this way knowledge of the entire "pre-history" of a wave perturbation enables us to establish the dispersion relation of a BGK wave, taking into account the contribution of the resonant structure that develops during the course of evolution along with the familiar terms. In essence this also happens in the linear theory. The contribution of the resonant particles is rigorously included by solving a Cauchy problem.[3]

The nonlinear dispersion relation which includes the contribution of the trapped electrons has long been known.[30,31] It can also be found in one form or another in Refs. 32–36 although it is difficult to say that it has been applied systematically. For a closed description of the perturbation wave dynamics the nonlinear dispersion relation is just as necessary as the conservation laws for particle number, momentum, and energy. It is convenient because it automatically removes the question of the size of the nonlinear phase shift,[37] which inevitably arises in connection with the approach of Refs. 14–18. The usefulness of the nonlinear dispersion relation is also demonstrated in the analysis of the satellite instability[33,38] and of the propagation of waves loaded with trapped particles in a weakly inhomogeneous plasma.[39]

2. FORMULATION OF THE PROBLEM AND METHODOLOGICAL COMMENTS

Consider the propagation of a Langmuir wave oscillating in time with frequency ω in a weakly inhomogeneous plasma with a longitudinal density gradient. Generally speaking, the plasma density profile depends on the mechanism which maintains the nonuniformity. Here, however, we avoid discussing the specific mechanism by assuming that the ions are motionless and the ion density profile is given. In the absence of the wave the unperturbed electron distribution function $f_0(V)$ is also assumed known, e.g., Maxwellian.

Figure 1 shows a typical formulation of the problem. A wave with prescribed initial values of the wave number k_0 and amplitude A_0 is incident on the region of nonuniformity from infinity ($x=-\infty$). We will be interested in the spatial dependence of these parameters, and also of the electron distribution function in the region of variation. In principle the approach described below allows one to consider arbitrary smooth plasma density profiles. However, we sacrifice a certain amount of generality and consider only the simplest of these, such as the ones shown in Fig. 1. Furthermore, we will assume that the initial phase velocity is much greater than the electron thermal velocity, $\omega/k_0 \gg v_T$. This formulation of

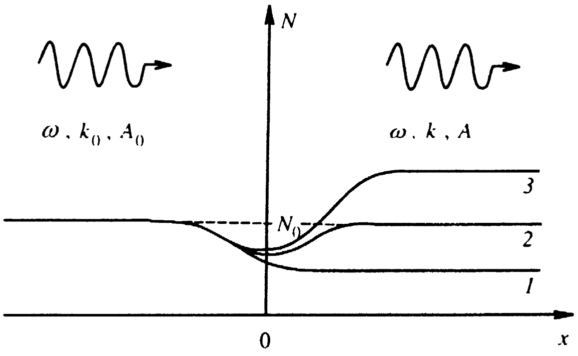

FIG. 1. Plasma density profiles with smooth gradients. The wave number k and wave amplitude A are also smooth functions of position.

the problem is attractive because there are no resonant particles for the initial wave (or more precisely, their number is exponentially small), and consequently there is no need to artificially specify an initial particle distribution in the neighborhood $V \simeq \omega/k_0$ of the resonance. The phase velocity drops as the wave propagates in the region of reduced density, and it interacts with the electrons in the tail of the distribution. The distribution function is naturally perturbed in the region of velocities $V \simeq u(x) = \omega/k(x)$ as a result of this interaction. If the phase velocity subsequently increases (see Fig. 1, traces *2* and *3*), some of the electrons are trapped and are accelerated by the wave.[16,17] Thus, this problem is also of interest in its own right in connection with the description of how the high-energy electron component forms.[40,41]

Regarding the adiabaticity of the wave–particle interaction, we assume that the local scale length $L = N/|dN/dx|$ of the plasma density variation is always so large that the wave evolves slowly in comparison with the phase-mixing of the resonant electrons, i.e., the wave variables change little over distances on the order of the spatial bounce period $l_b = 2\pi(\omega/k)\omega_b^{-1}$. Under these conditions, except for a discontinuous change in the nature of the motion in crossing the resonant region,[23,24] the particle dynamics can be described using the adiabatic approximation. In addition, at large L we will ignore the exponentially weak wave reflection.[42,43] If the reflected wave is neglected the total electron energy flux averaged over a wave period is conserved. This can easily be shown formally, starting from the Vlasov equation and using the periodicity of the wave. Consequently, in what follows we use the energy balance equation directly to determine the spatial dependence of the wave amplitude instead of calculating the local damping rate.[14,15,37] Then the analysis of energy exchange in the system simply reduces to identifying the contributions of the resonant and nonresonant particles in this equation. The fully self-consistent description is obtained by simultaneous solution of the Vlasov–Maxwell equations without any *a priori* assumptions (e.g., assumptions about the harmonicity of the oscillations) whatsoever. The restriction introduced by the requirement of wave periodicity implies that the parameters are interrelated through the nonlinear dispersion relation.

A series of studies designed to explain the mechanisms for the production of the so-called trigger radiation in the earth's magnetosphere[44–48] (see also the review in Ref. 49 and work cited there) are especially deserving of mention among treatments which have been responsible for making important contributions to the theory of wave–particle interactions in inhomogeneous plasmas. Generally speaking, the resonant electrons are treated by identifying their current density in the Maxwell equations (or in the equations for the wave amplitude and phase). The calculation of the perturbation in the resonant particle current is a fairly involved procedure, which in addition often includes non-self-consistent elements. In this connection the approach described in what follows differs from the majority of its predecessors in that the contribution of the resonant particles is identified in the solutions of the Vlasov–Maxwell equations rather than in the equations themselves. Finally, in order to avoid certain inaccuracies we will use rigorous expressions for the adiabatic invariants of the electron motion in the field of a wave whose parameters vary slowly. Closely related analysis of the time-dependent problem was carried out in Refs. 22 and 26.

Below we will use the following units to describe the physical variables:

$$[t]=\omega^{-1},\quad [x]=k_0^{-1},\quad [k]=k_0,$$

$$[V]=[u]=[v_T]=\omega/k_0,\quad [T]=m\omega^2/k_0^2,$$

$$[\Phi]=m\omega 2/ek02,$$

$$[E]=m\omega^2/ek_0,\quad [f]=n_{\rm cr}k_0/\omega,$$

$$n_{\rm cr}\equiv m\omega^2/4\pi e^2,\quad [n]=[N]=n_{\rm cr},\quad [j]=e\omega n_{\rm cr}/k_0,$$

$$[\mathscr{E}]=mn_{\rm cr}\omega^2/k_0^2,\quad [S]=mn_{\rm cr}\omega^3/k_0^3,$$

where $u=\omega/k(x)$ is the wave phase velocity, $T=mv_T^2/2$ is the electron temperature, E and Φ are the electric field and potential, f is the electron distribution function, n and N are the electron and ion densities, j is the electron current density, $\mathscr{E}$ is the energy density, S is the energy flux density, and e and m are the electron charge and mass. In these units the Vlasov–Maxwell equations assume the form

$$\frac{\partial f}{\partial t}+V\frac{\partial f}{\partial x}-\frac{\partial\Phi}{\partial x}\frac{\partial f}{\partial V}=0,\quad E=-\frac{\partial\Phi}{\partial x},$$

$$\frac{\partial^2\Phi}{\partial x\partial t}-j=0,\quad j=\int_{-\infty}^{\infty}dVVf,\tag{1}$$

$$\frac{\partial^2\Phi}{\partial x^2}=N-n,\quad n=\int_{-\infty}^{\infty}dVf.$$

Neglecting reflection, the space–time dependence of the physical variables is given by the following expressions:

$$\mathscr{F}=\mathscr{F}(\psi,x),\quad \psi=\int_{-\infty}^{x}dx'k(x')-t,\tag{2}$$

where we have written $k(x)=1/u(x)$ and ψ is the wave phase. The partial derivatives $\partial\mathscr{F}/\partial x\sim L^{-1}$ in the variables ψ and x are assumed small by virtue of the weak inhomogeneity. For averages over the wave period we use angle brackets:

$$\langle\mathscr{F}\rangle\equiv(2\pi)^{-1}\int_{-\pi}^{\pi}d\psi\mathscr{F}(\psi,x).$$

