Nonlocal effects in a model of the critical state

V. S. Gorbachev and S. E. Savel'ev

Moscow State Engineering-Physics Institute, 115409 Moscow, Russia
(Submitted 7 October 1994)
Zh. Éksp. Teor. Fiz. **107**, 1247–1268 (April 1995)

A new model of the critical state is presented in which nonlocal effects associated with the nonlocal relation between the magnetic induction and the vortex density are taken into account in a unified manner, together with the nonlocal relation between the vortex density and the vortex displacement in the pinning potential. Boundary effects (Meissner currents and vortex images) are taken into account exactly. In the general case the model comprises a system of three equations for three unknown functions of the coordinates—the magnetic induction, the vortex density, and the displacement of the vortices. The nonlinear process of the penetration of an oscillating magnetic field into a hard superconductor is investigated by means of the model. In the local limit the model goes over into the traditional model of the critical state. The differences between the new model and the local model of the critical state are discussed, together with the possible regions of applicability of the new model. © *1995 American Institute of Physics.*

1. INTRODUCTION

To describe a hard type-II superconductor in external magnetic fields one usually uses the critical-state model (see, e.g., Refs. 1–3). In this model it is assumed that Abrikosov vortices nucleate on the surface of the sample and move into the interior of the sample until the force created by the vortex gradient is balanced by the pinning force. The constitutive equation of the critical state has the form (see, e.g., Ref. 3)

$$\mathrm{curl} H_{eq}(B) = \frac{4\pi}{c} J_c(B), \tag{1}$$

where $H_{eq}(B)$ is the thermodynamic field (a function of the induction B) and J_c is the critical current density, determined both by the properties of the pinning centers and by the elastic properties of the vortex lattice (see, e.g., Refs. 4–6). Equation (1) must be supplemented by boundary conditions. In the simplest variants of the model it is assumed that, on the boundary, $H_{eq}=H_0$ (H_0 is the external field). A more correct procedure is to take into account the surface barriers H_{ent} and H_{exit} at the entrance and exit of vortices.[3] For a soft superconductor with a plane boundary expressions for H_{ent} and H_{exit} were obtained in Ref. 7.

The distribution of the induction in a hard superconductor is not determined uniquely by Eq. (1) and the boundary conditions. To solve this equation it is necessary to take the previous history into account. This is done as follows: When the external field changes, screening currents with current density equal to $\pm J_c$ are induced in the region near the surface. In the region further from the surface the induction remains as it was before the change of the field. The position of the boundary between these regions is determined from the condition that the induction be continuous. The current density experiences a discontinuity at this point.

In the above formulation of the critical-state model a number of effects are not taken into account. For example, this model is not applicable for the description of superconducting samples with sizes on the order of λ. In this case, as well as the uncertainty in the boundary conditions it is necessary to take into account the nonlocal relation between the magnetic induction B and the density n of Abrikosov vortices. This must be taken into account in bulk superconductors as well, in the case when large gradients of the vortex density ($\nabla n/n \sim 1/\lambda$) arise. A local relation between the magnetic induction and the vortex density is usually assumed:

$$B = n\Phi_0, \tag{2}$$

where Φ_0 is the quantum of magnetic flux. In addition, in the region near the boundary it is necessary to take into account the interaction of the vortices, their images, and the Meissner currents.

In addition, for superconducting samples of finite size an important role can be played by nonlocal effects associated with "reversible displacement of vortices" (see Refs. 4, 8, 9, and 11). The concept of the "reversible displacement of a vortex" near a pinning center was introduced in Ref. 8. In this case we take account of the fact that a small change of the field causes a vortex to be displaced from its equilibrium position. Here, the vortex moves reversibly in the pinning potential, and the current density is smaller than its critical value. When the displacement of the vortex becomes equal to the interaction length d_0, the vortex is depinned. According to Labusch,[4] d_0 is that displacement of a pinned vortex lattice for which the lattice loses its stability.

Nonlocal effects associated both with the nonlocal relation between B and n and with the reversible motion of vortices were considered in Ref. 10. In that paper, however, only the linear response of the system to an oscillating field of small amplitude, superposed on a large background uniform field rather than on the critical profile, was studied.

The critical state was considered with allowance for the reversible displacement of vortices in Ref. 12, in which, however, effects due to the nonlocal relation between the induction and the vortex density were not included.

A critical-state model that treats effects associated with the nonlocal relation between B and n but not the reversible

 1063-7761/95/040694-12$10.00

displacement of vortices was described briefly in Ref. 13.

In the present paper we present a unified approach that takes into account the nonlocal relation between B and n (i.e., the finiteness of the quantity λ) and also the nonlocal relation between n and the vortex displacement $\mathbf{u}$ (i.e., the reversible displacement of vortices). We stress immediately that, for simplicity, we are not treating effects associated with the difference of the thermodynamic field H_{eq} from the induction B (this is justified in fields $H \gg H_{c1}$). In addition, we do not intend to consider nonlocal effects associated with the collective character of the pinning of the vortices;[5] (see also the review Ref. 6). Including these effects changes the value of the critical-current density J_c, which in our analysis is a phenomenological parameter.

In Sec. 2 we formulate a general approach that makes it possible to obtain the equations of the nonlocal model of the critical state. This approach is applied in Sec. 3 to study the effects of the nonlocal relation between B and n. We consider the penetration of magnetic field into a superconducting half-space and into a plate of finite thickness. It is shown that when the external field varies nonmonotonically there is a range of fields in which a change in the field does not lead to hysteresis losses; the vortex-density distribution has discontinuities, and in the case of the superconducting plate there is always a vortex-free region in the center. All the effects listed vanish as $\lambda \to 0$, which corresponds to going over to the usual model of the critical state. In Sec. 4 we construct a model that takes into account not only the finiteness of λ but also the reversible displacement of vortices. Here, all the effects remain, but the characteristic quantities are renormalized. The penetration of an oscillating field of small amplitude superposed on the critical profile is investigated. It is found that in this case, in contrast to the case of a constant background field, the attenuation of the field has a nonexponential character. In the Conclusion we discuss the results and the conditions for applicability of the model.

2. THE GENERAL APPROACH

In the local critical-state model (1) there is a single phenomenological parameter—the critical current density J_c. According to Labusch,[4] J_c is related to the bulk pinning force P_c at which the vortex lattice becomes unstable. It may be said that when J becomes equal to J_c (or, equivalently, P becomes equal to P_c) in some region of the superconductor the vortices break away from the pinning centres and are displaced in an irreversible manner. We shall call this region critical, and denote it by Ω_{crit}. But if in some region the current density has still not reached its critical value, the displacement of the vortex lattice is reversible. We shall call this region subcritical, and denote it by Ω_{rever}. The current density in Ω_{rever} is smaller than J_c and depends on the magnitude of the displacement $\mathbf{u}$ of the vortices. The character of the division of the volume of the superconductor into a critical and a subcritical region depends on the boundary conditions and on the previous history.

We shall consider a hard type-II superconductor in the form of an arbitrary cylinder in an external field H_0 parallel to its generator. We shall assume $H_{c1} \ll H_0 \ll H_{c2}$, where H_{c1} and H_{c2} are the first and second critical fields, respectively. We shall also consider only those regions of the superconductor in which the magnetic field satisfies these inequalities. In this case, the average spacing a between the vortices satisfies the relation $\xi \ll a \ll \lambda$, where ξ is the coherence length.

