Noise-induced intermittency in systems with reactions and diffusion

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We solve the problem of the loss of stability of stationary states induced by external multiplicative noise in one-component systems with reactions and diffusion. We obtain the critical value of the noise level above which there is a transition, through intermittency, from one stable state of the system to another as well as the formation of new attractors for the system. © *1995 American Institute of Physics.*

1. INTRODUCTION

The study of reaction-diffusion equations with external noise plays a central role in problems dealing with combustion and detonation,[1] with the escape from a potential by means of noise,[2] with chemical kinetics,[3] with diffusion on a fluctuating surface,[4] in phenomena of the ballistic growth of crystals,[5] and in stochastic evolutionary models.[6] It is well known that external noise can significantly change the dynamic behavior of such systems, as a result of which one can observe noise-induced phase transitions.[7]

At the present time there exist rather powerful methods which make it possible to analyze processes in reaction-diffusion systems without noise (see, e.g., Ref. 8). In contrast to this, theoretical methods for studying stochastic reaction-diffusion processes are still not very universal.

Recently appreciable progress has been achieved by means of the so-called optimum fluctuation method in the study of one-component stochastic differential equations for a reaction-diffusion system of a general form.[9] The essence of this method consists in the following: a formal solution of the initial stochastic equation which describes a one-component reaction-diffusion system is approximated by the method of steepest descent. The system of two coupled deterministic reaction-diffusion equations obtained for the initial component and an auxiliary field as the result of this procedure enables us to describe both very rare events, when the behavior of the system is determined basically by a single trajectory, and the optimal, i.e., most probable, behavior of the system, averaged over an ensemble of realizations.

We consider a one-component system with reactions and diffusion; its behavior is described by the following stochastic partial differential equation:

$$\frac{\partial n(\mathbf{r},t)}{\partial t}=g(n)+D\Delta n(\mathbf{r},t)+h(n)\eta(\mathbf{r},t)+\xi(\mathbf{r},t), \qquad (1)$$

where $n(\mathbf{r},t)$ is the local density of some reactant R, $g(n)$ is a function describing the deterministic kinetics of R, $\Delta=\nabla^2$ is the Laplacian operator, and D is the diffusion constant. We assume that the concentrations of the other reagents A,B,C,... are constant. The effect of the external noise is described by the term $h(n)\eta(\mathbf{r},t)$, where $\eta(\mathbf{r},t)$ is Gaussian white noise of intensity σ with a zero average and a correlation function

$$s(\mathbf{r}-\mathbf{r}',t-t')=\langle \eta(\mathbf{r},t)\eta(\mathbf{r}',t')\rangle$$
$$=2\sigma\delta(\mathbf{r}-\mathbf{r}')\delta(t-t'). \qquad (2)$$

By definition $h(n)=1$ in the case of additive external noise. In the case of multiplicative noise the quantity $h(n)$ is a function of n describing the noise-induced deviations from the deterministic behavior of the system.

Intrinsic noise must satisfy the fluctuation-dissipation theorem connected with the requirement that $n(\mathbf{r},t)$ be conserved.[10] We show that the correlation function

$$s(\mathbf{r}-\mathbf{r}',t-t')=\langle \xi(\mathbf{r},t)\xi(\mathbf{r}',t')\rangle$$
$$=2\Gamma\nabla\nabla'\delta(\mathbf{r}-\mathbf{r}')\delta(t-t'), \qquad (3)$$

satisfies this requirement, where Γ is the noise intensity and in what follows $\langle\cdots\rangle$ denotes averaging over an ensemble of random realizations of the field $\xi(\mathbf{r},t)$.

The formal solution of Eq. (1) in terms of integrals over trajectories was given in Ref. 9. In fact, each trajectory $n=n(\mathbf{r},t)$ which is a solution of Eq. (1) has a well defined probability of being realized, given by the probability functional $P[n(\mathbf{r},t)]$:

$$P[n(\mathbf{r},t)]=\int Dp(\mathbf{r},t)\exp\left\{\int d\mathbf{r}\int dt\left(H-p\,\frac{\partial n}{\partial t}\right)\right\}, \qquad (4)$$

where

$$H=\sigma\int d\mathbf{r}dtd\mathbf{r}'dt's(\mathbf{r}-\mathbf{r}',t-t')[p(\mathbf{1})h(\mathbf{1})-\gamma h'(\mathbf{1})]$$
$$\times[p(\mathbf{2})h(\mathbf{2})-\gamma h'(\mathbf{2})]+\int d\mathbf{r}\,dt[-\gamma g'(\mathbf{1})+p(\mathbf{1})$$
$$\times(g(\mathbf{1})+D\Delta n)]. \qquad (5)$$