3. ELECTRON DISTRIBUTION FUNCTION

We solve the Vlasov equation using the adiabatic approximation, i.e., instead of the usual linearization we utilize the small ratio $\varepsilon=l_b/L\ll 1$, which allows us to consider waves of arbitrary intensity. As a first step we go over to the new independent variables x, ψ, W, where $W=v^2/2+\Phi(\psi,x)$ is the electron energy in the noninertial coordinate system moving with the local wave phase velocity $u=u(x)$. In this system the particle velocity is equal to $v=V-u=\pm\sqrt{2(W-\Phi)}$, where the signs $\pm$ correspond to particles overtaking the wave and falling behind it, respectively. We obtain the equation

$$\frac{1}{u}\frac{\partial f}{\partial\psi}+\left(1+\frac{u}{v}\right)\frac{\partial f}{\partial x}+\left[\frac{u}{v}\frac{\partial\Phi}{\partial x}-(u+v)\frac{du}{dx}\right]\frac{\partial f}{\partial W}=0.\tag{3}$$

We look for a solution of (3) by expanding in ε:

$$f=f_\pm(W,x)+\tilde{f}_\pm(W,x,\psi).\tag{4}$$

For the trapped electrons we use the relation $f_+=f_-\equiv F$ (see, e.g., Ref. 50) to integrate (3) along the closed particle trajectory. This yields an equation for the distribution to leading (zeroth) order in ε:

$$\frac{\partial F}{\partial x}\oint d\psi\left(1+\frac{u}{v}\right)+\frac{\partial F}{\partial W}\oint d\psi\left[\frac{u}{v}\frac{\partial\Phi}{\partial x}-(u+v)\frac{du}{dx}\right]$$

$$=-\oint d\psi\frac{\partial\tilde{f}_\pm}{\partial\psi}=0.\tag{5}$$

Without loss of generality we assume that the potential reaches its maximum value $\Phi_{\rm max}$ at $\psi=\pm\pi$ and its minimum $\Phi_{\rm min}$ at $\psi=0$. Then (5) assumes the form

$$u\frac{\partial F}{\partial x}\int_{-\psi_0}^{\psi_0}\frac{d\psi}{\sqrt{2(W-\Phi)}}+\frac{\partial F}{\partial W}\int_{-\psi_0}^{\psi_0}d\psi$$

$$\times\left[\frac{u}{\sqrt{2(W-\Phi)}}\frac{\partial\Phi}{\partial x}\right.$$

$$\left.-\sqrt{2(W-\Phi)}\frac{du}{dx}\right]=0,\tag{6}$$

where $\pm\psi_0(x,W)$ are the turning points determined by the relation $W=\Phi(\pm\psi_0,x)$. The solution of (6) is an arbitrary function of the adiabatic invariant

$$J=u\int_{-\psi_0}^{\psi_0}\frac{d\psi}{2\pi}[2(W-\Phi)]^{1/2}.\tag{7}$$

The procedure for determining the distribution function of the untrapped electrons is similar to that described above, except that for them we have $f_+\neq f_-$, and instead of matching the solution on a closed trajectory we use its periodicity in phase. Then we have

$$\frac{\partial f_\pm}{\partial x}\int_{-\pi}^{\pi}d\psi\left(1+\frac{u}{v}\right)+\frac{\partial f_\pm}{\partial W}\int_{-\pi}^{\pi}d\psi \times\left[\frac{u}{v}\frac{\partial\Phi}{\partial x}-(u+v)\frac{du}{dx}\right] =-\frac{1}{u}\int_{-\pi}^{\pi}d\psi\frac{\partial\tilde{f}_\pm}{\partial\psi}=0.$$

From this it follows that $f_\pm=f_\pm(I_\pm)$, where

$$I_\pm=u^2/2+W\pm u\langle\sqrt{2(W-\Phi)}\rangle. \tag{8}$$

It is easy to find the first correction $\tilde{f}_\pm\sim\varepsilon$, which is responsible for the exchange of energy between the electrons and the electric field. Among other things this determines the rate of energy exchange $\langle jE\rangle$. However, the result of the wave–particle interaction is described even in zeroth order in ε, which makes the application of the conservation laws especially convenient. Thus, we actually work with the distribution function averaged over phases, which is an analog of the ergodic equation.[8] In a weakly inhomogeneous plasma this component of the electron distribution varies smoothly as a function of position, like the other wave properties.

The specific form of the distribution function depends on the initial conditions, the wave parameters at infinity ($x=-\infty$), and the form of the unperturbed distribution f_0. The uniqueness of $f(W,x)$ is closely related to the possibility of following the entire evolution of the wave, and allows us to distinguish those BGK solutions which can actually be realized. Note also that Eqs. (7) and (8) imply that the electron distribution is a functional of $\Phi(\psi,x)$, i.e., generally speaking it depends on both the amplitude and the shape of the wave potential, which must be determined self-consistently using the Poisson equation. This last consideration is frequently disregarded in the treatment of time-independent nonlinear waves.

In our particular case we have $u\gg v_T$ at the onset of the evolution. Then, neglecting the exponentially small number of "tail" electrons, as initial conditions at $x=-\infty$ we assume

$$f_+=F=0,\quad f_-(I_-)=f_0(I_-). \tag{9}$$

In the limit of infinitely small amplitudes f_- goes over to the prescribed unperturbed distribution f_0. In particular, if in the absence of a wave f_0 is Maxwellian, then we have

$$f_-(I_-)=N_0/\sqrt{2\pi T}\exp(-I_-/T), \tag{10}$$

where $N_0\equiv N(x=-\infty)$ (Fig. 1).

In order to find the distribution function for arbitrary x, we use the identity $\langle j\rangle=0$ [cf. Eq. (1)], which ensures conservation of particle number and momentum in the system. Going over to integration with respect to W and then to integration with respect to $J, I_\pm$ in the integral for the current density we find

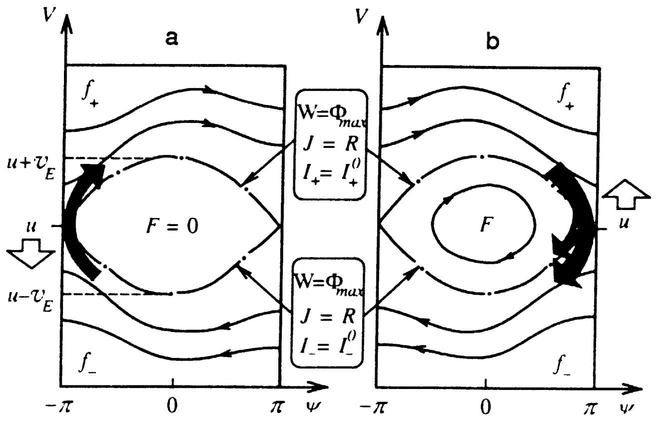

FIG. 2. Qualitative form of the phase plane close to resonance. Electron trajectories are shown in the wave field with (a) decreasing and (b) increasing phase velocity, along with the separatrix (chain curve) dividing the particles according to their motion into three classes. Particle transitions from one class to another (shown schematically by heavy arrows) close to the hyperbolic singular points $\psi=\pm\pi$, $V=u$ are accompanied by short-time scale violation of the adiabaticity of the motion.

$$\langle j\rangle=\left\langle\int_\Phi^\infty dW\left[f_+\left(\frac{u}{\sqrt{2(W-\Phi)}}+1\right)+f_-\left(\frac{u}{\sqrt{2(W-\Phi)}}-1\right)\right]\right\rangle =\int_{I_+^0}^\infty dI_+f_+(I_+)+2\int_0^R dJF(J) +\int_{I_{\min}}^{I_-^0}dI_-f_-(I_-)-\int_{I_{\min}}^\infty dI_-f_-(I_-)=0, \tag{11}$$

where R and $I_\pm^0$ are the limiting values of the adiabatic invariants (7) and (8) on the separatrix $W=\Phi_{\max}$ which separates the phase plane into three regions as shown in Fig. 2. Here

$$R=R(x)=u\langle\sqrt{2(\Phi_{\max}-\Phi)}\rangle, \tag{12}$$

$$I_\pm^0=I_\pm^0(x)=I_0\pm R, \tag{13}$$

$$I_0=I_0(x)=u^2/2+\Phi_{\max}\simeq u^2/2. \tag{14}$$

The two integrals with respect to I_- in Eq. (11) appear because the relation between W and I_- is multivalued. They correspond to the two intervals where the function $I_-=I_-(W)$ is monotonic, and in the small-amplitude limit they describe the contributions of particles with $\langle V\rangle>0$ and $\langle V\rangle<0$. This fact, however, is of little importance for analyzing the balance of the resonant particle number, which always satisfies $\langle V\rangle>0$.