The microscopic field h in the superconductor is determined by the London equation

$$h+\lambda^2 \operatorname{curl}\operatorname{curl} h=\Phi_0 \sum_i \delta(\boldsymbol{\rho}-\boldsymbol{\rho}_i)=\Phi_0 N, \tag{3}$$

where $\boldsymbol{\rho}$ is the two-dimensional radius vector in the plane perpendicular to the magnetic field, the summation is performed over all the vortex filaments, $N=\Sigma_i \delta(\boldsymbol{\rho}-\boldsymbol{\rho}_i)$ is the microscopic density of vortices, and Φ_0 is the quantum of magnetic flux. Any solution of (3) can be represented in the form of a sum $h=h_m+h_v$ of the Meissner field and vortex field (see, e.g., Ref. 14), which satisfy the equations

$$h_m+\lambda^2 \operatorname{curl}\operatorname{curl} h_m=0, \tag{4}$$

$$h_v+\lambda^2 \operatorname{curl}\operatorname{curl} h_v=\Phi_0 N, \tag{5}$$

with the boundary conditions $h_m=H_0$ and $h_v=0$. The solution of (5) can be represented in the form

$$h_v=\int d^2\boldsymbol{\rho}_1 N(\boldsymbol{\rho}_1) h_1(\boldsymbol{\rho},\boldsymbol{\rho}_1), \tag{6}$$

where $h_1(\boldsymbol{\rho},\boldsymbol{\rho}_1)$ is the field, at the point $\boldsymbol{\rho}$, of one vortex situated at the point $\boldsymbol{\rho}_1$.

In a macroscopic description of the superconductor it is necessary to average the microscopic field h over scales d much greater than the spacing a between the vortices. If $d \gg \lambda$ we obtain the macroscopic equation (1) and the local relation (2). To construct a nonlocal model we shall perform the averaging over scales $a \ll d \ll \lambda$.

We shall average Eqs. (4) and (5) over scales $a \ll d \ll \lambda$. Then the Meissner field remains unchanged ($B_m=h_m$), and the average vortex field

$$B_v(\boldsymbol{\rho})=\int d^2\boldsymbol{\rho}_1 n(\boldsymbol{\rho}_1) h_1(\boldsymbol{\rho},\boldsymbol{\rho}_1), \tag{7}$$

satisfies the equation

$$B_v+\lambda^2 \operatorname{curl}\operatorname{curl} B_v=\Phi_0 n. \tag{8}$$

In the case of soft type-II superconductors the field $B_v(\boldsymbol{\rho})$ is uniform over distances much greater than λ from the boundaries, and Eq. (8) gives the local relation (2) between the induction and the vortex density. In the case of hard superconductors the local relation remains valid only far from the boundary and for not very strong pinning, when the magnetic induction varies over distances greater than λ. It is this relation that is used in the traditional model of the critical state, but in the case of characteristic sizes comparable to λ, or in the case of sufficiently strong pinning, it is necessary to use the more general relation between $B_v(\boldsymbol{\rho})$ and $n(\boldsymbol{\rho})$ given by Eq. (8).

We now take into account the nonlocal effects associated with the reversible displacement of vortices in the subcritical region Ω_{rever}. In this region, when the external field changes the vortices are displaced by a distance $\mathbf{u}(\boldsymbol{\rho})$, and this changes the vortex density n from the old density n_{old}. Integrating over time the continuity equation $\partial n/\partial t = -\mathrm{div}(n\mathbf{v})$, where $\mathbf{v}$ is the velocity of displacement of the vortices, we obtain

$$n = n_{\text{old}} - \mathrm{div}(n_{\text{old}}(\mathbf{u} - \mathbf{u}_{\text{old}})). \tag{9}$$

In the right-hand side of this expression we have assumed that $n \approx n_{\text{old}}$; this corresponds to retaining only terms quadratic in u in the expression for the energy (see below).

The critical-state equation (1) is the equation for the balance of forces acting on a vortex. In macroscopic form this equation can be formulated as the condition for an extremum of the Gibbs free energy G (for such an approach, see, e.g., Ref. 2). This formulation is convenient for the construction of a nonlocal generalization of the critical-state model. We stress that as a result we obtain all possible metastable states of the system. The choice of a particular state is determined not only by the equations that arise but also by the previous history of the system.

We shall determine the specific form of the Gibbs energy G as a functional of the vortex density n in the critical region Ω_{crit} and of the displacement $\mathbf{u}$ in the subcritical region Ω_{rever}. We shall confine the analysis to an isotropic superconductor (certain nonlocal anisotropic effects are considered in Ref. 15). In this case G can be written as the sum of the following terms: the electromagnetic energy $G_{e/m}$, the work G_p^{crit} of the pinning forces in the critical region, and the energy G_p^{rever} of the vortices in the pinning potential in the subcritical region.

First we shall consider the electromagnetic part of the Gibbs energy:

$$G_{e/m} = \frac{1}{8\pi} \int d^2\boldsymbol{\rho}(h^2 + \lambda^2(\mathrm{curl}\, h)^2 - 2H_0 h). \tag{10}$$

By means of the relations (4) and (5) we can represent this expression in the form[14]

$$G_{e/m} = \frac{1}{8\pi} \int d^2\boldsymbol{\rho}(\Phi_0 N h_v - 2H_0 h_v), \tag{11}$$

where the energy is calculated relative to that of the Meissner state. To obtain the Gibbs energy expressed in terms of the macroscopic variables n and B_v it is necessary to average (11) (see Appendix 1). We obtain

$$G_{e/m} = \frac{1}{8\pi} \int d^2\boldsymbol{\rho}(\Phi_0 n B_v - 2H_0 B_v), \tag{12}$$

where B_v is the average vortex field. The latter expression is not exact. As shown in Appendix 1, it is calculated with relative accuracy $\sim H_{c1}/H \ll 1$, which corresponds in fields $H \gg H_{c1}$ to neglect of the difference between the induction B and the field strength $H(B)$. All the calculations below will be performed to this accuracy.

We now consider G_p^{crit}, defined as the work of the pinning forces upon penetration of vortices into the superconductor in the critical region. We shall confine ourselves to the simplest case of an isotropic superconductor with $P_c = \text{const}$. Suppose that when the external field changes by δH_0 the density in Ω_{crit} changes by $\delta n(\boldsymbol{\rho})$. The associated work δA performed by the pinning forces is

$$\delta A = \int_{\Omega_{\text{crit}}} d^2\boldsymbol{\rho} P_c l(\boldsymbol{\rho})\, \delta n(\boldsymbol{\rho}),$$

where $l(\boldsymbol{\rho})$ is the shortest distance to the boundary. The same can be obtained from variation of

$$G_p^{\text{crit}} = \int_{\Omega_{\text{crit}}} d^2\boldsymbol{\rho} P_c l(\boldsymbol{\rho}) n(\boldsymbol{\rho}). \tag{13}$$

The energy of a vortex in Ω_{rever} depends on the displacement $\mathbf{u}$ of the vortex. We shall confine ourselves to the simplest approximation (see Refs. 10 and 11), in which the restoring force satisfies $\mathbf{F} \sim \mathbf{u}$ for $u < d_0$, where d_0 is the interaction length. Finally, we obtain for the Gibbs energy

$$G_p^{\text{rever}} = \int_{\Omega_{\text{rever}}} d^2\boldsymbol{\rho} n_{\text{old}} \frac{P_c}{2d_0} \mathbf{u}^2. \tag{14}$$

This expression corresponds to keeping only terms quadratic in u in the energy, and therefore it has been assumed that $n \approx n_{\text{old}}$.

Variation of (12)–(14) with allowance for (8) and (9) gives the possibility of obtaining the equations of a nonlocal model of the critical state. Here, as already mentioned above, these equations should be supplemented by an algorithm for taking the previous history into account. These questions are discussed in the next section for the simplest variant of the model, without allowance for the reversible displacements of the vortices in the subcritical region.

3. NONLOCAL MODEL OF THE CRITICAL STATE WITH NEGLECT OF THE REVERSIBLE MOTION OF VORTICES

In this section we shall neglect the reversible displacement of vortices in the subcritical region, i.e., we shall assume that in Ω_{rever}

$$n = n_{\text{old}}. \tag{15}$$

In this case we can disregard the term G_p^{rever}, and we therefore obtain for the Gibbs energy

$$G = \frac{1}{8\pi} \int_{\Omega} d^2\boldsymbol{\rho}(\Phi_0 n B_v - 2H_0 B_v) + \int_{\Omega_{\text{crit}}} d^2\boldsymbol{\rho} P_c \ln(\boldsymbol{\rho}). \tag{16}$$

Varying G with respect to n (see Appendix 2), we obtain, in the critical region,

$$B = H_0 - (4\pi/\Phi_0) P_c l.$$

In this equation we have $P_c = |P_c|$ when the external field increases and $P_c = -|P_c|$ when it decreases. Using the definition $J_c = (c|P_c|)/(\Phi_0)$, we finally obtain

$$B = H_0 \pm \frac{4\pi}{c} J_c l. \tag{17}$$

This expression coincides with that obtained in the local model of the critical state. We stress, however, that in the derivation of (17) the Meissner field was taken into account exactly, and therefore is valid near the boundary of the superconductor, at distances much smaller than λ. (Of course, this is not so in a surface layer $\sim a$.)