Here $p(r,t)$ is an auxiliary real field and we have introduced the notation $p(\mathbf{1})\equiv p(\mathbf{r},t)$, $h(\mathbf{2})\equiv h(n(\mathbf{r}',t'))$, $h'=\partial h/\partial n$, and so on. For generality we have used a constant discretization γ $(0\leqslant\gamma\leqslant 1)$ which in the case of the Stratonovich interpretation of Eq. (1) is equal to 1/2 and in the case of the Itoh interpretation is equal to zero.[7,11] It is clear that the description of the quantities which are averaged over an ensemble of realizations should not depend on the choice of this factor. We shall show that our approach assumes a well defined choice of γ.

To select the trajectory making the largest contribution to the probability functional (4) we use the saddle-point ap-

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proximation procedure.[1,9] The most probable (optimal) trajectory then satisfies the following variational equation:

$$\delta\left\{\int d\mathbf{r}dt\left(H-p\,\frac{\partial n}{\partial t}\right)\right\}=0, \tag{6}$$

or, more precisely,

$$\frac{\partial n}{\partial t}=\frac{\delta H}{\delta p},$$
$$\frac{\partial p}{\partial t}=-\frac{\delta H}{\delta n}. \tag{7}$$

Substituting Eq. (5) into (7) we get

$$\frac{\partial n}{\partial t}=g(n)+D\Delta n+2\sigma h(n)\int d\mathbf{r}'dt's(\mathbf{r}-\mathbf{r}',t-t')\times[p(\mathbf{1})h(\mathbf{1})-\gamma h'(\mathbf{1})],$$
$$-\frac{\partial p}{\partial t}=pg'-\gamma g''+D\Delta p+2\sigma(ph'-\gamma h'')\int d\mathbf{r}'dt's\times(\mathbf{r}-\mathbf{r}',t-t')[p(\mathbf{1})h(\mathbf{1})-\gamma h'(\mathbf{1})]. \tag{8}$$

The substitution of (3) into Eq. (8) gives the following deterministic model:

$$\frac{\partial n}{\partial t}=g(n)+D\Delta n. \tag{9}$$

Intrinsic noise thus does not lead to noise-induced transitions[12] and in what follows we therefore discuss effects induced by purely external noise.

Substituting (2) into (8) we get the following set of equations which describe the effect of external noise:

$$\frac{\partial n}{\partial t}=g(n)+D\Delta n+2\sigma h(n)(ph-\gamma h'),$$
$$-\frac{\partial p}{\partial t}=pg'-\gamma g''+D\Delta p+2\sigma(ph'-\gamma h'')(ph-\gamma h'). \tag{10}$$

One should, however, emphasize that the use of the saddle-point approximation procedure (6)–(8) requires in the justification of the procedure, obvious from a physical point of view, the choice of a region in which the noise plays a significant role. For a correct interpretation of the set of Eqs. (10) we must be guided by the fact that the solution of this system is the most probable (optimal) trajectory of the random process described by the stochastic differential equation (1). The most probable behavior then is that the system makes a transition from the initial state $n(\mathbf{r},0)$ into an attractor $n_0=A^{(0)}$ which lies close to it in configuration space and which is determined by the condition $g(n_0)=0$. As σ occurs explicitly in Eqs. (10) it is possible, firstly, to choose the noise level as a control parameter and, secondly, to study the response of the system to the action of the noise near a stationary attractor. In the low-noise limit this problem was studied in Ref. 13.

It has been shown by a number of authors (see, e.g., Ref. 14) that the behavior of some dynamical systems can be changed appreciably when the control parameter λ exceeds a critical value λ_c. For instance, it was found in Ref. 15 that for a system with external noise of intensity σ the response of the system to a change in the control parameter depends on the ratio $|\lambda-\lambda_c|/\sigma$.

Since the attractor $A^{(0)}$ possesses a potential of the attractive kind which has an effective barrier

$$g'(n_0)=-w<0, \tag{11}$$

one may expect that when one exceeds some critical value of the control parameter $\sigma=\sigma_c$ this barrier will be surmounted and one will observe in the system a noise-induced transition. Such behavior was observed[16] in the computer simulation of the action of external multiplicative noise on a system described by the two-dimensional Swift–Hohenberg equation. A method for obtaining critical values of the control parameter σ based on the stability in the linear approximation and a few assumptions about the behavior of the statistical moments was proposed in a recent paper.[17] However, in the case of white noise (2) which is δ-correlated in space the expressions for σ_c given in Ref. 17 contain singular terms. We propose an approach free of this deficiency which presents explicit solutions of Eq. (1). The basis of our method is a procedure for selecting a small parameter which is directly connected with the noise. This enables us to introduce a consistent renormalization procedure and a perturbation theory into the body of the optimal trajectory method (4)–(8). As a result the dynamics of the system (1) is split into motions in two mutually orthogonal subspaces and the behavior of the solutions lying in the "noise-induced" subspace is a solution of the Ginzburg–Landau equation. We describe the observed intermittency pattern in terms of the density peaks and troughs which appear and we find the conditions on the noise level under which new, noise-induced attractors are formed in the system.