Treating I_0 as the independent variable and differentiating (11) we obtain an equation which expresses the balance of the number of particles that intersect the separatrix:

$$2F(R)\,\frac{dR}{dI_0}-\left(1+\frac{dR}{dI_0}\right)f_+(I_0+R)+\left(1-\frac{dR}{dI_0}\right)\times f_-(I_0-R)=0. \tag{15}$$

Note the important difference in the behavior of the resonant electrons when the wave is slowing down, $du/dx<0$ ($dN/dx<0$), and when it is speeding up, $du/dx>0$ ($dN/dx>0$). Ultimately this difference causes irreversible changes in the distribution function. As the phase velocity decreases, the bulk of the electrons, including the thermal electrons $V\approx v_T$, remain nonresonant, $V<u$. These particles do not cross the separatrix, and their distribution function obviously is always equal to $f_-=f_0(I_-)$ [cf. Eq. (9)]. On the other hand, part of the tail electrons which become resonant cross the separatrix and enter the ranks of those that overtake the wave $V>u$, $f_-=f_+$. It is important to note that not one of them can remain for any length of time in the trapping region $W<\Phi_{\max}$, since the trapped electrons, even if there happen to be any, have a tendency to spill out. Specifically, the constancy of J [cf. Eq. (7)] implies that if u decreases then W grows. Hence for $dN/dx<0$ in the region of phase space corresponding to the trapped particles a distinctive electron "void" develops, drifting along with the wave in the direction of smaller velocities, as shown in Fig. 2a (BGK structures having the form of electron voids were considered in Refs. 51 and 52). Substituting $F=0$, $f_-=f_0$ in (15) we find the form of the function f_+:

$$f_+(I_0+R)=f_0(I_0-R)(1-dR/dI_0)/(1+dR/dI_0). \tag{16}$$

If the wave amplitude is small, then we have $R\ll I_0$ and (16) can be expanded. We retain only the first correction $\sim R$, which is proportional to the square root of the amplitude. Then for $dN/dx<0$ we find

$$\begin{aligned}
&f_-(I_-)=\begin{cases} f_0(I_-), & V<0,\\ f_0(I_-), & V>0,\quad I_-<I_-^0,\end{cases}\\
&f_+(I_+)=f_0(I_+)+g_-(I_+),\quad I_+>I_+^0,\\
&g_-(I_0)\equiv-2(d/dI_0)[R_-(I_0)f_0(I_0)],\\
&F=0,
\end{aligned} \tag{17}$$

where the subscript "$-$" indicates that the function $R_-(I_0)$ given parametrically by the expressions (12) and (14) is determined by the wave variables in the range $dN/dx<0$.

At the density minimum $N_m=N(x_m)$ the phase velocity also attains a minimum $u_m=u(x_m)$. After passing the minimum (see Fig. 1, traces *2* and *3*) it begins to increase for $dN/dx>0$. Now a particle crossing the separatrix $W=\Phi_{\max}$ in the reverse direction can be trapped by the wave, since according to (7) the trapped electrons descend to the bottom of the wave potential well as u increases (Ref. 39; cf. Fig. 2b). Thus, where the density gradient is positive, part of the electrons which overtake the wave and go into resonance with it then fall behind, and part of them are trapped by the wave, $f_+\to f_-$, F. The increments in particle number in the three regions of phase space are proportional to the corresponding changes in the phase volume.[15,24] Then from (15) as before, using the small amplitude of the wave, we find

$$\begin{aligned}
&f_-(I_-)=\begin{cases} f_0(I_-), & V<0,\\ f_0(I_-), & V>0,\quad I_-<I_m\equiv I_-^0(x_m),\\ f_0(I_-)+g_+(I_-), & V>0,\quad I_m<I_-<I_-^0,\end{cases}\\
&g_+(I_0)\equiv-2(d/dI_0)[R_-(I_0)f_0(I_0)]+2R_+(I_0)\times(df_0(I_0)/dI_0),\\
&f_+(I_+)=f_0(I_+)+g_-(I_+),\quad I_+>I_+^0,\\
&F(J)=\begin{cases} 0, & J<R_m\equiv R(x_m),\\ f_0[I_0(R_+)]|_{R_+=J}, & R_m<J<R.\end{cases}
\end{aligned} \tag{18}$$

The subscript "+" on $R_+(I_0)$ indicates that in Eqs. (12)–(14) we must use the wave parameters in the region $dN/dx>0$. The subscript "m" is affixed to quantities at the point x_m where the density is a minimum. Expressions (17) and (18) can also be derived by calculating the probabilities for a resonant particle to pass from one region of phase space to another, as was done in Refs. 22 and 24 in treating the time evolution of a spatially periodic wave.

If the phase velocity again becomes large, $u\gg v_T$, as N increases, the function f_+ becomes exponentially small everywhere, and as in the beginning of the evolution we can assume $f_+=0$. Thus, as a result of traversing the region of the plasma density gradient the wave leaves a "trace" in the form of a perturbation in the electron distribution in the range of velocities where it passed. Moreover, a small fraction of the particles is trapped and entrained by the wave to large velocities. Even if the plasma density profile is absolutely symmetric as in Fig. 1 (trace *2*), when it leaves the gradient region a beam of trapped particles develops. For $u\gg u_m$ the trapped electrons, after dropping to the bottom of the potential well, form phase-synchronized bunches. From Eqs. (7), (10), and (18) their distribution in the phase plane assumes the form of a thin ring with a sharp interior boundary, corresponding to the onset of the capture process at the point x_m and a rapid decay in the direction of the separatrix corresponding to the monotonic decrease of $f_0(V)$.

4. MOMENTS OF THE DISTRIBUTION FUNCTION

To close the description of the spatial evolution of the wave it suffices to calculate the current density j (or the density n) and the average energy flux $\langle S\rangle$. Nevertheless, to complete the picture it is useful to also find expressions for the average electron density $\langle n\rangle$ and average energy density $\langle\mathscr{E}\rangle$ in the system. In this section we calculate these basic moments of the distribution function.

First of all, we write the electric potential as the sum of the electrostatic part $\Phi_s(x)$, which in a weakly inhomogeneous plasma is generally nonzero but small due to the quasineutrality of the plasma, and the wave perturbation $\tilde\Phi(x,\psi)$, $\Phi=\Phi_s+\tilde\Phi$, where $\langle\Phi\rangle=\Phi_s$, $\langle\tilde\Phi\rangle=0$. In what follows it is convenient to write $\tilde\Phi$ in the form

$$\tilde\Phi=2A(\alpha-p),\quad p\equiv\langle\alpha\rangle, \tag{19}$$

where $A(x)\equiv(\tilde\Phi_{\max}-\tilde\Phi_{\min})/2$ is the wave amplitude and $\alpha=\alpha(x,\psi)$ is the function describing the shape of the wave.

