Time-dependent Schrödinger equation admitting an exact solution

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The one-dimensional Schrödinger equation with a time-dependent potential of the form $\alpha_1\delta(x-vt)+\alpha_2\delta(x+vt)$, which admits an exact analytical solution, is studied. © *1995 American Institute of Physics.*

1. Quantum-mechanical systems that admit exact solutions are of great interest and attract the attention of theoreticians (see, e.g., Ref. 1). Systems with a compact exact solution in the case of a Hamiltonian that depends nontrivially on time are especially rare. They include the well-known one-dimensional system with variable frequency,[2] for which a solution can be obtained without making any approximations (see also Ref. 3).

Zhdanov and Chikhachev[4] studied the solution of a system consisting of oppositely propagating δ-function potentials, i.e., a Hamiltonian with a potential of the form $\alpha(\delta(x-vt)+\delta(x+vt))$. Solutions were obtained explicitly, also without making use of any approximations.

In Ref. 5 it was shown that the Schrödinger equation can be solved exactly with a considerably more general potential of the form $\alpha_1\delta(x-vt)+\alpha_2\delta(x+vt)$, i.e., for a system that models oppositely propagating potential centers with different depths of the single bound state. The solutions obtained in Ref. 5 were used to describe ionization and charge-exchange phenomena. The work of Däppen[6] is also noteworthy. There the results of Ref. 4 were rederived in part, and it was shown in addition that besides the symmetric bound state in a system of oppositely propagating δ-function potentials there also exists an antisymmetric state. The results of Ref. 6 were also used to describe ionization and charge-exchange phenomena.

It is probably hard to find any nontrivial time-dependent system admitting an exact solution other than those mentioned—the oscillator with a variable frequency and oppositely propagating δ-function potentials.

The present work is devoted to the study of the solutions of the Schrödinger equation with oppositely propagating δ-function potentials. We will find, first, a solution corresponding to an oscillatory asymptotic form for the equation with $\alpha_1=\alpha_2$, and secondly, a solution of the Cauchy problem in the case $\alpha_1\neq\alpha_2$ for δ-function potentials moving apart from different points. These topics were not treated in Refs. 4–6.

2. Consider the Schrödinger equation in the following form:

$$\left\{ i\frac{\partial}{\partial t}+\frac{1}{2}\frac{\partial^2}{\partial x^2}+\alpha[\delta(x-vt)+\delta(x+vt)]\right\}\Psi(xt)=0. \tag{1}$$

Here x and t are the space and time coordinates, Ψ is the wave function of the particle, and the constant α characterizes the depth of the single bound state of a δ-function potential at rest.

Throughout we will use a system of units in which $e=\hbar=m=1$.

In the case $v=0$ there is a complete system of eigenfunctions (see, e.g., Ref. 7):

$$\psi_0=\sqrt{2\alpha}\,\exp(-2\alpha|x|)\,\exp(2i\alpha^2t), \tag{2}$$

$$\psi_k=\frac{1}{\sqrt{2\pi}}\left[\exp(ikx)+\frac{2i\alpha}{|k|-2i\alpha}\exp(i|k||x|)\right]\times\exp\left(-\frac{ik^2t}{2}\right), \tag{3}$$

where k is the wave number characterizing the state of a particle which is not bound.

In Refs. 4 and 6 solutions of (1) were studied corresponding to the bound state which goes over to (2) in the limit $v\to0$. In this section we show that there exist solutions with an oscillatory asymptotic form that go over to (3) in the limit $v\to0$.

Note that the solution (3) of Eq. (1) for $v=0$ can be found using a representation in the form of an incident plane wave and an outgoing wave.

We proceed similarly in the present case of a time-dependent problem ($v\neq0$), i.e., we set

$$\psi_k=\tilde\psi_k+\frac{1}{\sqrt{2\pi}}\exp\left(ikx-\frac{ik^2t}{2}\right). \tag{4}$$

For the outgoing wave $\tilde\psi_k$ we find the following equation:

$$i\frac{\partial\tilde\psi_k}{\partial t}+\frac{1}{2}\frac{\partial^2\tilde\psi_k}{\partial x^2}+\alpha(\delta(x-vt)+\delta(x+vt))\tilde\psi_k+\frac{\alpha}{\sqrt{2\pi}}\exp\left(-\frac{ik^2t}{2}\right)[\delta(x-vt)\exp(ikvt)+\delta(x+vt)\exp(-ikvt)]=0. \tag{5}$$