2. PERTURBATION THEORY

We introduce a small parameter

$$\varepsilon=\pm\left(\frac{|\sigma-\sigma_c|}{\sigma}\right)^m,\quad m\geqslant 1, \tag{12}$$

where the choice of the values of m enables us to "maintain" small values of ε for the description of the behavior of the system both in the immediate vicinity of σ_c ($m\sim 1$) and also far from the critical value of the noise level ($m\gg 1$). Assuming that the deviations εn_1 induced by the multiplicative noise are small we introduce the following functional forms:

$$h(n)=\varepsilon n_1, \tag{13}$$

$$\begin{cases} n=n_0+\varepsilon n_1,\\ p=\varepsilon^{-1}\rho. \end{cases} \tag{14}$$

Substituting Eqs. (13) and (14) into Eqs. (10) we get

$$\frac{\partial n_1}{\partial t}=-wn_1+D\Delta n_1+2\sigma n_1(\rho n_1-\gamma\varepsilon),$$
$$-\frac{\partial \rho}{\partial t}=-w\rho-\gamma g''\varepsilon+D\Delta\rho+2\sigma\rho(\rho n_1-\varepsilon\gamma). \tag{15}$$

Neglecting terms which are small as $\varepsilon \to 0$ we get a set of equations describing the behavior of the system near its attractor:

$$\frac{\partial n_1}{\partial t} = -wn_1 + 2\sigma p n_1^2 + D\Delta n_1,$$
$$-\frac{\partial p}{\partial t} = -wp + 2\sigma p^2 n_1 + D\Delta p. \tag{16}$$

Returning now to the initial variable p we note that as a result we have obtained the same set of equations as if we had chosen from the start the Itoh interpretation of Eq. (1). This means[7] that in the phase transition point $\sigma=\sigma_c$ the instantaneous fluctuations are uncorrelated with the state of the system at the same time. In other words, in the vicinity of the critical point the phase space of the $\{n,p\}\in N$ system is in actual fact a direct sum of two mutually orthogonal subspaces:

$$N = N_0 \oplus N_1, \tag{17}$$

where all attractors of the deterministic Eq. (9) belong to the subspace N_0: $n_0 \in N$, and the noise-induced subspace N_1 is a direct product: $n_1 \otimes p = N_1$. In this sense our approach and the approach proposed in Ref. 18 to describe the reaction kinetics with a limited diffusion perpendicular to the diffusion coordinate have a certain similarity.

We can greatly simplify the set of Eqs. (16) which we have obtained, reducing it to a single equation by means of the following substitutions:

$$\tau = -2\sigma i t, \quad \mathbf{x} = \mathbf{r}\sqrt{2\sigma/D}^{1/2},$$
$$\psi = n_1 e^{wt}, \quad \psi^* = \pm p e^{-wt}. \tag{18}$$

Equations (16) then go over into a nonlinear Schrödinger equation ("with attraction" in the case of the plus sign and "with repulsion" in the case of the minus sign, respectively):

$$i\frac{\partial \psi}{\partial \tau} + \Delta_{\mathbf{x}}\psi \pm |\psi|^2\psi = 0. \tag{19}$$

Carrying out the inverse substitutions we get solutions of the initial set of Eqs. (16):

$$n = n_0 + \varepsilon[W(\mathbf{r},t)]^{\mathrm{real}},$$
$$p = \varepsilon^{-1}[W^*(\mathbf{r},t)]^{\mathrm{real}}, \tag{20}$$

where $W(r,t)$ satisfies the Ginzburg–Landau equation

$$\frac{\partial W}{\partial t} = -wW + D\Delta_{\mathbf{r}}W \pm 2\sigma|W|^2W. \tag{21}$$

The operation $[\cdots]^{\mathrm{real}}$ in Eqs. (20) indicates that the solution of Eq. (21) must be written in a real form by means of an analytical continuation of the free parameters satisfying the initial set of equations (16).