In the subcritical region Ω_{rever} the vortex density n does not change when the field changes. In the local model this also implies constancy of the magnetic induction B. When nonlocal effects are taken into account this is not so. The magnetic induction in the subcritical region obeys the equation

$$B+\lambda^2 \text{curl curl } B=\Phi_0 n_{old} \tag{18}$$

and matching conditions on the boundary with the critical region. We shall examine these conditions. The first is the continuity of the magnetic induction. The second can be obtained from Eq. (8). Integrating (8) over a small region near the boundary and letting the size of this region tend to zero, we find that the current density $J=(c/4\pi)\text{curl } B$ should also be continuous. Moreover, this is true even if the vortex density n experiences a discontinuity at the boundary. (Only in the case of a δ-function singularity of n is it not true.) These two conditions also permit us to find the position of the boundary of the critical and subcritical region. We stress that in the local model the current density has a discontinuity on the boundary of the critical and subcritical regions. In the nonlocal model this discontinuity appears in the limit $\lambda\to 0$.

As noted above, the variational approach makes it possible to obtain all the metastable states of the system, and to find each particular state it is necessary to take the previous history of the system into account. We shall consider in more detail how the previous history must be taken into account in the nonlocal model of the critical state. Suppose that we know the distributions of the magnetic induction B_{old} and n_{old} corresponding to some value of the external field H_0. Let the external field change. Then, in the general case, a critical region arises near the surface. In this region the vortex density n changes from n_{old}, but the current density J is a known quantity, equal in magnitude to the critical current density J_c. This makes it possible to find the induction in this region, using Eq. (17). Knowing the induction, we can find the new vortex density from (8). In the subcritical region the vortex density is a known quantity $n=n_{old}$, and the current density, which is smaller than the critical density, changes. This new current density and, correspondingly, the new induction can be determined from Eq. (18) and the matching conditions.

Thus, a nonlocal model of the critical state has been constructed above. We now consider features of this model for the simple special case of the penetration of the field into a superconducting half-space $x>0$. Suppose that the external field H_0 has first increased to H_{max}, and has then begun to vary periodically in the range (H_{min}, H_{max}). The problem is effectively one-dimensional, and the basic equations (17) and (18) can be written in the form

$$\frac{dB}{dx}=\frac{4\pi}{c}J, \tag{19}$$

$$B-\lambda^2\frac{d^2B}{dx^2}=\Phi_0 n. \tag{20}$$

In the critical region we must substitute $J=\pm J_c$ into (19), and the vortex density $n(x)$ here is an unknown function. In the subcritical region the vortex density is known $[n(x)=n_{old}(x)]$, the distribution of the induction is found from (20), and the current density must be determined from (19). In advance we know only that $|J|\leqslant J_c$.

When the field increases from 0 to H_{max} the vortices penetrate to a depth b into the superconductor. In the interval $0<x<b$ a critical region appears, in which the current density is $J=-J_c$. The magnetic induction in this case is

$$B(x)=H_{max}-\frac{4\pi}{c}J_c x. \tag{21}$$

The region $x>b$ is subcritical with $n_{old}=0$. Taking this into account, from (20) we obtain

$$B(x)=C\exp\left(-\frac{x-b}{\lambda}\right), \tag{22}$$

where C is a certain constant. The conditions for continuity of the induction and current density at the point $x=b$ give two equations, and the two unknown constants that are determined from these equations are b and C. We obtain

$$b=\frac{cH_{max}}{4\pi J_c}-\lambda, \tag{23}$$

$$C=\frac{4\pi}{c}J_c\lambda. \tag{24}$$

Substituting the function (21) into Eq. (20), we find the vortex density $n(x)$ in the critical region $(0<x<b)$:

$$n(x)=\frac{1}{\Phi_0}\left(H_{max}-\frac{4\pi}{c}J_c x\right). \tag{25}$$

In the subcritical region $x>b$ we have $n(x)=n_{old}=0$. But the induction in this region is nonzero [see (22)]. Nevertheless, it is natural to call b the penetration depth; strictly, it signifies the depth of penetration of the vortices, and gives the characteristic scale of the penetration of the field.

We note an important property of the solution obtained: At the point b the density vanishes discontinuously. For the size of the discontinuity we have

$$\Delta n(b)=\frac{4\pi}{c}\frac{J_c\lambda}{\Phi_0}. \tag{26}$$

Physically, the density discontinuity appears at the point b because a vortex can only advance into the interior of the superconductor if the finite pinning force P_c is overcome. The vortex advances under the action of the electromagnetic force created by the Meissner field and the field of the other vortices. It is found that if the vortex density is smaller than $(4\pi/c)(J_c\lambda/\Phi_0)$, and, consequently, the spacing between the vortices is sufficiently large, the electromagnetic force acting on our outer vortex is smaller than P_c. For the same reason, for $H_{max}<(4\pi/c)J_c\lambda$ it is completely impossible for vortices to enter into the superconductor. In this case the Meissner-current density

$$J_m=\frac{c}{4\pi}\frac{H_{max}}{\lambda}\exp-\left(\frac{x}{\lambda}\right)$$

is smaller than J_c throughout the superconductor. Only for $H_{max}>(4\pi/c)J_c\lambda$ do we have $J_m>J_c$, near the surface. In this case the force exerted by the Meissner currents on the first vortex nucleated at the surface exceeds P_c, and the vortex moves into the interior of the superconductor.

We stress that, because in the derivation of the equations of the model we neglected fields $\sim H_{c1}$, everything that has been said above is valid, strictly speaking, only if $(4\pi/c)J_c\lambda \gg H_{c1}$.

When, having reached H_{max}, the external field H_0 begins to decrease, in fields satisfying $H_{max}-\Delta H<H_0<H_{max}$ the vortex density remains constant throughout the superconductor. The pinning force acting on a vortex near the boundary changes from P_c to $-P_c$. The magnetic induction then also changes, since the Meissner component h_m changes. Since the vortex field remains unchanged for $H_{max}-\Delta H<H_0<H_{max}$, it is easy to find the interval ΔH in which the current density at the surface changes from $-J_c$ to J_c:

$$\Delta H=\frac{8\pi}{c}J_c\lambda. \tag{27}$$

It may be said that in this interval of fields no critical region arises, and the vortex density remains everywhere as formed during the increase of the field:

$$n_{old}(x)=\begin{cases}\dfrac{1}{\Phi_0}\left(H_0-\dfrac{4\pi}{c}J_cx\right) & \text{for } 0<x<b,\\ 0 & \text{for } x>b.\end{cases} \tag{28}$$

To determine the induction it is necessary to solve Eq. (20) with the vortex density $n(x)=n_{old}(x)$ given by the expression (28), with the boundary condition $B(0)=H_0$. The solution is presented in Fig. 1b.

In the range $H_{max}-\Delta H<H_0<H_{max}$ the hysteresis losses in the superconductor are equal to zero. The quantity ΔH is proportional to $J_c\lambda$ and depends on the geometry of the system. Below, ΔH is also calculated for a plate. When we go over to the local model, $\Delta H\to 0$.