Then we write $z_\pm=|x\pm vt|$ and, as in Refs. 4 and 6 we set

$$\tilde\psi_k=\exp\left(\frac{iv^2t}{2}\right)\sum_s\{C_s^{(1)}e^{ivx}\varphi_s(z_-,t)+C_s^{(2)}e^{ivx}\varphi_s(z_+,t)e^{-ivx}\}. \tag{6}$$

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Substituting (6) in (5) and using the expression $\varphi_s(z,t)=\exp\{-a_s z+ia_s^2 t/2\}$ we arrive at the following system of equations:

$$\sum_s \left\{C_s^{(1)}(\alpha-a_s)\exp\left(\frac{ia_s^2 t}{2}\right)+C_s^{(2)}\alpha\exp\left[\frac{it}{2}(a_s+2iv)^2\right]\right\}$$

$$=-\frac{\alpha}{\sqrt{2\pi}}\exp\left[-\frac{it}{2}(k-v)^2\right],$$

$$\sum_s \left\{C_s^{(1)}\alpha\exp\left[\frac{it}{2}(a_s+2iv)^2\right]+C_s^{(2)}(\alpha-a_s)\exp\left(\frac{ia_s^2 t}{2}\right)\right\}$$

$$=-\frac{\alpha}{\sqrt{2\pi}}\exp\left[-\frac{it}{2}(k+v)^2\right]. \qquad (7)$$

Equations (7) differ from the analogous system of equations studied in Refs. 4 and 6, which describe bound states, only in being inhomogeneous.

We transform (7) as follows: rewrite the summation index $s\to s-1$ in the first term of the second equation and the second term of the first equation. We find the system

$$\sum_s \left\{C_s^{(1)}(\alpha-a_s)\exp\left(\frac{ia_s^2 t}{2}\right)+C_{s-1}^{(2)}\alpha\exp\left[\frac{it}{2}(a_{s-1}+2iv)^2\right]\right\}=-\frac{\alpha}{\sqrt{2\pi}}\exp\left[-\frac{it}{2}(k-v)^2\right],$$

$$\sum_s \left\{C_{s-1}^{(1)}\alpha\exp\left[\frac{it}{2}(a_{s-1}+2iv)^2\right]+C_s^{(2)}(\alpha-a_s)\times\exp\left(\frac{ia_s^2 t}{2}\right)\right\}=-\frac{\alpha}{\sqrt{2\pi}}\exp\left[-\frac{it}{2}(k+v)^2\right].$$

Since these relations hold for arbitrary t, we must have the recurrence relation $a_s=a_{s-1}+2iv$. Hence we find $a_s=2ivs+\text{const}$, where the constant is determined from the condition that the right-hand sides of the equations be consistent.

We set

$$a_s=i[-|k|+v(2s-1)]. \qquad (8)$$

Then we have $a_0=-i(|k|+v)$, $a_1=-i(|k|-v)$. Consider the case $k>0$.

The system (7) can be rewritten in the form

$$\sum_s [C_s^{(1)}(\alpha-a_s)+C_{s-1}^{(2)}\alpha]\exp\left(\frac{it}{2}a_s^2\right)=-\frac{\alpha}{\sqrt{2\pi}}\times\exp\left[-\frac{it}{2}(k-v)^2\right],$$

$$\sum_s \left[C_{s-1}^{(1)}\alpha+C_s^{(2)}(\alpha-a_s)\exp\left(\frac{it}{2}a_s^2\right)\right]=-\frac{\alpha}{\sqrt{2\pi}}\times\exp\left[-\frac{it}{2}(k+v)^2\right]. \qquad (9)$$

In order to get convergent series we must restrict ourselves to solutions $C_s^{(1)}C_s^{(2)}$ that vanish for negative s. We set $C_{-1}^{(1)}=C_0^{(1)}=0$.

Only two equations of the system (9) are inhomogeneous:

$$C_1^{(1)}(\alpha-a_1)+C_0^{(2)}\alpha=-\frac{\alpha}{\sqrt{2\pi}},$$

$$C_{-1}^{(1)}\alpha+C_0^{(2)}(\alpha-a_0)=-\frac{\alpha}{\sqrt{2\pi}}.$$

Hence we find

$$C_0^{(2)}=-\frac{\alpha}{\sqrt{2\pi}}\frac{1}{\alpha+i(k+v)},$$

$$C_1^{(2)}=-\frac{\alpha}{\sqrt{2\pi}}\frac{i(k+v)}{\alpha+i(k+v)}\frac{1}{\alpha+i(k-v)}.$$