This function is even in ψ to leading order in ε (to which order we will restrict ourselves everywhere) and varies over the range $0 \leqslant \alpha \leqslant 1$ (for more details see Sec. 5). Going over to the integration variable $w = W - \Phi_s + 2Ap = (V-u)^2/2 + 2A\alpha$, we find the desired moments in the form

$$j = \int_{-\infty}^{\infty} dVVf = \int_{2A\alpha}^{\infty} \frac{dw}{|v|}[(u+|v|)f_+ + (u-|v|)f_-],$$

$$|v| \equiv \sqrt{2(w-2A\alpha)},$$

$$\langle n\rangle = \left\langle \int_{-\infty}^{\infty} dVf \right\rangle = \left\langle \int_{2A\alpha}^{\infty} \frac{dw}{|v|}(f_+ + f_-) \right\rangle,$$

$$2\langle \mathcal{E}\rangle = \left\langle \int_{-\infty}^{\infty} dVV^2 f \right\rangle = \left\langle \int_{2A\alpha}^{\infty} \frac{dw}{|v|}[(u+|v|)^2 f_+ + (u-|v|)^2 f_-] \right\rangle, \tag{20}$$

$$2\langle S\rangle = \left\langle \int_{-\infty}^{\infty} dVV^3 f \right\rangle = \left\langle \int_{2A\alpha}^{\infty} \frac{dw}{|v|}[(u+|v|)^3 f_+ + (u-|v|)^3 f_-] \right\rangle.$$

4.1. In the region of negative gradients $dN/dx < 0$ we use (17) to find $j = j_0 + j_1$, where

$$j_0 = \int_{2A}^{\infty} \frac{dw}{|v|}[(u+|v|)f_0(I_+) + (u-|v|)f_0(I_-)],$$

$$j_1 = \int_{2A}^{\infty} \frac{dw}{|v|}(u+|v|)g_-(I_+), \tag{21}$$

$$I_\pm = u^2/2 + \Phi_s + w - 2Ap \pm u\langle|v|\rangle. \tag{22}$$

Here we have omitted the simple but rather lengthy procedure of evaluating the integrals, although clearly a number of comments should be made about the technique employed. In considering small-amplitude waves it is convenient to break up the range of integration into two parts, e.g., $2A < w < \sqrt{A}$ and $\sqrt{A} < w < \infty$. In the nonresonant region both the distribution function and $|v| = \sqrt{2(w-2A\alpha)}$ can be expanded in powers of $\sqrt{A}$. In evaluating the resonant contribution we make use of the narrowness of the range of integration and expand f_0. Then the terms in j_0 proportional to $A^{(2m+1)/4}$ $(m=1,2,3,\ldots)$, associated with the artificially introduced boundary $w = \sqrt{A}$, cancel one another. Finally, in calculating j_1 it is necessary to take into account the fact that $g_- \sim \sqrt{A}$ is small. For small values of A, restricting ourselves to the leading terms, we find

$$j_0 = 2Au(\alpha - p)P - 4u\sqrt{A}f_0(u)\sqrt{1-\alpha},$$

$$j_1 = 4u\sqrt{A}f_0(u)\frac{\langle\sqrt{1-\alpha}\rangle}{\sqrt{1-\alpha}\langle 1/\sqrt{1-\alpha}\rangle}, \tag{23}$$

$$P \equiv \int_{-\infty}^{\infty} \frac{dV}{V-u}\frac{\partial f_0}{\partial V}, \tag{24}$$

where $f_0(u)$ is the value of the unperturbed distribution function at the resonance and the integral (24) is understood to be a principal value.

The average moments of the distribution function (20) are calculated in a similar way. Neglecting small corrections due to the weak anharmonicity of the wave (see Sec. 5), we can further simplify the calculations by assuming that the wave is sinusoidal $\alpha = (1/2)(1-\cos\psi)$, $p = 1/2$. Then for the perturbations of the moments we find

$$\langle n\rangle - \int_{-\infty}^{\infty} dVf_0 = \frac{A^2}{4}\frac{\partial}{\partial u}\frac{P-Q}{u} - \frac{8}{\pi}\int_u^1 \frac{dVf_0(V)}{V}\sqrt{A_-(V)} + \frac{64}{9\pi}\frac{A^{3/2}}{u}\frac{\partial f_0}{\partial u}, \tag{25}$$

$$\langle \mathcal{E}\rangle - \int_{-\infty}^{\infty} dV\frac{V^2}{2}f_0 = -\frac{A^2}{8}\frac{\partial}{\partial u}(uP) + \frac{4}{\pi}\int_u^1 dVVf_0(V)\sqrt{A_-(V)} - \frac{32}{9\pi}A^{3/2}u\frac{\partial f_0}{\partial u}, \tag{26}$$

$$\langle S\rangle = -\frac{A^2}{4}\frac{\partial}{\partial u}(u^2 P) + \frac{8}{\pi}\int_u^1 dVV^2 f_0(V)\sqrt{A_-(V)} - \frac{64}{9\pi}A^{3/2}u^2\frac{\partial f_0}{\partial u}, \tag{27}$$

where we have written

$$Q \equiv \int_{-\infty}^{\infty} \frac{dV}{V}\frac{\partial f_0}{\partial V}$$

and $A_- \equiv A_-(u)$ is the wave amplitude as a function of phase velocity in the region $dN/dx < 0$ determined by the closed set of equations describing the wave evolution (see Sec. 6), where $A_0 \equiv A_-(1)$ is the initial amplitude for $x = -\infty$, $u = 1$. The first terms on the right-hand sides of Eqs. (25)–(27) constitute the contribution of the bulk of the nonresonant electrons. The second terms describe the contribution $g_-(I_+)$ of the trace left by the wave in the tail of the distribution and associated with the electrons that have been resonant. The third terms are the contribution of $f_0(I_\pm)$ in the narrow resonant region. All three sets of terms are the leading contributions in the expansion of the corresponding quantities in half-integral powers of the amplitude.

For a Maxwellian distribution $f_0(V) = N/\sqrt{2\pi T}\exp(-V^2/2T)$, $N(x) = N_0\exp(-\Phi_s/T)$ the conditions for Eqs. (25)–(27) to be applicable take the form $A \ll T$, which is equivalent to $E_0^2/8\pi nT \ll (kr_D)^2$ in the generally accepted notation ($r_D = \sqrt{T/4\pi e^2 n}$), or $v_E \ll v_T$, where v_E is the half-width of the trapping region (Fig. 2). Assuming $(kr_D)^2 = T/u^2 \ll 1$ we can easily find the approximate relations

$$P=\frac{N}{u^2}\left(1+\frac{3T}{u^2}\right),\quad Q=-\frac{N}{T},$$

$$\frac{\partial}{\partial u}\frac{P-Q}{u}\simeq-\frac{\partial}{\partial u}\frac{Q}{u}=-\frac{N}{Tu^2},\quad \frac{\partial}{\partial u}(uP)=-\frac{N}{u^2},\quad (28)$$

$$\frac{\partial}{\partial u}(u^2P)=-\frac{6NT}{u^3}.$$

To interpret Eqs. (25)–(27) we supplement these relations with an expression for the mean energy density $\mathscr{E}_F=A^2/4u^2$ of the field of a harmonic wave. The perturbation in the energy density of the nonresonant component (below we use the superscript N) consists of the perturbation associated with the redistribution of the density (subscript D) in response to ponderomotive forces and the transported energy (subscript S). Similarly, we represent the contribution of the resonant particles (subscript R). Then the perturbation introduced by the wave can be written in the form of a sum of five terms $\mathscr{E}_W=\mathscr{E}_F+\mathscr{E}_D^{(N)}+\mathscr{E}_S^{(N)}+\mathscr{E}_D^{(R)}+\mathscr{E}_S^{(R)}$. Using Eqs. (25)–(28) we find

$$\mathscr{E}_D^{(N)}=Tn^N/2=-A^2N/8u^2,\quad n^{(N)}=-A^2N/4Tu^2,$$
$$S^{(N)}=v_g(\mathscr{E}_F+\mathscr{E}_S^{(N)})=3A^2NT/2u^3,\quad (29)$$
$$\mathscr{E}_S^{(N)}=A^2N/4u^2,\quad v_g\equiv 3T/u.$$

Thus, allowing for the contribution of the resonant particles, we can calculate the transported energy and the energy flux using well-known formulas (see, e.g., Refs. 53 and 54):

$$\mathscr{E}=\frac{\partial}{\partial\omega}(\omega\varepsilon_0)\frac{E_0^2}{16\pi},\quad S=-\omega\frac{\partial\varepsilon_0}{\partial k}\frac{E_0^2}{16\pi},$$

where ε_0, the dielectric function, satisfies

$$\varepsilon_0=1-\frac{4\pi e^2}{mk^2}\int_{-\infty}^{\infty}\frac{dV}{V-\omega/k}\frac{df_0}{dV}=0\quad (30)$$

the Vlasov dispersion equation.[55] Although this is quite natural, it is worth noting that by no means all distribution functions $f(W)$ invoked in constructing BGK solutions[27] guarantee this identity.