When the field decreases further a critical region $(0<x<x_0)$ arises. As before, the boundary x_0 of the critical and subcritical regions is not known in advance, and is determined from the matching conditions in the process of the solution. The difference from the solution considered previously is that, in the subcritical region $x>x_0$, the vortex density $n_{old}(x)$ is not equal to zero, but is given by the expression (28). We have (see Fig. 1b)

$$B(x)=\begin{cases}H_0+\dfrac{4\pi}{c}J_cx & \text{for } 0<x<x_0,\\ H_{max}-\dfrac{4\pi}{c}J_cx-\dfrac{8\pi}{c}J_c\lambda\exp\left(-\dfrac{x-x_0}{\lambda}\right) & \text{for } x_0<x<b,\\ \dfrac{4\pi}{c}J_c\lambda\exp\left(-\dfrac{x-b}{\lambda}\right)-\dfrac{8\pi}{c}J_c\lambda\exp\left(-\dfrac{x-x_0}{\lambda}\right) & \text{for } x>b,\end{cases} \tag{29}$$

$$n(x)=\begin{cases}\dfrac{1}{\Phi_0}\left(H_0+\dfrac{4\pi}{c}J_cx\right) & \text{for } 0<x<x_0,\\ \dfrac{1}{\Phi_0}\left(H_{max}-\dfrac{4\pi}{c}J_cx\right) & \text{for } x_0<x<b,\\ 0, & \text{for } x>b,\end{cases} \tag{30}$$

where

$$x_0=\frac{c(H_{max}-H_0)}{8\pi J_c}-\lambda. \tag{31}$$

On the boundary $x=x_0$ between the critical and subcritical regions the density experiences a discontinuity

$$\Delta n(x_0)=\frac{8\pi}{c}\frac{J_c\lambda}{\Phi_0}. \tag{32}$$

We note that the appearance of a density discontinuity on the boundary between the critical and subcritical regions is a general consequence of the proposed model. In fact, Eqs. (17) and (18), together with the conditions that the induction and current density be continuous on the boundary of the critical and subcritical regions, make it possible to determine all the unknown quantities [in particular, the vortex density $n(\boldsymbol{\rho})$] everywhere. Here, no matching conditions for the vortex density are used in the construction of the solution, and this, generally speaking, should lead (and usually does lead) to a density discontinuity. The magnitude of the discontinuity depends on the geometry of the system, and is also calculated below for a plate. When we go over to the local model $(\lambda\to 0)$ the density discontinuities disappear.

From a physical point of view, the reason for the appearance of the density discontinuities is as follows. In the example considered, the vortex density $n_{old}(x)$ in the subcritical region $x>x_0$ was formed during the penetration of vortices from the boundary into the interior of the superconductor. As the field decreases however, in the critical region $0<x<x_0$ the vortices move in the opposite direction, toward the boundary of the superconductor. Here, in the one case the force acting on the vortex is equal to P_c, and in the other it is equal to $-P_c$. For the electromagnetic force acting on a vortex at the point x_0 to change by $2P_c$, it is necessary that the induction $B(x_0)$ change by ΔH (27). This implies the existence of the density discontinuity (32). The simplest way to see this is to note that in the case J_c=const, in the critical region, the simple relation $B(x)=n(x)\Phi_0$ holds. (This is connected with the linearity of the critical profile, and is not

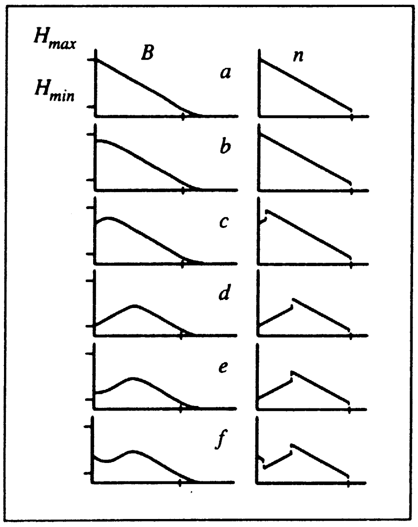

FIG. 1. Evolution (a–f) of the spatial distributions of the magnetic induction B and vortex density n in the half-space $x>0$ for periodic variation of the external magnetic field.

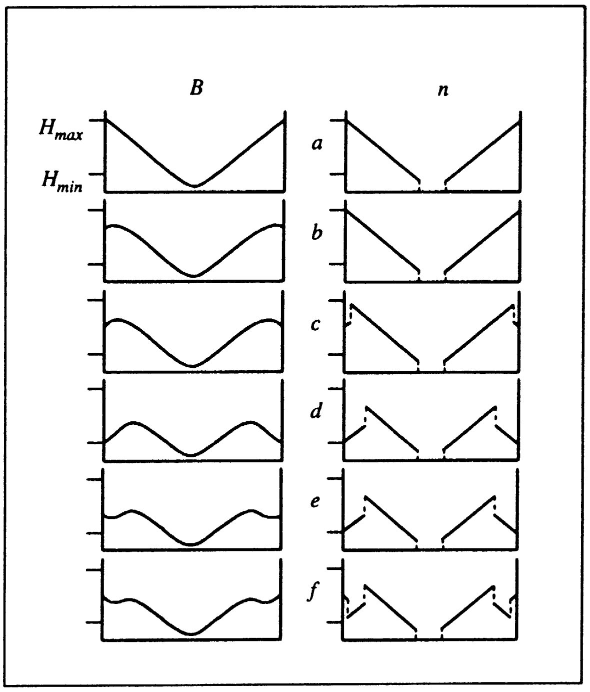

FIG. 2. The same as in Fig. 1, but for a plate $-d/2<x<d/2$.

true in the subcritical region.) At the moment of formation of the density $n_{\rm old}$ (28) the corresponding critical profile of the field is determined by Eq. (21) with $J=-J_c$. As the external field decreases the vortex density at the point $x=x_0$ does not change at first; nevertheless, the induction at this point decreases, and the current density changes from $-J_c$ to J_c. Here, the relation $B(x_0)=n(x_0)\Phi_0$ is not valid. When the induction decreases by ΔH, the current density becomes equal to J_c. To the left of the point x_0 a new critical state will be formed, for which the relation $B(x)=n(x)\Phi_0$ is again valid. From this it follows immediately that the new vortex density (to the left of the point x_0) differs from the old (to the right of the point x_0) by $\Delta n=\Delta H/\Phi_0$.

When the field increases from $H_{\min}$ to $H_{\max}$ the calculations are completely analogous. The only expression to change is that for $n_{\rm old}(x)$, which in this case is given by Eq. (30). The profiles of the induction and density for this case are presented in Figs. 1d–f.

A series of field and vortex-density distributions for penetration of the field into a plate $-d/2<x<d/2$ is given in Fig. 2. The principal features here are the following.

The penetration depth b^{pl} during the initial increase of the external field is given implicitly by the expression

$$b^{pl}+\lambda\,\coth\left(\frac{d-2b^{pl}}{2\lambda}\right)=\frac{cH_{\max}}{4\pi J_c}. \tag{33}$$

We stress that in the nonlocal model the vortex-density and field distributions during the initial increase of the external field are always similar to those given in Fig. 2a. In this, the nonlocal model differs from the local model of the critical state, in which the field distributions are qualitatively different in fields $H_{\max}<H_p$ and $H_{\max}>H_p$, where $H_p=(2\pi/c)J_cd$ is the penetration field. In Fig. 3 we plot the penetration depth b^{pl} as a function of the external field $H_{\max}$ for different values of the quantity λ/d. It can be seen that for finite values of λ/d the penetration depth is always smaller than $d/2$, i.e., for any external field there exists a vortex-free region at the center of the plate. (Of course, this conclusion is valid if the size of the vortex-free region is greater than the spacing a between the vortices. In sufficiently strong fields this is not so, and the vortex-free region disappears.) Only in the limiting case $\lambda/d\rightarrow 0$ does a break appear in $b(H_{\max})$, and the penetration depth is described by the usual formula