For $s\neq 0$, 1 it follows from Eqs. (9) that

$$C_s^{(1)}(\alpha-a_s)+C_{s-1}^{(2)}\alpha=0,$$

$$C_{s-1}^{(1)}\alpha+C_s^{(2)}(\alpha-a_s)=0. \qquad (10)$$

Using (10) we find

$$C_2^{(2)}=-\frac{i(k+v)}{\sqrt{2\pi}\alpha}\frac{\alpha}{\alpha+i(k+v)}\frac{\alpha}{\alpha+i(k-v)}\frac{\alpha}{\alpha+i(k-3v)},$$

$$C_3^{(1)}=\frac{i(k+v)}{\sqrt{2\pi}\alpha}\frac{\alpha}{\alpha+i(k+v)}\frac{\alpha}{\alpha+i(k-v)}\frac{\alpha}{\alpha+i(k-3v)}\times\frac{\alpha}{\alpha+i(k-5v)}.$$

By virtue of the condition $C_{-1}^{(1)}=C_0^{(1)}=0$ the coefficients $C_s^{(1)}$ with even s and all $C_s^{(2)}$ with odd s vanish. For $C_s^{(1)}$ with odd s and $C_s^{(2)}$ with even s we can find without difficulty from (10) that

$$C_s^{(1)}=\left(\frac{\alpha}{2iv}\right)^{s-3}\frac{C_3^{(1)}\Gamma\left(4-\frac{k+v}{2v}-\frac{\alpha}{2iv}\right)}{\Gamma\left(s+1-\frac{k+v}{2v}-\frac{\alpha}{2iv}\right)},$$

$$C_s^{(2)}=\left(\frac{\alpha}{2iv}\right)^{s-2}\frac{C_2^{(2)}\Gamma\left(3-\frac{k+v}{2v}-\frac{\alpha}{2iv}\right)}{\Gamma\left(s+1-\frac{k+v}{2v}-\frac{\alpha}{2iv}\right)}. \qquad (11)$$

Thus, expression (4) together with (6) and the coefficients given by (11) provide the required solution.

For negative k ($k<0$) we must exchange $C_s^{(1)}\leftrightarrow C_s^{(2)}$.

It is of interest to note that in Eq. (6), which when added to the plane wave represents the solution of (1), terms inevitably appear corresponding to waves converging toward the moving δ-function centers. The characteristic momentum $p_s=|k|-v(2s-1)$ for large s is negative.

In the limit $v\to 0$, however, the solution only corresponds to a single diverging wave. This solution goes over to

(3) in the limit $v \to 0$, which can be seen most easily by setting $a_s = -i|k|$. This leads to expressions for $C_s^{(1)}$, $C_s^{(2)}$ such that the series (6) is easily summed.

We note further that it is of interest to clarify the question about the completeness of the system of functions found in this section, along with the functions corresponding to the bound state (see Ref. 6).

In the case $v=0$, as is shown in Ref. 7, the system of functions (2) and (3) is complete.

3. In this section we consider an equation more general than (1):

$$\left[i\frac{\partial}{\partial t}+\frac{1}{2}\frac{\partial^2}{\partial x^2}+\alpha_1\delta(x-vt)+\alpha_2\delta(x+vt)\right]\Psi(x,t)=0. \tag{12}$$

As we have noted, this equation models a system of two oppositely propagating single-level δ-function potential centers whose depths may differ.

To specify the Cauchy problem we must provide a value of Ψ at the initial time:

$$\Psi(x,t)|_{t=t_0>0}\equiv\varphi(x). \tag{13}$$

The problem consisting of Eqs. (12) and (13) is a generalization of that solved in Ref. 4. Here, however, we have $\alpha\neq\alpha_2$ and $t_0\neq 0$, i.e., the δ-functions are moving apart but not from the origin, and finally in the present case $\varphi(x)$ need not be symmetric: $\varphi(x)\neq\varphi(-x)$.

It is very convenient to look for a solution in the form:

$$\Psi(x,t)=\Psi_0(x,t)+\Psi_1(x,t)+\Psi_2(x,t), \tag{14}$$

where

$$\Psi_0(x,t)=\frac{1}{\sqrt{2\pi i(t-t_0)}}\exp\left[\frac{ix^2}{2(t-t_0)}\right]\int_0^\infty ds \times\exp\left[\frac{is^2}{2(t-t_0)}\right]\left[\varphi(s)\exp\left(-\frac{ixs}{t-t_0}\right)+\varphi(-s)\exp\left(\frac{ixs}{t-t_0}\right)\right], \tag{15}$$