The Ginzburg–Landau Eq. (21) naturally arises in many problems when one describes small deviations from criticality; it is a so-called amplitude equation to which a broad class of partial differential equations can be reduced.[8]

3. SOLUTION OF THE GINZBURG–LANDAU EQUATION

Equation (21) possesses an important property: the temporal evolution of its solution has the following gauge form:[19]

$$\frac{\partial W}{\partial t} = -\frac{\delta F}{\delta W^*}, \quad \frac{\partial W^*}{\partial t} = -\frac{\delta F}{\delta W}, \tag{22}$$

$$F = \int d\mathbf{r}[w|W|^2 + D|\nabla W|^2 \mp \sigma|W|^4]. \tag{23}$$

One can easily show that

$$\frac{dF}{dt} = -2\int d\mathbf{r}\left|\frac{\partial W}{\partial t}\right|^2 \leqslant 0. \tag{24}$$

This means that the functional F has a minimum on the attractors of the Ginzburg–Landau Eq. (21) and is a Lyapunov functional for this equation.

Since for a Ginzburg–Landau equation with repulsion the functional F takes on only positive values, it follows from (24) that the only possible solution in the limit as $t\to\infty$ is $W=0$, i.e., the dynamics of the system is purely relaxational.

In what follows we restrict ourselves for simplicity to a one-dimensional space. Taking the solution of the corresponding nonlinear Schrödinger equation (19) with repulsion in the form[20]

$$\psi = \sqrt{\Omega}\,\frac{\partial}{\partial x}\left\{\ln\left[\cosh\left(x\sqrt{\frac{\Omega}{2}}\right)\right]\right\}e^{-i\Omega\tau} \tag{25}$$

and returning to the original variables we obtain the solutions which satisfy the Ginzburg–Landau equation with repulsion:

$$W = \frac{\sqrt{qD}}{2\sigma}\,\frac{\partial}{\partial r}\left\{\ln\left[\cosh\left(r\sqrt{\frac{(q}{2D}}\right)\right]\right\}e^{-(w+q)t}, \tag{26}$$

where Ω and q are the free parameters of the solutions (25) and (26).

In the case of a Ginzburg–Landau equation with attraction the functional F can take on both positive and negative values. As a result the solutions can, depending on the initial values of $W(r,0)$, describe both purely relaxational dynamics and the phenomenon of explosive growth.[19] It was shown in Ref. 19 that soliton solutions of the Ginzburg–Landau equation with attraction in the form

$$w = \sqrt{\frac{w}{\sigma}}\,\frac{1}{\cosh(r\sqrt{w/D})} \tag{27}$$

are "limit" solutions of (21) in the sense that if the functionals F of (23) and

$$I = \sigma\int |W|^4 dr \tag{28}$$

satisfy the following conditions:

$$F|_{t=0} < F_s = \frac{4}{3\sigma}\sqrt{w^3D}, \quad I > F_s, \tag{29}$$

where the index "s" refers to the soliton solution (27), the cubic term will dominate in the Ginzburg–Landau equation

(21) with attraction and its solutions will describe the explosive growth effect. Similarly we obtain the solution satisfying Eq. (21) with attraction:

$$W=\sqrt{\frac{q}{\sigma}}\frac{e^{(q-w)t}}{\cosh(r\sqrt{q/D})}. \tag{30}$$

Substituting (30) into (23) and (28) we can easily show that the condition (29) for explosive growth corresponds to $q>w$.

4. APPLICATION OF THE METHOD TO THE PROBLEM WITH MONOMOLECULAR REACTIONS

We assume that in the medium the substance R decays and undergoes autocatalytical multiplication. The decay of R is then a monomolecular process with a constant rate which is independent of r and t and the multiplication occurs solely inside certain multiplication centers which appear at random moments in random positions in the medium. The multiplication centers may be generated by some external stochastic action, for instance, by laser irradiation.[21]

The rate of change of the concentration $n(r,t)$ of the substance R in such a reactive medium in the case of a one-dimensional space is described by the following stochastic differential equation:

$$\frac{\partial n(r,t)}{\partial t}=[-w+\eta(r,t)]n+D\Delta n(r,t)+\xi(r,t), \tag{31}$$

where $w>0$ is the average value of the difference in the rate constants of the decay and the multiplication, $\eta(r,t)$ is the fluctuating part of this difference, and $\xi(r,t)$ describes the fluctuations in the rate of change of the concentration of R caused by the intrinsic noise in the system.

We approximate $\eta(r,t)$ by Gaussian noise with the following correlation function:

$$\langle\eta(r,t)\eta(r',t')\rangle=2\sigma s(r-r')\delta(t-t'), \tag{32}$$

where $s(r-r')$ describes the spatial correlation between the randomly appearing separate multiplication centers. It is shown in Ref. 3 that when the diffusion is sufficiently large, namely, for $D\gg 2\sigma r_0^2$ (where 2σ is the strength of the fluctuating field $\eta(r,t)$ and r_0 the characteristic size of a separate multiplication center) the function $s(r-r')$ can be represented by a δ-function.