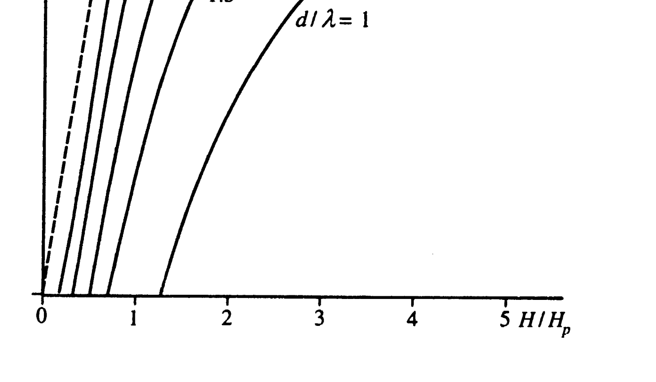

FIG. 3. Dependence of the penetration depth b^{pl} on the external field H_0 for different values of the parameter d/λ. The dashes denote the local limit (34).

$$b^{pl}=\begin{cases} cH_{\max}/4\pi J_c & \text{for } H_{\max}<H_p, \\ d/2 & \text{for } H_{\max}>H_p. \end{cases} \tag{34}$$

The magnitude of the barrier for the plate is modified in comparison with (27):

$$\Delta H^{pl}=\frac{8\pi}{c} J_c\lambda \coth\left(\frac{d}{\lambda}\right). \tag{35}$$

The magnitudes of the density discontinuities on the boundary between the critical and subcritical regions are calculated in an analogous way:

$$\Delta n^{pl}(x_0^{pl})=\frac{8\pi}{c}\frac{J_c\lambda}{\Phi_0} \coth\left(\frac{x_0^{pl}}{\lambda}\right), \tag{36}$$

where x_0^{pl} is determined implicitly by the expression

$$\frac{d}{2}-x_0^{pl}+\lambda \coth\left(\frac{x_0^{pl}}{\lambda}\right)=\frac{c(H_{\max}-H_0)}{4\pi J_c}. \tag{37}$$

Distinctive features of the surface impedance of a plate of thickness $d\sim\lambda$ in the given model are discussed in Ref. 13 (see also the Conclusion).

4. NONLOCAL MODEL OF THE CRITICAL STATE WITH ALLOWANCE FOR REVERSIBLE DISPLACEMENT OF VORTICES

In Sec. 3 we disregarded the reversible displacement of vortices in the subcritical region. However, as will be seen below, in strong magnetic fields the effects associated with this displacement are often comparable to the nonlocal effects associated with the finiteness of λ. In this section we present a critical-state model that takes these two factors simultaneously into account.

The Gibbs free energy in this case has the form

$$G=\frac{1}{8\pi}\int_{\Omega} d^2\boldsymbol{\rho}(\Phi_0 nB_v-2H_0B_v)+\int_{\Omega_{\text{crit}}} d^2\boldsymbol{\rho} lP_c\, n(\boldsymbol{\rho})$$
$$+\int_{\Omega_{\text{rever}}} d^2\boldsymbol{\rho} n_{\text{old}}\frac{P_c}{2d_0}\mathbf{u}^2. \tag{38}$$

Variation of G gives

$$\delta G=\delta G_{\text{crit}}+\frac{1}{8\pi}\int_{\Omega_{\text{rever}}} d^2\boldsymbol{\rho}\bigg(2\Phi_0(B_v+h_m-H_0)\delta n$$
$$+\frac{P_c}{d_0}\mathbf{u} n_{\text{old}}\delta\mathbf{u}\bigg), \tag{39}$$

where δG_{crit} is the variation of G over the critical region, which coincides with that considered in Sec. 3. It follows from (9) that $\delta n=-\text{div}(n_{\text{old}}\delta u)$. Using this relation, we obtain

$$\delta G=\delta G_{\text{crit}}+\frac{\Phi_0}{4\pi}\int_S (H_0-B)n_{\text{old}}\delta\mathbf{u}d\mathbf{l}$$
$$+\frac{\Phi_0}{4\pi}\int_{\Omega_{\text{rever}}} d^2\boldsymbol{\rho} n_{\text{old}}\delta\mathbf{u}\left([\mathbf{e}_0,\text{curl }\mathbf{B}]+\frac{4\pi}{c}\frac{J_c}{d_0}\mathbf{u}\right), \tag{40}$$

where S signifies integration over the boundary, $d\mathbf{l}$ is a length differential perpendicular to the boundary and pointing into the critical region, and $\mathbf{e}_0$ is the unit vector in the direction of $\mathbf{B}$.

Equating the integrand in the volume integral in (40) to zero, and multiplying by $\mathbf{e}_0$, we obtain

$$\text{curl }\mathbf{B}=\frac{4\pi}{c}J_c\left[\mathbf{e}_0,\frac{\mathbf{u}}{d_0}\right]. \tag{41}$$

From the condition that the current density be continuous on the boundary of the critical and subcritical regions, it follows that here we have $u=d_0$ (and, correspondingly, $\delta\mathbf{u}=0$), and therefore the surface integral in (40) is equal to zero.

Thus, we can now formulate the system of equations of a nonlocal model of the critical state with allowance for reversible displacement of vortices. It consists of three equations for the three unknown functions $\mathbf{B}(\boldsymbol{\rho})$, $n(\boldsymbol{\rho})$, and $\mathbf{u}(\boldsymbol{\rho})$:

$$\mathbf{B}+\lambda^2 \text{ curl curl }\mathbf{B}=\Phi_0 n\mathbf{e}_0, \tag{42}$$

$$\text{curl }\mathbf{B}=\frac{4\pi}{c}J_c\left[\mathbf{e}_0,\frac{\mathbf{u}}{d_0}\right], \tag{43}$$

$$n=n_{\text{old}}-\text{div}(n_{\text{old}}(\mathbf{u}-\mathbf{u}_{\text{old}})). \tag{44}$$

In the critical region the displacement is equal to the maximum possible value, $u=\pm d_0$. In this case, only the first two equations of this system are used, and the system goes over into the one described in Sec. 3. In the subcritical region we have $|u|<d_0$, and self-consistent solution of all three equations is necessary.

As in Sec. 3, to find a particular solution it is necessary to take the previous history into account. Now, when the external field changes the density varies not only in the critical but also in the subcritical region. In the latter region, knowledge of the previous history implies knowledge not only of the old density n_{old} but also of the old displacement $\mathbf{u}_{\text{old}}$, which in the critical region is equal to $\pm d_0$.

Using this model we shall consider a specific problem—the penetration of the field into the half-space $x>0$. The formulation of the problem is identical to that considered in Sec. 3. Eliminating the vortex density n from Eqs. (42)–(44), and writing them in one-dimensional form, we obtain

$$\frac{dB}{dx}=-\frac{4\pi}{c}J_c\frac{u}{d_0}, \tag{45}$$

$$B-\lambda^2\frac{d^2B}{dx^2}=\Phi_0\left[n_{\text{old}}-\frac{d}{dx}(n_{\text{old}}(u-u_{\text{old}}))\right]. \tag{46}$$

During the increase of the field from 0 to $H_{\max}$ there is no subcritical region, and, therefore, in this case, the solution coincides with the corresponding solution from Sec. 3. When the field is subsequently reduced from $H_{\max}$, a range of fields $H_{\max}-\Delta H^d<H_0<H_{\max}$ also arises in which only the sub-

critical region exists. The magnetic induction here is determined by Eq. (46), in which we must set $u_{\rm old}=d_0$. (Note that if we were considering the penetration of an oscillating field superposed on a background uniform field, $u_{\rm old}$ would be equal to zero.) Solving this equation and joining it with the exponentially decaying solution in the region $x>b$ (see Appendix 2), we obtain (for simplicity it is assumed that $d_0=\text{const}\ll\lambda$)

$$B(x)=\begin{cases} H_{\max}-\dfrac{4\pi}{c}J_c x+(H_0-H_{\max})\sqrt{\dfrac{\lambda_{\rm eff}^d(0)}{\lambda_{\rm eff}^d(x)}}\exp\left(\dfrac{-2(\lambda_{\rm eff}^d(0)-\lambda_{\rm eff}^d(x))}{d_0}\right) & \text{for } x<b,\\ \left[\dfrac{4\pi}{c}J_c\lambda+(H_0-H_{\max})\sqrt{\dfrac{\lambda_{\rm eff}^d(0)}{\lambda}}\exp\left(\dfrac{-2(\lambda_{\rm eff}^d(0)-\lambda)}{d_0}\right)\right]\exp\left(-\dfrac{(x-b)}{\lambda}\right) & \text{for } x>b, \end{cases} \tag{47}$$

where

$$\lambda_{\rm eff}^d=\sqrt{\lambda^2+d_0\,\frac{H_{\max}-4\pi/cJ_c x}{4\pi/cJ_c}}. \tag{48}$$

The expressions obtained solve the problem of the penetration of a field disturbance into a superconductor superposed on a nonuniform critical profile. A similar problem was solved in Ref. 10, except that there the penetration of a disturbance superposed on a uniform field was considered. In contrast to Ref. 10, in which the disturbance decays exponentially, in our case the attenuation has a nonexponential character. This is due to the nonuniform vortex-density distribution. It may be said that in the case of a uniform density distribution our results coincide with those of Ref. 10.