For small amplitudes the second term dominates among the resonances in Eqs. (25)–(27). Under the conditions $A\ll T$, $T/u^2\ll 1$, when the damping of the wave is small, we can take the function $\sqrt{A_-(V)}\simeq\sqrt{A(u)}$ out from under the integral and find

$$\mathscr{E}_D^{(R)}=n^{(R)}u^2/2=-(4/\pi)T\sqrt{A(u)}f_0(u),$$
$$n^{(R)}=-(8/\pi)(T/u^2)\sqrt{A(u)}f_0(u),$$
$$S^{(R)}=u\mathscr{E}_S^{(R)}=(8/\pi)Tu\sqrt{A(u)}f_0(u),\quad (31)$$
$$\mathscr{E}_S^{(R)}=(8/\pi)T\sqrt{A(u)}f_0(u).$$

Thus, we see that the energy of the wave is transported in the form of two components $\langle S\rangle=S^{(N)}+S^{(R)}$. The part associated with the wave motion of the thermal electrons propagates with the group velocity v_g. The other part, the resonant component due to the electron void, moves with the phase velocity. A similar division of the energy into two "blocks" for a wave packet was noted in Refs. 56 and 57, although there perturbation theory was used, i.e., an essentially different limiting case $\gamma/\omega_b\gg 1$ was being considered.

The question regarding the contribution of the resonant particles to the wave energy has been discussed in the literature.[58–60] A Van Kampen wave[4] can be regarded as the linear analog of a BGK wave. Nevertheless, it is not entirely clear how adequately the Van Kampen mode, a product of linear theory, describes a real physical object, i.e., a stationary (or quasistationary) periodic wave. Moreover, it is quite nontrivial to calculate nonlinear quantities using the theory of Van Kampen modes[56,61,62] and a number of troublesome questions remain.[63,64] In this connection the BGK method has the advantage that it allows us to determine the structure of the resonant electron distribution in all its details and to find the nonlinear quantities by direct calculation. As for the phenomena we have treated, they cannot be described at all in linear theory, inasmuch as all physical variables associated with particle resonances are connected one way or another with the bounce frequency and are proportional to half-integral powers of the amplitude.

The expressions we have found for the moments admit a simple physical interpretation (see also Ref. 14). An elementary event in which electrons interact resonantly with a wave at a time such that $u=V$ holds can be treated as a discontinuous increase in the particle velocity by an amount on the order of the width of the resonant region $\sim v_E=2\sqrt{A}$. Then, using the constancy of the average particle flux $f_+\simeq f_-V/(V+v_E)$, we can derive expressions for the perturbations of all moments of the distribution function through simple qualitative estimates. If in addition we assume that the discontinuity has magnitude $(4/\pi)v_E$ we find not just qualitative but quantitative agreement with rigorous calculations.

4.2. Now we consider the moments of the distribution function in the region where the density gradient is positive (see Fig. 1). We will assume that the wave phase velocity has reached some minimum value $u=u_m$ and then increased again to the values $u\gg v_T$, u_m. Then, setting $f_+=0$ in Eq. (18), we find using (20) that

$$j=2Au(\alpha-p)P-\frac{8}{\pi}\int_{u_m}^{u}dVf_0(V)\frac{\partial}{\partial V}[V\sqrt{A_+(V)}]$$
$$+2u\int_{2A\alpha}^{2A}dwF/\sqrt{2(w-2A\alpha)},\quad (32)$$

where the first term is the contribution of the bulk electrons, the second is due to the trace left by the wave in the tail of the distribution, and the third is the contribution of the beam of trapped particles. The function $A_+=A_+(u)$ is determined by the dependence of the amplitude on the phase velocity in the region $dN/dx>0$. Acceleration of trapped electrons is accompanied by the occurrence of recoil momentum in the plasma since $\langle j\rangle=0$ must hold, and the mean current $\langle j_T\rangle$ of the trapped particles is exactly balanced by the second term in (32), as can easily be shown by direct calculation. Thus we have

$$\mu\equiv\langle j_T\rangle=\frac{8}{\pi}\int_{u_m}^{u}dVf_0(V)\frac{\partial}{\partial V}[V\sqrt{A_+(V)}],\quad (33)$$

as a result of which (32) can be rewritten in the form

$$j=2Au(\alpha-p)P+2u\left[\int_{2A\alpha}^{2A}\frac{dwF}{\sqrt{2(w-2A\alpha)}}-\left\langle\int_{2A\alpha}^{2A}\frac{dwF}{\sqrt{2(w-2A\alpha)}}\right\rangle\right]. \tag{34}$$

The perturbations of the remaining moments of (20) are equal to [cf. Eqs. (25)–(27)]

$$\langle n\rangle-\int_{-\infty}^{\infty}dVf_0(V)=\frac{A^2}{4}\frac{\partial}{\partial u}\frac{P-Q}{u}-\frac{8}{\pi}\int_{u_m}^{u}dVf_0(V)\left[\frac{\sqrt{A_-}}{V}+\frac{\partial}{\partial V}\sqrt{A_+}\left(1-\frac{V}{\underline{u}}\right)\right], \tag{35}$$

$$\langle\mathscr{E}\rangle-\int_{-\infty}^{\infty}dV\frac{V^2}{2}f_0=-\frac{A^2}{8}\frac{\partial}{\partial u}(uP)+\frac{4}{\pi}\int_{u_m}^{u}dVf_0\times\left[V\sqrt{A_-}+\frac{\partial}{\partial V}\sqrt{A_+}V(\underline{u}-V)\right], \tag{36}$$

$$\langle S\rangle=-\frac{A^2}{4}\frac{\partial}{\partial u}(u^2P)+\frac{4}{\pi}\int_{u_m}^{u}dVf_0\times\left[2V^2\sqrt{A_-}+\frac{\partial}{\partial V}\sqrt{A_+}V(\underline{u}^2-V^2)\right], \tag{37}$$

where the terms describing the contribution of the trapped electrons have been underlined. For a Maxwellian distribution f_0 the main contribution to the integral comes from small velocities $V\simeq u_m$, where the number of particles is not too small and the energy exchange between waves and particles is strongest. For this region, when $u\gg v_T$, u_m holds, the upper limit of the integral is irrelevant and it can be replaced by ∞. By assuming $V\simeq u_m\ll u$ we can estimate the relative contributions of the trapped electrons and of the trace left by the wave in the tail of the distribution. Comparing the terms in square brackets we see that the contribution of the trapped particles to the density perturbation is small, whereas in the expressions for the energy density and the flux the underlined terms dominate. Thus, the energy balance in the subsequent evolution of the wave is determined by the interaction with the beam of trapped electrons pulled out of the tail of the distribution and entrained by the wave. The division into two components of the energy flux in the system consisting of the wave and the trapped particles is even more obvious than in the case of an electron void.

5. NONLINEAR DISPERSION RELATION

For a closed description of the spatial evolution of a wave it is necessary to determine the shape of the self-consistent potential $\alpha(\psi,x)$ [cf. Eq. (19)]. Using Eqs. (2) and (19) and also the continuity equation $\partial n/\partial t+\partial j/\partial x=0$ we can show that to leading order in ε the second and third equations of Eqs. (1) both yield

$$2A\,\partial^2\alpha/\partial\psi^2+uj=0. \tag{38}$$

5.1. In the region $dN/dx<0$ we can use Eqs. (21)–(23) to arrive at a nonlinear oscillator equation of the form

$$\frac{1}{C}\frac{\partial^2\alpha}{\partial\psi^2}+(\alpha-p)-\left(\frac{3b}{4}\right)\times\left(\sqrt{1-\alpha}-\frac{2}{\sqrt{1-\alpha}}\right)=0. \tag{39}$$

$$p\equiv\langle\alpha\rangle,\quad z\equiv\langle\sqrt{1-\alpha}\rangle/\langle 1/\sqrt{1-\alpha}\rangle, \tag{40}$$

where we have written $C\equiv u^2P$ [cf. Eq. (24)] and $b\equiv 8u^2f_0(u)/3\sqrt{A}C$. The term proportional to in Eq. (39) describes the contribution of an electron void. The presence of this term implies that there is another nonlinear phenomenon associated with resonant particles: the potential profile deviates from sinusoidal. If the anharmonicity parameter satisfies $b=0$, which holds, e.g., at the start of the evolution when $f_0(u)$ is negligible, then the solution of (39) is trivial: $\alpha=(1/2)(1-\cos\psi)$, $p=1/2$, where the condition that the waves vary periodically in time yields the Vlasov dispersion relation[55] [cf. Eq. (30)] $C=1$. The difficulties in solving Eq. (39) for $b\neq 0$ arise because the coefficients p and z, which are functions of α, are not known in advance. The problem simplifies somewhat in the case of weak anharmonicity $b\ll 1$, since in the limit $b\rightarrow 0$ the functional z must also go to zero because $\langle(1-\alpha)^{-1/2}\rangle$ diverges for a harmonic wave. Nevertheless, we cannot completely disregard the term proportional to z, because it is singular for $\alpha=1$. This singularity is closely related to the behavior of the distribution function and the wave potential close to the hyperbolic singular points $\psi=\pm\pi$, $V\equiv u$ in the phase plane (Fig. 2).