$$\Psi_1(x,t)=\frac{1}{\sqrt{2\pi i(t-t_0)}}\exp\left[\frac{i(x-vt_0)^2}{2(t-t_0)}\right]\int_0^\infty dy \times\int_0^\infty ds\exp\left[\frac{i(s+iy)^2}{2(t-t_0)}+\frac{i(s+iy)}{t-t_0}|x-vt|\right] \times\{G_{11}(s,y)\chi_1(s)+G_{12}(s,y)\chi_2(s)\}, \tag{16}$$

$$\Psi_2(x,t)=\frac{1}{\sqrt{2\pi i(t-t_0)}}\exp\left[\frac{i(x+vt_0)^2}{2(t-t_0)}\right]\int_0^\infty dy \times\int_0^\infty ds\exp\left[\frac{i(s+iy)^2}{2(t-t_0)}+\frac{i(s+iy)}{t-t_0}|x+vt|\right] \times\{G_{21}(s,y)\chi_1(s)+G_{22}(s,y)\chi_2(s)\}. \tag{17}$$

Here

$$\chi_1(s)=\varphi(s)\exp\left(-\frac{ivts}{t-t_0}\right)+\varphi(-s)\exp\left(\frac{ivts}{t-t_0}\right),$$

$$\chi_2(s)=\varphi(s)\exp\left(\frac{ivts}{t-t_0}\right)+\varphi(-s)\exp\left(-\frac{ivts}{t-t_0}\right).$$

In addition to the variables of integration y,s the dependent functions G_{ik} can also depend on t and t_0, and of course on the parameters α_1, α_2, v.

It can easily be shown next that

$$\left(i\frac{\partial}{\partial t}+\frac{1}{2}\frac{\partial^2}{\partial x^2}\right)\Psi_0(x,t)=0,$$

$$\left(i\frac{\partial}{\partial t}+\frac{1}{2}\frac{\partial^2}{\partial x^2}\right)\Psi_1(x,t) =\frac{\delta(x-vt)}{\sqrt{2\pi i(t-t_0)}}\exp\left(iv^2\frac{t-t_0}{2}\right)\int_0^\infty ds\left\{\exp\left[\frac{is^2}{2(t-t_0)}\right] \times[G_{11}(s,0)\chi_1(s)+G_{12}(s,0)\chi_2(s)] +\int_0^\infty dy\exp\left[i\frac{(s+iy)^2}{2(t-t_0)}\right]\left[\frac{dG_{11}(s,y)}{dy}\chi_1(s) +\frac{dG_{12}(s,y)}{dy}\chi_2(s)\right]\right\},$$

$$\left(i\frac{\partial}{\partial t}+\frac{1}{2}\frac{\partial^2}{\partial x^2}\right)\Psi_2(x,t) =\frac{\delta(x+vt)}{\sqrt{2\pi i(t-t_0)}}\exp\left(iv^2\frac{t-t_0}{2}\right)\int_0^\infty ds\left\{\exp\left[\frac{is^2}{2(t-t_0)}\right] \times[G_{21}(s,0)\chi_1(s)+G_{22}(s,0)\chi_2(s)] +\int_0^\infty dy\exp\left[i\frac{(s+iy)^2}{2(t-t_0)}\right]\left[\frac{dG_{21}(s,y)}{dy}\chi_1(s) +\frac{dG_{22}}{dy}\chi_2(s)\right]\right\}.$$

Thus, by substituting Eqs. (14)–(17) in (12), equating the coefficients of $\delta(x\pm vt)$, and using the arbitrariness of the functions $\chi_{1,2}(s)$ we arrive at the following equations for G_{ik}:

$$-i\frac{dG_{11}}{dy}=\alpha_1G_{11}+\alpha_1\exp\left(\frac{2iv^2tt_0}{t-t_0}+2ivt\frac{s+iy}{t-t_0}\right)G_{21},$$

$$-i\frac{dG_{21}}{dy}=\alpha_2G_{21}+\alpha_2\exp\left(\frac{2iv^2tt_0}{t-t_0}+2ivt\frac{s+iy}{t-t_0}\right)G_{11},$$

$$-i\frac{dG_{12}}{dy}=\alpha_1G_{12}+\alpha_1\exp\left(\frac{2iv^2tt_0}{t-t_0}+2ivt\frac{s+iy}{t-t_0}\right)G_{22},$$

$$-i\frac{dG_{22}}{dy}=\alpha_2G_{22}+\alpha_2\exp\left(\frac{2iv^2tt_0}{t-t_0} +2ivt\frac{s+iy}{t-t_0}\right)G_{12}, \tag{18}$$

where the following boundary conditions should be applied:

$$G_{11}|_{y=0}=\alpha_1 i\exp\left(\frac{iv^2tt_0}{t-t_0}+\frac{iv^2t}{2}\right),$$

$$G_{12}|_{y=0}=G_{21}|_{y=0}=0,$$

$$G_{22}|_{y=0}=\alpha_2 i\ \exp\left(\frac{iv^2tt_0}{t-t_0}+\frac{iv^2t}{2}\right). \quad (19)$$