The expression for the barrier ΔH^d is modified in comparison with (27). As in Sec. 3, we obtain

$$\Delta H^d=\frac{8\pi}{c}J_c\lambda_{\rm eff}^d(0). \tag{49}$$

We note that the latter formula contains an effective penetration depth $\lambda_{\rm eff}^d(0)$, which can be much greater than the London depth λ.

When the external field is lowered below $H_{\max}-\Delta H^d$ a critical region arises. Joining the solutions in the critical and subcritical regions, we obtain

$$B(x)=\begin{cases} H_0+\dfrac{4\pi}{c}J_c x & \text{for } 0<x<x_0,\\ H_{\max}-\dfrac{4\pi}{c}J_c x-\dfrac{8\pi}{c}J_c\lambda_{\rm eff}^d(x_0)\sqrt{\dfrac{\lambda_{\rm eff}^d(x_0)}{\lambda_{\rm eff}^d(x)}}\exp\left(\dfrac{-2(\lambda_{\rm eff}^d(x_0)-\lambda_{\rm eff}^d(x))}{d_0}\right) & \text{for } x_0<x<b,\\ \left[\dfrac{4\pi}{c}J_c\lambda-\dfrac{8\pi}{c}J_c\lambda_{\rm eff}^d(x_0)\sqrt{\dfrac{\lambda_{\rm eff}^d(x_0)}{\lambda}}\exp\left(\dfrac{-2(\lambda_{\rm eff}^d(x_0)-\lambda)}{d_0}\right)\right]\exp\left(-\dfrac{(x-b)}{\lambda}\right) & \text{for } x>b, \end{cases} \tag{50}$$

where x_0 is the boundary of the critical and subcritical regions, which can be determined from the relation

$$H_{\max}-H_0=\frac{8\pi}{c}J_c(x_0+\lambda_{\rm eff}^d(x_0)). \tag{51}$$

Here, the density discontinuity on the boundary of the critical and subcritical regions is given by the expression

$$\Delta n(x_0)=\frac{8\pi}{c}\frac{J_c\lambda}{\Phi_0}\frac{\lambda}{\lambda_{\rm eff}^d(x_0)}. \tag{52}$$

In Figs. 4a–c we give profiles of the induction $B(x)$ and density $n(x)$ as the field is decreased from $H_{\max}$ to $H_{\min}$, with allowance for reversible displacements of vortices in the subcritical region. It can be seen that, in contrast to the results of Sec. 3, the vortex density changes in the subcritical region as well, and this leads to a decrease of the density discontinuity at the point x_0. We stress the following interesting feature. Allowance for the reversible displacements leads to a renormalization of the quantities ΔH and Δn. Moreover, if in the expression (27) for the barrier this renormalization consists in the replacement of λ by $\lambda_{\rm eff}$, the density discontinuity (32) is multiplied by $\lambda/\lambda_{\rm eff}$. It may be said that when reversible displacements are taken into account ΔH increases and Δn decreases.

We now discuss the question of how the system will behave as the field is increased from $H_{\min}$ to $H_{\max}$. All the results of this section remain valid in this case too. In all the formulas, only the expression for the effective penetration depth is modified, because of the decrease of the vortex density near the boundary. In place of $\lambda_{\rm eff}^d$ we must substitute

$$\lambda_{\rm eff}^i=\sqrt{\lambda^2+d_0\,\frac{H_{\min}+(4\pi/c)J_c x}{(4\pi/c)J_c}}. \tag{53}$$

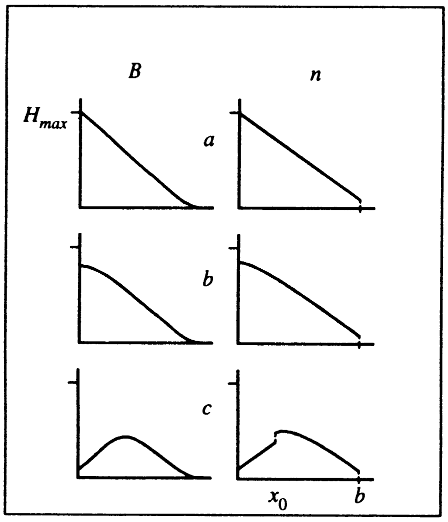

FIG. 4. The same as in Fig. 1, but with allowance for the reversible displacements of vortices in the subcritical region. Profiles for the first half-period of the variation of the external field are given.

To conclude this section, we shall discuss the question of how this model and the local model of the critical state are related. When the formal limit $\lambda \rightarrow 0$ is taken in the formulas of this section, we obtain the local model of the critical state with allowance for the reversible displacement of vortices. The barrier ΔH then remains, but the density discontinuities Δn vanish. If, in addition, we let $d_0 \rightarrow 0$, keeping $u < d_0$, we obtain the usual local model of the critical state.

5. CONCLUSION

The model of the critical state is a rather crude approximation. In particular, it is found not to be valid for the description of a whole group of phenomena in systems with characteristic sizes commensurate with the London field-penetration depth λ. In this case, it is necessary to take nonlocal effects into account.

Nonlocal behavior in the critical-state model is usually taken into account in different versions of the theory of collective pinning[5] (see also the review Ref. 6). The main aim of this theory is to calculate (starting from microscopic phenomenological characteristics of the superconductor such as the coherence length, London depth, pinning potential, etc.) the macroscopic observable characteristics—the critical-current density, the depinning temperature, the melting temperature of the vortex lattice, etc. The finiteness of the London depth λ is taken into account, e.g., in the calculation of the elastic response of the vortex lattice.[15,16] All these questions lie outside the scope of this paper. Here we assume that we know the bulk critical-current density J_c, which is a constant and does not depend on the magnitude of the magnetic field, the vortex density, the distance from the boundary of the superconductor, etc. This assumption is rather crude, and in a more systematic model it would be necessary to calculate J_c and the distributions of vortex density and induction self-consistently with allowance for nonlocal and boundary effects. However, this is a considerably more complicated problem than that considered here.

In recent years a detailed analysis of the penetration of electromagnetic waves into a type-II superconductor has been carried out with allowance for nonlocal and boundary effects (Meissner currents and vortex images), bulk pinning, and creep.[10,17,18] However, the linear response was considered. The penetration of an oscillating field superposed on a uniform constant field was studied.

In Refs. 4, 8, 9, and 11, field penetration into a superconductor was investigated in the framework of the local Bean model with allowance for reversible displacements of vortices near pinning centres.

In this paper we have investigated the nonlinear response and the formation of the critical state with allowance for nonlocal effects. We have presented a unified approach that takes into account both the nonlocal relation between B and n (i.e., the finiteness of the quantity λ) and also the nonlocal relation between n and the vortex displacement $\mathbf{u}$ (i.e., the reversible displacement of vortices).

A model in which the equilibrium displacement of vortices near pinning centres is neglected and it is assumed that the vortex density $n(x)$ changes only when vortices break away from pinning centers has been considered in Sec. 3. This model was described briefly in Ref. 13. We shall consider certain distinctive features of this model.

First, at the matching points the vortex density distribution exhibits a discontinuity. The appearance of the discontinuities signifies that the characteristic scale of the variation of the density near the matching points (and near the boundary) is a, and not λ as might have been expected. We note that an analogous fact was discovered previously for soft superconductors in Ref. 14, in which it was shown that the spacings between vortices do not change right up to the boundary of the superconductor.