The procedure for solving the nonlinear oscillator equation is well known (cf., e.g., Refs. 22, 26, and 27). Multiplying (39) by $\partial\alpha/\partial\psi$ and integrating with respect to ψ we find

$$C^{-1}(\partial\alpha/\partial\psi)^2+\alpha(\alpha-2p)+b\sqrt{1-\alpha}\times(1-\alpha-z)=H=\text{const}. \tag{41}$$

Noting that $\partial\alpha/\partial\psi=0$ holds at the minimum $\alpha=0$, $\psi=0$ and maxima $\alpha=1$, $\psi=\pm\pi$ of the potential, we find the relation

$$H=b(1-z)=1-2p, \tag{42}$$

and eliminating p and H by means of these relations we reduce (41) to the form

$$C^{-1/2}\frac{\partial\alpha}{\partial\psi}=\begin{cases}-\sqrt{U}, & -\pi\leq\psi\leq 0,\\ \sqrt{U}, & 0\leq\psi\leq\pi,\end{cases} \tag{43}$$

with an effective nonlinear oscillator potential

$$U(\alpha)=\alpha(1-\alpha)\left[1+b\left(\frac{1-\sqrt{1-\alpha}}{\alpha}\right)\times\left(1+\frac{z}{\sqrt{1-\alpha}}\right)\right]. \tag{44}$$

The condition for periodicity in ψ and expressions (40) additionally enable us to write

$$\pi\sqrt{C}=\int_0^1\frac{d\alpha}{\sqrt{U}},\quad p=\int_0^1\frac{d\alpha\alpha}{\sqrt{U}}\Big/\int_0^1\frac{d\alpha}{\sqrt{U}},$$

$$z=\int_0^1\frac{d\alpha\sqrt{1-\alpha}}{\sqrt{U}}\Big/\int_0^1\frac{d\alpha}{\sqrt{U}\sqrt{1-\alpha}}. \tag{45}$$

Now the search for a solution reduces to determining the functions $p(b)$ and $z(b)$ satisfying (42) and (45). Analysis of the behavior of the integrals for small values $b \ll 1$ reveals $z\ln(8/bz) \simeq 1$, $p \simeq (1-b)/2$, and $\sqrt{C} \simeq 1 - 4b/\pi$. The last relation, which determines the resonant correction, allows us to write the nonlinear dispersion relation

$$1-u^2\int_{-\infty}^{\infty}\frac{dV}{V-u}\frac{\partial f_0}{\partial V}-\frac{64}{3\pi}\frac{u^2 f_0(u)}{\sqrt{A}}=0, \tag{46}$$

where the third term is the contribution of the electron void. Using (43) we can easily show that the wave profile $\alpha(\psi)$ deviates little from sinusoidal in shape everywhere except a small neighborhood of the potential hill $|\psi-\pi| \lesssim bz$, where instead of the quadratic dependence $\alpha = 1-(\pi-\psi)^2/4$ for a sine curve we have $\alpha = 1-[(\pi-\psi)/4]^{4/3}(bz)^{2/3}$.

Equation (46) can also be treated as the leading terms in an expansion in $\sqrt{A} \ll 1$ of the nonlinear dielectric function

$$\varepsilon_N = 1-u^2\int_{-\infty}^{\infty}\frac{dVf_0(V)}{(V-u)^2+9(\pi/8)^4A}=0,$$

omitting here the symbol for a principal value integral, although it is difficult to judge how useful this formula is in the case of large amplitudes.

5.2. Now we consider the solution of (38) in the region of positive gradients $dN/dx>0$. Substituting j from (34) we find

$$C^{-1}\frac{\partial^2\alpha}{\partial\psi^2}+(\alpha-p)+\frac{u^2}{AC}\left[\int_{2A\alpha}^{2A}\frac{dwF}{\sqrt{2(w-2A\alpha)}}\right.$$
$$\left.-\left\langle\int_{2A\alpha}^{2A}\frac{dwF}{\sqrt{2(w-2A\alpha)}}\right\rangle\right]=0.$$

The distribution function for the trapped electrons is given by (18). The sequence of operations described above again leads to (43), but now with an effective potential

$$U(\alpha)=\alpha(1-\alpha)\left\{1+\frac{u^2}{CA^2\alpha(1-\alpha)}\right.$$
$$\times\left[\int_{\max(w_0,2A\alpha)}^{2A}dwF\sqrt{2(w-2A\alpha)}\right.$$
$$\left.\left.-(1-\alpha)\int_{w_0}^{2A}dwF\sqrt{2w}\right]\right\}, \tag{47}$$

where the minimum value w_0 of the particle energy is determined by the condition (7) that the adiabatic invariant be constant [cf. also Eqs. (12) and (18)],

$$u\int_{-\psi_*}^{\psi_*}\frac{d\psi}{2\pi}\sqrt{2(w_0-2A\alpha)}=R_m=2u_m\sqrt{A_m}$$
$$\times\langle\sqrt{1-\alpha(\psi,x_m)}\rangle,$$
$$w_0=2A\alpha(\pm\psi_*,x), \tag{48}$$

and corresponds to particles trapped at $u=u_m$ at the point x_m where the density minimum N_m is located.

In contrast to the case of an electron void, the nonlinear correction to U due to a beam of trapped electrons [the second term in basis in Eq. (47)] is small for all values $0 \leqslant \alpha \leqslant 1$ if the number of trapped particles is not too large. It can easily be shown that the anharmonicity of the wave is always small if the current density of the beam [cf. Eq. (33)] satisfies the inequality

$$u\mu/A\psi_* \ll 1. \tag{49}$$

Physically Eq. (49) means that the repulsive Coulomb self-field in the electron bunch is small in comparison with the electric field of the wave. This condition is most stringent for trapped electrons concentrated close to the bottom of the potential well $\psi=0$ where the wave field is also small. When (49) fails to hold the electron bunch begins to spread out, "pushing apart" the walls of the potential well, which causes the wave potential $\alpha(\psi)$ to be deformed.

Thus, under condition (49) the wave is almost sinusoidal, $\alpha \simeq (1/2)(1-\cos\psi)$. We can find the correction to the wave profile without difficulty using (43). However, we are primarily interested in the contribution of the trapped particles to the dispersion relation. Substituting (47) in the first of Eq. (45), expanding $\sqrt{U}$, and integrating with respect to α, we arrive at the relation

$$C^{1/2}=1+\frac{2u^2}{\pi A^{3/2}C}\int_{w_0}^{2A}dwF(J)[2E(\kappa)-K(\kappa)], \tag{50}$$

where $K(\kappa)$ and $E(\kappa)$ are complete elliptic integrals of the first and second kind with modulus $\kappa \equiv \sqrt{w/2A}$. It is convenient to transform to the integration variable J [cf. Eq. (7)]. Using the expressions

$$J=(4/\pi)u\sqrt{A}[E(\kappa)-(1-\kappa^2)K(\kappa)],$$
$$R=(4/\pi)u\sqrt{A},\quad R_m=(4/\pi)u_m\sqrt{A_m}, \tag{51}$$

in the conditions for weak anharmonicity and setting $C \simeq 1$ in the nonlinear correction, we find

$$C^{1/2}=1+\frac{2u}{A}\int_{R_m}^{R}dJF(J)[2E(\kappa)/K(\kappa)-1]. \tag{52}$$

Since $F(J)$ falls off exponentially as J increases above R_m (cf. Sec. 3), Eq. (52) can be further simplified by removing the factor

$$2E(\kappa)/K(\kappa)-1 \simeq g(\kappa_0) \equiv 2E(\kappa_0)/K(\kappa_0)-1,$$
$$\kappa_0 \equiv \sqrt{w_0/2A}, \tag{53}$$

from underneath the integral and noting that the remaining integral is equal to half the average current density μ of the trapped particles. Finally, we find $\sqrt{C} = 1 + \mu u g(\kappa_0)/A$ or

$$1-u^2\int_{-\infty}^{\infty}\frac{dV}{V-u}\frac{\partial f_0}{\partial V}+\frac{2\mu u}{A}g(\kappa_0)=0, \tag{54}$$

where the energy level w_0 of the trapped electrons is determined according to (48), (51), and (53) by the expressions

$$u\sqrt{A}r(\kappa_0)=u_m\sqrt{A_m},\quad r(\kappa) \equiv E(\kappa)-(1-\kappa^2)K(\kappa). \tag{55}$$

To conclude this section we note that using Eqs. (8)–(10) and the calculations described above, we can also easily find the correction to the dispersion relation associated with the nonlinearity of the nonresonant part of the component. If there are no resonant particles the nonlinear dispersion relation takes the form $N=1\sim(3T/u^2)(1+2T/u^2+A^2/u^4)$, from which it follows that for weak waves $A/u^2\ll 1$ the nonlinear correction $\sim TA^2$ is always much smaller than the thermal correction (see also Refs. 65 and 66).