The system of equations (18) and (19) splits into two independent systems of equations. The solution can be found explicitly after some simple manipulations and written using Bessel functions:

$$G_{11}=\xi^{\mu}[A_1J_\nu(z)+A_2J_{-\nu}(z)], \quad (20)$$

$$G_{21}=\xi^{\mu}i\sqrt{\frac{\alpha_1}{\alpha_2}}\,[A_1J_{\nu-1}(z)-A_2J_{-\nu+1}(z)], \quad (21)$$

where

$$\xi=\exp[2iv_1(vt+s+iy)],\quad v_1=\frac{vt}{t-t_0},$$

$$\mu=\frac{1}{2}-\frac{i(\alpha_1+\alpha_2)}{4v_1},$$

$$\nu=\frac{1}{2}+i\,\frac{\alpha_1-\alpha_2}{4v_1},\quad z=\frac{\sqrt{\alpha_1\alpha_2}}{2v_1}\,\xi$$

The boundary conditions (19) yield the following expressions for the constants $A_{1,2}$:

$$A_2=A_1\,\frac{J_{\nu-1}(z_0)}{J_{-\nu+1}(z_0)},\quad z_0=z|_{y=0}=\frac{\sqrt{\alpha_1}\alpha_2}{2v_1}$$
$$\times\exp[2iv_1(vt+s)],$$

$$A_1=i\alpha_1J_{-\nu+1}(z_0)\,\frac{\pi z_0}{2\sin\pi z_0}\exp\left[-\frac{iv^2t_0^2}{2(t-t_0)}-iv_1s\right.$$
$$\left.-(vt+s)\,\frac{\alpha_1+v_2}{2}\right].$$

Similarly we find

$$G_{22}=\xi^{\mu}[A_3J_{1-\nu}(z)+A_4J_{\nu-1}(z)], \quad (22)$$

$$G_{12}=\xi^{\mu}i\sqrt{\frac{\alpha_1}{\alpha_2}}\,[A_3J_{-\nu}(z)-A_4J_\nu(z)],$$

$$A_4=A_3\,\frac{J_{-\nu}(z_0)}{J_\nu(z_0)},\quad A_3=\alpha_2 iJ_\nu(z_0)\,\frac{\pi z_0}{2\sin\pi z_0}$$

$$\times\exp\left[-\frac{iv^2t_0^2}{2(t-t_0)}-iv_1s-(vt+s)\,\frac{\alpha_1+\alpha_2}{2}\right]. \quad (23)$$

Thus, Eqs. (14)–(17) together with (20)–(23) completely solve the problem at hand.

This solution holds for $t\geqslant t_0\geqslant 0$. If these conditions are violated, then the quantity $v_1=vt/t-t_0$ can become negative, as a result of which ξ increases exponentially as a function of y and the factor $\exp[i(s+iy)^2/2(t-t_0)]$ becomes inadequate to ensure the convergence of the integration $\int^{\infty}...dy$, because the Bessel functions of complex argument can grow too rapidly. Nevertheless, this solution can be used even in the case of converging δ-function potentials ($t_0<0$). For this it suffices to set $v'=-v>0$, i.e., to take the "separation" velocity negative. In this case we have $v_1=-v't/t-t_0\geqslant 0$ for $0\geqslant t\geqslant t_0$. That is, these relations will hold for $t\leqslant 0$. To evaluate $\Psi(x,t)|_{t>0}$ we must calculate $\Psi(x,t)|_{t=0}$ and then use the relations obtained above in this section, which simplify drastically for $t_0=0$.

To study ionization and charge-exchange processes in a number of cases we should set $t_0=0$ or $t_0=-\infty$, which results in the considerable simplifications used in Ref. 5.

We note finally that in the simplest case $\alpha_1=\alpha_2$, $t_0=0$ and $\varphi(x)\equiv\varphi(-x)$ the Bessel function can be expressed in terms of elementary functions, as a result of which after a number of transformations we find an equation which is identical with that obtained previously in Ref. 4.

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