Second, when the external field varies in the interval $H_{\max} - \Delta H < H < H_{\max}$ the density of vortices in the plate does not change anywhere. If the amplitude of the oscillating field is smaller than ΔH, the hysteresis losses in the plate are equal to zero.

Third, for any value of $H_{\max}$ there always exists a region in the center of the plate where the vortex density satisfies $n=0$. This is due to mutual repulsion of the Abrikosov vortices at the center of the plate. (Of course, the vortex density could be nonzero at the center of the plate if, e.g., magnetic field was frozen in the plate during the transition to the superconducting state.) This result diverges qualitatively from the predictions of the traditional critical-state model, in which a region with $n=0$ exists only in a field smaller than the penetration field $H_p = (2\pi/c) J_c d$.

Note that the magnitude of the vortex-density discontinuity, the barrier ΔH, and the size of the region with $n=0$ are proportional to λ, and vanish when we take the local limit $\lambda \rightarrow 0$.

In Sec. 4 we present a nonlocal model of the critical state with allowance for the reversible displacement of vortices. Here, as in Sec. 3, in the case of periodic variation of the field on the surface a range of fields appeared in which the hysteresis losses vanish and the vortex density is discontinuous in the superconductor. However, the magnitudes of this interval of fields and of the density discontinuities change, and begin to depend on the previous history of the system. In the formulas the London depth λ is now replaced by an effective penetration depth λ_{eff}, which can be much greater than λ.

The model presented is valid in fields $H_{c1} \ll H_0 \ll H_{c2}$. Here, we have not taken into account effects associated with the difference of the thermodynamic field H_{eq} from the induction B; this is justified in fields $H \gg H_{c1}$. The averaging method given in Appendix 1 makes it possible, in principle, to treat these effects. Then, with the same accuracy, it is necessary to perform a more correct analysis of the surface effects.

We now discuss some possible ranges of applicability of the model presented.

It is obligatory in the description of superconductors with characteristic sizes comparable to the penetration depth. In particular, in Ref. 13 distinctive features of the surface impedance of a plate of a hard superconductor of thickness $d \sim \lambda$ are discussed using the example of a textured ceramic high-temperature superconductor with characteristic granule thicknesses ~ 10 μm. It is well known that the surface resistance $\mathscr{R}$ of a superconducting plate has a maximum when the depth of penetration of an oscillating magnetic field is on the order of the sample thickness d (see, e.g., Ref. 20). According to the local model of the critical state, the height of this maximum, normalized to the surface reactance $\chi_n = 2\pi\omega d/c^2$ of the sample in the normal state, does not depend on any physical parameters:

$$R'_{\max} = \mathscr{R}_{\max}/\chi_n = 3/4\pi. \tag{54}$$

The result (54) does not depend on the manner in which the penetration depth of the oscillating field varies—the impedance can be investigated as a function of the constant field, the temperature, the amplitude of the oscillating field, etc. In certain cases, however, $R'_{\max}$ is found to be appreciably smaller than $3/4\pi$. In Ref. 13 the temperature dependence of the dimensionless surface resistance $R' = \mathscr{R}/\chi_n$ is given for various values of the amplitude of the oscillating field. All the plots contain the maxima under discussion, the height of which at low temperatures agrees with the predictions of the local model of the critical state. However, as the temperature approaches the critical value the height of the maximum decreases. As shown in Ref. 13, this behavior of the impedance is related to the increase of the penetration depth λ, which can be comparable to the characteristic geometric dimensions of the system. The surface resistance of a plate in an oscillating external field $H(t) = H_0 + h_0 \cos \omega t$, calculated by means of the proposed nonlocal model, has a maximum whose height $R'_{\max}$ decreases as the parameter λ/d increases. We note that the fused samples of high-temperature superconductor in this experiment had a laminar structure in which the characteristic thickness of the crystallites was about 10 μm, which is comparable to λ. The role of the parameter d in this experiment was played by the thickness of the crystallites, and not by the total thickness of the superconducting sample. Another possible application may be to low-temperature microcomposite superconductors with superconducting strands of radius $\sim \lambda$ (Refs. 12, 21).

In bulk superconductors with scales much larger than the penetration depth, it is necessary to use the proposed model if the critical-current density is sufficiently large. In addition, a nonlocal model is necessary in the study of surface effects, when it is necessary to take the Meissner currents into account explicitly.

The model presented is also necessary in the investigation of the penetration of an oscillating field of small amplitude into the bulk of a hard superconductor superposed on a large stationary field. In this case, differences from the local model appear even in the case of superconductors of large size. In the article, it is shown that when the external field varies in an interval $H_{\max} - \Delta H < H < H_{\max}$ the vortex density in the superconductor does not change anywhere (only the Meissner component of the field changes). If the amplitude of the oscillating field is smaller than ΔH, the hysteresis losses are equal to zero. The quantity ΔH is directly proportional to the bulk critical-current density J_c and the London depth λ. In the Bean model, $\Delta H = 0$. In experiment we should expect a considerable decrease of the hysteresis losses (in comparison with the values that are predicted by the local model of the critical state) if the amplitude of the oscillating field is comparable to ΔH.

The authors thank I. F. Voloshin, L. M. Fisher, and V. A. Yampol'shiĭ, in coauthorship with whom the work in Ref. 13 was carried out. We should also like to thank the referee for careful study of the paper and useful comments that facilitated improvement of the article.

APPENDIX 1

In this Appendix we discuss the transition from the microscopic description (by means of the vortex coordinates) to the macroscopic description (by means of the density n and the induction B of the field). For this it is necessary to average the Gibbs energy (11) and the microscopic field (6) over scales much larger than the London depth λ and much smaller than the spacing a between the vortices. We introduce the s-particle distribution functions $F_s(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2, \ldots, \boldsymbol{\rho}_s)$, which are the probability of finding the first vortex in the volume element $d\boldsymbol{\rho}_1$, the second in the volume element $d\boldsymbol{\rho}_2$, etc. We shall normalize these distribution functions to the total number N of vortices. We note that in this case $F_1(\boldsymbol{\rho})$ coincides with the vortex density $n(\boldsymbol{\rho})$ on those parts where $F_1(\boldsymbol{\rho})$ is varying slowly: $\nabla(F_1)/F_1 \ll 1/a$. The functions F_s and F_{s-1} are related as follows:

$$F_{s-1}(\boldsymbol{\rho}_1, \ldots \boldsymbol{\rho}_{s-1}) = \frac{1}{N-s+1} \int_V d\boldsymbol{\rho}_s F_s(\boldsymbol{\rho}_1, \ldots, \boldsymbol{\rho}_{s-1}, \boldsymbol{\rho}_s). \tag{A1.1}$$

Multiplying the Gibbs energy (11), which depends on the vortex coordinates, by the N-particle distribution function $F_N(\boldsymbol{\rho}_1, \ldots, \boldsymbol{\rho}_N)$, and using (A1.1), we obtain

$$G[F_1,F_2]=\frac{\Phi_0}{8\pi}\int d^2\boldsymbol{\rho} d^2\boldsymbol{\rho}_1 F_2(\boldsymbol{\rho},\boldsymbol{\rho}_1)h_1(\boldsymbol{\rho},\boldsymbol{\rho}_1)$$
$$+\frac{\Phi_0}{8\pi}\int d^2\boldsymbol{\rho} F_1(\boldsymbol{\rho})h_1(\boldsymbol{\rho},\boldsymbol{\rho})$$
$$-\frac{H_0}{4\pi}\int d^2\boldsymbol{\rho} d^2\boldsymbol{\rho}_1 F_1(\boldsymbol{\rho}_1)h_1(\boldsymbol{\rho},\boldsymbol{\rho}_1). \quad (A1.2)$$

We represent the two-particle distribution function in the form

$$F_2(\boldsymbol{\rho},\boldsymbol{\rho}_1)=F_1(\boldsymbol{\rho})F_1(\boldsymbol{\rho}_1)+F_1(\boldsymbol{\rho})g(\boldsymbol{\rho},\boldsymbol{\rho}_1). \quad (A1.3)$$

The form of the conditional-probability density $g(\boldsymbol{\rho},\boldsymbol{\rho}_1)$ is determined below.