6. SPATIAL EVOLUTION OF THE WAVE

The description of the wave evolution in a weakly inhomogeneous plasma reduces to finding the spatial dependence of the amplitude A and the phase velocity u (wave number $k=u^{-1}$). To this end we use the closed system consisting of the energy balance equation and the nonlinear dispersion relation. These two conditions determine $A=A(N)$ and $u=u(N)$ as functions of the local plasma density and thus solve the specified problem, regardless of the specific form of the density profile $N=N(x)$. Consider a typical case $|N-1|\ll 1$ in which the thermal and resonant directions in the dispersion relation are small. To be specific we assume that the unperturbed distribution function f_0 is Maxwellian.

6.1. In the region where the gradient is negative, $dN/dx<0$ (see Fig. 1), expanding the integral and retaining the first thermal correction in the limit $T/u^2\ll 1$ ($k^2r_D^2\ll 1$) [cf. Eq. (28)], we can write Eq. (46) in the form

$$N=1-3T/u^2-(64/3\pi)u^2f_0(u)/\sqrt{A}. \tag{56}$$

Note that the sign of the resonant correction is the same as that of the thermal correction. Thus, in the process by which a void forms the resonant electrons bunch mainly in the vicinity of the potential maximum $\alpha=1$, $\psi=\pm\pi$, in phase with the thermal particles.

The damping of a small-amplitude wave on an electron void is described according to Eqs. (27) and (28) by the energy balance equation

$$\frac{3TA^2}{2u^3}+\frac{8}{\pi}\int_u^1 dVV^2f_0(V)\sqrt{A_-(V)}=\langle S\rangle=\langle S_0\rangle$$

$$=\frac{3}{2}TA_0^2, \tag{57}$$

where we have written $A=A_-(u)$, and $u=1$, $A(1)=A_0$ holds at infinity ($x=-\infty$).

In the initial stage $u\lesssim 1$ we can ignore the damping, since the number of resonant electrons is exponentially small, $f_0(1)\simeq 0$. As N decreases the phase velocity falls off in accordance with the linear dispersion relation and we have $A\simeq A_0u^{3/2}$. By virtue of the exponential dependence of $f_0(V)$ the wave experiences a perceptible damping if the phase velocity $u(N)$ drops to values on the order of a few times the thermal velocity $v_T\ll 1$. Regarding u as an independent variable and differentiating (57) we arrive at the equation

$$(d/du)[A^2(u)/u^3]=(16/3\pi)u^2f_0(u)\sqrt{A(u)}/T, \tag{58}$$

which with certain provisos is the same as the similar expression derived in Ref. 17 by a different method. The approximate solution (58) takes the form

$$A=A_0u^{3/2}\left[1-\left(\frac{4}{\pi}\right)u^{7/4}\frac{f_0(u)}{A_0^{3/2}}\right]. \tag{59}$$

According to (59), damping develops for $u^2f_0(u)/A^{3/2}\simeq 1$, or in ordinary notation for $(\gamma/\omega)(kr_D)^2\simeq(\omega_b/\omega)^3$, where γ is the linear damping rate. Note, however, that the connection with Landau damping in this case is somewhat illusory, since the damping of a wave on an electron void is determined by the value of the distribution function itself at the resonance, rather than its derivative, and so occurs even for $\partial f_0/\partial u=0$. Nevertheless, for a Maxwellian distribution it is sometimes convenient in estimates to use $\gamma/\omega=(\pi/2)u^3f_0(u)/T$ instead of $f_0(u)$.

Equation (59) says that for some value $u=u_H$ the amplitude formally goes to zero, i.e., the wave is totally damped (see also Ref. 37). For small amplitudes the contribution of an electron void to the dispersion relation (56) can be considerable. For $\gamma\simeq\omega_b$ it is comparable with the thermal correction. One should also keep in mind that for a specified density gradient of length L the adiabaticity condition must fail sooner or later as the amplitude decreases, so that the damping goes over to the Landau regime.[14] But if the value of L is very large, then, generally speaking, it is desirable to analyze the instability of this BGK wave. In any case, the damping implies that the wave cannot have a velocity below the value u_H determined by the relation $u_H^{7/4}f_0(u_H)=\pi A_0^{3/2}/4$, if this value is reached for the specified density profile, i.e., $u_H>u_m=u(x_m)$.

6.2. Now we assume that $u_m>u_H$ holds for the given density profile, and the wave does not reach the strong-damping point. Then en route from the density minimum at x_m it traps a certain fraction of the resonant particles. Many of the electrons are trapped near the minimum x_m, where the wave amplitude satisfies $A_m\simeq A_0u_m^{3/2}$. From (33) the current density of the trapped particles is approximately equal to

$$\mu=(14/\pi)\sqrt{A_m}Tf_0(u_m)/u_m=(28/\pi^2)(\gamma_m/\omega)$$

$$\times(\omega_{bm}/\omega)(k_mr_D)^4u_m. \tag{60}$$

In practical calculations it is also convenient to use the ratio of this quantity to the total electron current density in the range of velocities traversed by the wave, $V\geq u_m$:

$$\mu/\int_{u_m}^1 dVVf_0(V)\simeq\frac{14}{\pi}\frac{\sqrt{A_m}}{u_m}=\frac{14}{\pi}\frac{\omega_{bm}}{\omega}.$$

From (54) in analogy with (56) we find the dispersion relation

$$N=1-3T/u^2+2\mu ug(\kappa_0)/A. \tag{61}$$

As the phase velocity increases for $u\gg v_T$, u_m the conservation of the adiabatic invariant (55) implies that the trapped particles settle to the bottom of the potential well $\kappa_0\to 0$, $r\to(\pi/4)\kappa_0^2$, $g\to 1$. In contrast to the case of the electron void, they bunch up at the antiphase ($\psi=0$) with respect to the thermal electrons; this is also shown by the plus sign of the resonant correction. Because of this, in particular, the wave loaded with the trapped electrons can penetrate into the supercritical regions of the plasma $N>1$ (Refs. 30, 31, 35, and 50), contrary to the conclusions of Ref. 16.

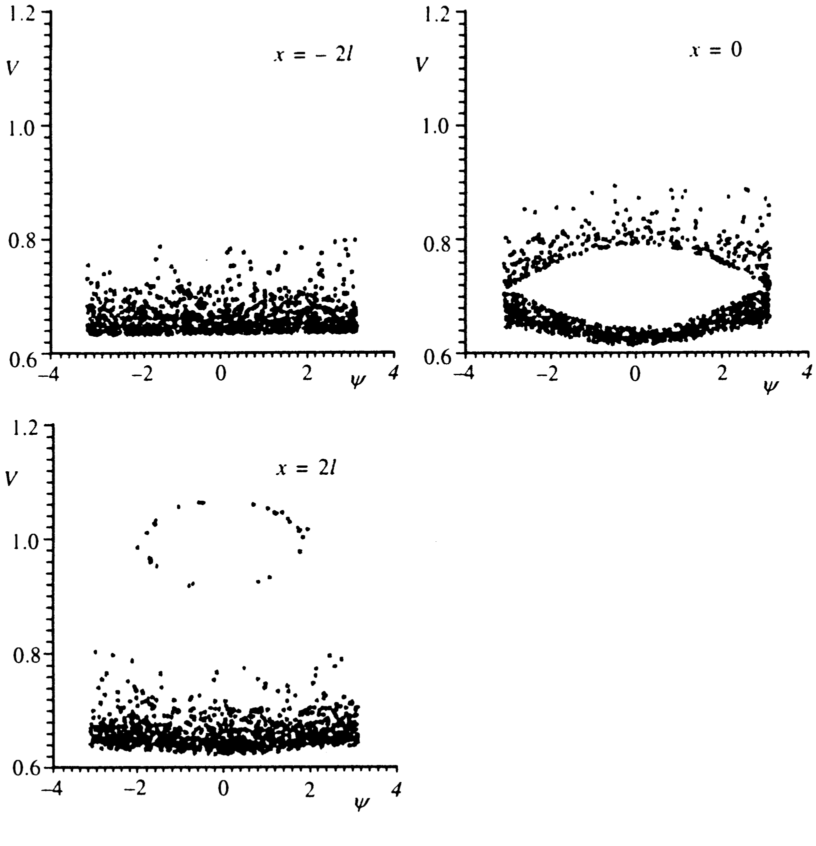

FIG. 3. Spatial evolution of the electron distribution in the (ψ, V) phase plane. For each position x one wave period $-\pi \leqslant \psi \leqslant \pi$ is shown.

The subsequent evolution of the wave as the plasma density increases is described using Eqs. (53), (55), and (61) and the equation for the conservation of the average energy flux [cf. Eqs. (28), (33), (37), and (60)]:

$$3TA^2/2u^3+\mu u^2/2=\langle S\rangle \simeq \langle S_0\rangle = 3TA_0^2/2 \simeq 3TA_m^2/2u_m^3. \quad (62)$$

From (62) we see that because of the damping on the trapped particles the phase velocity of the wave cannot exceed the value $u_B = \sqrt{3TA_0^2/\mu}$. The ratio of the maximum to the minimum velocity can be regarded as a dimensionless parameter which characterizes the strength of the wave,

$$\frac{u_B}{u_m}=a\equiv\sqrt{\frac{3TA_0^2}{\mu u_m^2}}=\sqrt{\frac{3TA_m^2}{\mu u_m^5}} = \left[\frac{3\pi^2}{28}\left(\frac{\omega_{bm}}{\omega}\right)^3\left(\frac{\gamma_m}{\omega}\right)^{-1}\frac{1}{(k_m r_D)^2}\right]^{1/2} \gg 1.$$

As the amplitude decreases near the point $u=u_B$ the trapped electrons rise from the bottom of the well, and for $A/A_m=(u_m/u)^2 \simeq a^{-2}$ they spill out of the potential wells $\kappa_0=1$, $r(\kappa_0)=1$ by virtue of Eq. (55). As in the case of an electron void, a substantial drop in the amplitude can destroy adiabaticity and hence cause the spilling to occur earlier. If the waves are sufficiently intense, $\omega_{bm}/\omega > a^{-3/2}$, then even before the spilling occurs the nonlinearity of the wave dispersion can be manifested for $A/A_m \lesssim (\omega_{bm}/\omega)^2 a$, i.e., the contribution of the trapped particles to (61) becomes comparable with the thermal correction. In connection with propagation over long distances the criteria for stability of the wave[13,33,38] become important also when energy can be exchanged with the beam of trapped electrons.

Thus, for a specified plasma density profile, depending on the wave intensity A_0, two basic "scenarios" are possible for the evolution. A small-amplitude wave $a \ll 1$ is absorbed by the resonant particles without reaching the density minimum at the point $u=u_H \gg u_m$. In the opposite limit $a \gg 1$ a small group of resonant electrons are trapped near the minimum, and the subsequent acceleration in these particles causes the wave to be absorbed at the point where $u=u_B \gg u_m$ holds.

The formation of a beam of accelerated trapped particles constitutes a clear example of the occurrence of irreversibility in a dissipationless system, and stems from the violation of adiabaticity in the motion when particles cross the narrow region of phase space adjacent to the separatrix (Fig. 2).[2,24] This effect, which appears somewhat unexpected, especially in the case of a symmetric density profile (Fig. 1b), is illustrated by the results of numerical simulation. Figure 3 displays the electron distribution in phase space found by numerical integration of the equations of motion for 1150 particles in the field of a wave with phase velocity varying according to $u(x)=1-\delta\exp(-x^2/l^2)$. The main purpose of the calculations is to exhibit the processes by which resonant structures develop. The problem was therefore solved in a non-self-consistent formulation and approximately describes the real situation if the resonant particles exert a weak effect on the wave, i.e., $A=A_0u^{3/2}$ and $N=1-T/u^2$. Furthermore, only the dynamics of the electrons in the tail of the Maxwellian distribution $V>V_{\min} \lesssim u_m$ was examined. The initial particle phases and velocities at $x=-2l$ were prescribed using a random-number generator. The parameters of the calculation were $v_T=0.2$, $A_0\equiv2.24\cdot10^{-3}$ ($A_m=1.36\cdot10^{-3}$), $l=3.2\cdot10^3$, $\delta=0.284$ ($u_m=0.716$), $V_{\min}=0.632$. The figure clearly shows the resonant structures we have been considering: an electron void at the point $x=0$, $u=u_m$ and a ring of trapped electrons at the exit from the region of density variation $x=2l$, $u\simeq1$, even though we used a fairly small characteristic scale for the variation of the phase velocity $l/l_b = l\sqrt{A}/2\pi u^2 \simeq 30$ in order to reduce running time.

7. CONCLUSIONS

Because of the rapid phase mixing of the resonant particles, the collisionless damping of plasma waves in a weakly inhomogeneous plasma is conveniently described as the smooth spatial evolution of a steady BGK wave. The presence of a small parameter ε makes it natural to use the adiabatic approximation to determine the electron distribution function, and allows us to avoid the customary linearization of the original equations. This yields an algorithm for the self-consistent solution of the nonlinear problem for an arbitrary smooth plasma density profile. This method permits a variety of nonlinear plasma wave effects to be described with considerable rigor, ranging from the effect of resonant particles on the dispersion of the wave to anharmonicity in the oscillations and the effects of the ponderomotive force. It is also worth noting that for a fully collisionless plasma it includes strongly irreversible processes by which structures develop in the resonant region of phase space. Since it is based on the conservation laws, this approach yields as a by-product simple expressions for the perturbations of all the basic moments of the distribution function in the form of sums of the contributions of resonant particles and the nonresonant component of the plasma, which facilitates a deeper understanding of the physics of these phenomena. In particular, it becomes clear that the wave–particle adiabatic interaction is quite similar to the energy exchange in a beam–plasma system, and the wave possesses properties of such a system close to a BGK equilibrium.

The final product of our analysis is a closed system of equations describing the amplitude and phase velocity of the wave as functions of the local plasma parameters, in the form of an energy balance equation and a nonlinear dispersion relation. Although the mathematical operations constituting a detailed derivation of this system can be somewhat lengthy, the final equations are simple and convenient to use.

In the examples we have treated, for a Langmuir wave propagating in the direction of decreasing plasma density in the region of phase space corresponding to trapped particles, a void develops which moves with decreasing phase velocity. Although the phase volume of the void decreases, the wave damping becomes stronger, since the number of untrapped resonant electrons surrounding the void is growing. Next, depending on the form of the plasma density profile and the shape of the unperturbed distribution function, two scenarios are possible for the evolution of the wave. The wave may reach the points where damping is strong, with a probable transition to the Landau regime, or if this point is not reached, it can pass through the density minimum and trap some of the electrons from the tail of the distribution. Then as the phase velocity increases on the positive density gradient the trapped electrons undergo acceleration, extracting energy from the wave. This process can play the role of an effective mechanism for the formation of high-energy particles in a plasma. The result of this energy exchange depends on the specific form of the density profile and can easily be estimated using the equations we have derived and the conditions for confinement of the trapped particles in the potential wells of the wave. In particular, we can determine the energy spectrum of the resulting fast particles.

In conclusion we note that the techniques used in the present calculations can easily be extended to other kinds of waves, e.g., circularly polarized waves in a magnetized plasma. This generalization is also possible for spatially bounded quasi-monochromatic wave packets. Finally, if certain difficulties of a computational nature are overcome, this algorithm is suited also for more intense waves than those treated above.

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