We shall consider first the regions where $\nabla(F_1)/F_1 \ll 1/a$. Here, $F_1=n$, and the functional (A1.2) has the form

$$G_{e/m}=\frac{1}{8\pi}\int d^2\boldsymbol{\rho}(\Phi_0 nB_v-2H_0B_v)+\Delta G, \quad (A1.4)$$

where

$$\Delta G=\frac{\Phi_0}{8\pi}\int d^2\boldsymbol{\rho} n(\boldsymbol{\rho})h_1(\boldsymbol{\rho},\boldsymbol{\rho})$$
$$+\frac{\Phi_0}{8\pi}\int d^2\boldsymbol{\rho} d^2\boldsymbol{\rho}_1 n(\boldsymbol{\rho})g(\boldsymbol{\rho},\boldsymbol{\rho}_1)h_1(\boldsymbol{\rho},\boldsymbol{\rho}_1). \quad (A1.5)$$

We have $g(\boldsymbol{\rho},\boldsymbol{\rho}_1)\to 0$ in the limit $|\boldsymbol{\rho}-\boldsymbol{\rho}_1|\to\infty$, since the particle correlations vanish at infinity. On the other hand, it is obvious that by virtue of the mutual repulsion of the vortices we have $g\to -n(\boldsymbol{\rho})$ as $|\boldsymbol{\rho}_1-\boldsymbol{\rho}_2|\to 0$. In addition, it can be shown that g is nonzero in a region of the order of the spacing a between the vortices. We adopt the following step approximation, which satisfies all the conditions enumerated above and the normalization condition $\int g(\boldsymbol{\rho},\boldsymbol{\rho}_1)d^2\boldsymbol{\rho}=-1$:

$$g=\begin{cases} -n(\boldsymbol{\rho}) & \text{for } |\boldsymbol{\rho}-\boldsymbol{\rho}_1|<\dfrac{1}{\sqrt{\pi n(\boldsymbol{\rho})}}, \\ 0 & \text{for } |\boldsymbol{\rho}-\boldsymbol{\rho}_1|>\dfrac{1}{\sqrt{\pi n(\boldsymbol{\rho})}}. \end{cases} \quad (A1.6)$$

Using this approximation, from (A1.5) we obtain

$$\Delta G=\frac{1}{8\pi}\int d^2\boldsymbol{\rho}\Phi_0 nH_{c1}\frac{\ln(H_{c2}/\Phi_0 n)}{\ln\kappa}. \quad (A1.7)$$

We stress that this expression has been obtained with logarithmic accuracy, i.e., in the argument of the logarithm coefficients of the order of unity have not been retained. To this accuracy the answer does not depend on the form of the function g. It can be seen that in fields $H_0\gg H_{c1}$ the term ΔG in the latter expression can be neglected.

The regions where $\nabla(F_1)/F_1\leqslant 1/a$ holds are surface regions and regions of density discontinuity. It can be show that in fields $H_0\gg H_{c1}$ the contribution to (A1.2) from these regions is of the same order as ΔG.

In the case of a uniform vortex-density distribution n=const, allowance for (A1.7) in the Gibbs energy (A1.4) gives the well known formula for the equilibrium induction of a regular lattice:

$$B=H_{eq}-\frac{\Phi_0}{8\pi\lambda^2}\ln\left(\frac{H_{c2}}{\Phi_0 n}\right). \quad (A1.8)$$

The approximate approach proposed here is valid for an arbitrary distribution of vortices, and gives the possibility of taking systematic account, in the model of the critical state, of effects associated with the difference of the thermodynamic field H_{eq} from the induction B. However, consideration of these questions goes beyond the scope of this article.

APPENDIX 2

In this Appendix we derive the critical-state equation (17), in which the contributions from the Meissner fields are taken into account explicitly. We shall perform the analysis for a superconducting cylinder of arbitrary (not necessarily circular) cross section in a magnetic field parallel to the generator of the cylinder.

To obtain the critical-state equation we vary G with respect to n:

$$\delta G=\frac{1}{8\pi}\int_\Omega d^2\boldsymbol{\rho}(\Phi_0B_v\delta n+\Phi_0 n\delta B_v-2H_0\delta B_v)$$
$$+\int_{\Omega_{\text{crit}}} d^2\boldsymbol{\rho} P_c l\delta n(\boldsymbol{\rho}). \quad (A2.1)$$

Taking into account that $\delta B_v=\int d^2\boldsymbol{\rho} h_1\delta n$, we obtain

$$\Phi_0B_v+\Phi_0\int_\Omega d^2\boldsymbol{\rho}_1 h_1(\boldsymbol{\rho}_1,\boldsymbol{\rho})n(\boldsymbol{\rho}_1)+8\pi P_c l$$
$$-2H_0\int_\Omega d^2\boldsymbol{\rho}_1 h_1(\boldsymbol{\rho}_1,\boldsymbol{\rho})=0. \quad (A2.2)$$

The second integral in this expression is equal to $-2H_0\Phi_0(1-h_m(\boldsymbol{\rho})/H_0)$ (see Appendix 4 in Ref. 19). To calculate the first integral we make use of the symmetry $h_1(\boldsymbol{\rho}_1,\boldsymbol{\rho})=h_1(\boldsymbol{\rho},\boldsymbol{\rho}_1)$ of the Green's function of the London equation (3). The result is that the first integral is equal to Φ_0B_v, and the vortex field is

$$B_v(\boldsymbol{\rho})=H_0\left(1-\frac{h_m(\boldsymbol{\rho})}{H_0}\right)+\frac{4\pi}{\Phi_0}P_c l. \quad (A2.3)$$

Finally, for the total field $B=h_m+B_v$ we obtain Eq. (17). Note that in the derivation we used the condition P_c=const.

APPENDIX 3

In this Appendix we give the solution of the system (45), (46) with $u_{\text{old}}=d_0$ and n_{old} determined by (28). Eliminating u from the system under consideration, and representing $B(x)$ in the form

$$B(x)=H_{\max}-\frac{4\pi}{c}J_c x+\beta(x), \quad (A3.1)$$

we obtain an equation for $\beta(x)$:

$$\beta-\lambda^2\frac{d^2\beta}{dx^2}=-d_0\frac{d\beta}{dx}-d_0x\frac{d^2\beta}{dx^2}+\frac{H_{\max}cd_0}{4\pi J_c}\frac{d^2\beta}{dx^2} \quad (A3.2)$$

with the boundary condition $\beta(0)=H_0-H_{\max}$.

This equation, after the substitution

$$y=\frac{2\lambda}{d_0}\sqrt{1+\frac{d_0}{\lambda}\frac{cH_{\max}}{4\pi J_c\lambda}-\frac{d_0x}{\lambda^2}}=\frac{2\lambda_{\rm eff}}{d_0}, \qquad (A3.3)$$

takes the form

$$y^2\frac{d^2\beta}{dx^2}+y\frac{d\beta}{dx}-y^2\beta=0. \qquad (A3.4)$$

Its general solution is $\beta(y)=C_1I_0(y)+C_2K_0(y)$. Note that in this equation $y\gg 1$ always holds (here we have used $d_0\sim a\ll\lambda$). Therefore, the following asymptotic representation of the general solution is always valid:

$$\beta(x)=C_1\sqrt{\frac{1}{2\pi y}}\exp(y)+C_2\sqrt{\frac{\pi}{2y}}\exp(-y). \qquad (A3.5)$$

Matching this solution with the exponentially decaying solution $\beta(x>b)=C_3\exp(-(x-b)/\lambda)$ and neglecting fields $\sim 4\pi J_c d_0/c$, we obtain (47). Note that, to this accuracy, we can disregard the change induced in the penetration depth b by the reversible displacement of vortices, and assume that b is given by (23).

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Translated by P. J. Sheperd