We note that the assumption that the deviations are small, which was introduced to linearize the "reaction term" $g(n)$ and the functional $h(n)$ in the general case of Eq. (1), is here superfluous because of the linearity of $g(n)=-wn$ and of $h(n)=n$. However, in order to remain independent of the interpretation of Eq. (31) (choice of the factor γ) we retain the functional forms (13) and (14) with the small parameter ε of (12).

We have $w>0$, so $A^{(0)}=n_0=0$ is an attractor of the system and the function $h(n)$ describing the action of the external noise has the form

$$h(n)=n=|\varepsilon|n_1, \tag{33}$$

which is the same as (13) when there is a plus sign the definition (12) of ε. The problem with monomolecular reactions is then mathematically equivalent to the problem of the escape from a fluctuating potential which was formulated earlier; in what follows (see also §6) this enables us to use the results of Ref. 6, which were obtained for Eq. (31), and apply them to the study of Eqs. (16).

According to (20), (26), and (30) the solutions of the stochastic differential equation (31) are the following solutions of the Ginzburg–Landau equation with repulsion and with attraction, respectively:

$$\begin{cases} n=\dfrac{\varepsilon\sqrt{qD}}{2\sigma}\dfrac{\partial}{\partial r}\left\{\ln\left[\cosh\left(r\sqrt{\dfrac{q}{2D}}\right)\right]\right\}e^{-(w+q)t}, \\ \rho=\dfrac{\sqrt{qD}}{2\sigma}\dfrac{\partial}{\partial r}\left\{\ln\left[\cosh\left(r\sqrt{\dfrac{q}{2D}}\right)\right]\right\}e^{(w+q)t}, \end{cases} \tag{34}$$

$$n=\varepsilon\sqrt{\frac{q}{\sigma}}\frac{e^{(q-w)t}}{\cosh(r\sqrt{q/D})},\quad \rho=\sqrt{\frac{q}{\sigma}}\frac{e^{-(q-w)t}}{\cosh(r\sqrt{q/D})}. \tag{35}$$

We note[22] that the nonlinear Schrödinger equations (19) which generate the solutions (34) and (35) are invariant under the Galilean transformation $\hat{G}$ with a free transformation parameter U':

$$\psi'=\psi(x-U'\tau,\tau)\exp\left\{\frac{iU'}{2}(x-U'\tau)+\frac{iU'^2\tau}{4}\right\}. \tag{36}$$

We act with the group $\hat{G}$ on the corresponding solutions of the nonlinear Schrödinger equation and analytically continue the transformation parameter U':

$$V=-iU'\sqrt{2\sigma D}. \tag{37}$$

From the "fixed" solution (34) we then get ($\zeta = r - Vt$)

$$\begin{aligned} n&=\frac{\varepsilon\sqrt{qD}}{2\sigma}\frac{\partial}{\partial\zeta}\left\{\ln\left[\cosh\left(\zeta\sqrt{\frac{q}{2D}}\right]\right\}\right. \\ &\times\exp\left\{-\left[q+w+\frac{V^2}{4D}\right]t-\frac{V\zeta}{2D}\xi\right\}, \\ \rho&=\frac{\sqrt{qD}}{2\sigma}\frac{\partial}{\partial\zeta}\left\{\ln\left[\cosh\left(\zeta\sqrt{\frac{q}{2D}}\right)\right]\right\} \\ &\times\exp\left\{\left[q+w+\frac{V^2}{4D}\right]t+\frac{V\zeta}{2D}\right\}, \end{aligned} \tag{38}$$

and from the fixed solution (35)

$$\begin{aligned} n&=\frac{\varepsilon\sqrt{q/\sigma}}{\cosh(\zeta\sqrt{q/D})} \\ &\times\exp\left\{\left(q-\frac{V^2}{4D}-w\right)t-\frac{V\zeta}{2D}\right\}, \\ \rho&=\frac{\sqrt{q/\sigma}}{\cosh(\zeta\sqrt{q/D})} \\ &\times\exp\left\{-\left[\left(q-\frac{V^2}{4D}-w\right)t-\frac{V\zeta}{2D}\right]\right\}. \end{aligned} \tag{39}$$

We note that the "moving" solutions obtained as the result of acting with the group $\hat{G}$ on the fixed solutions (34) and (35) go over into their fixed counterparts for $V=0$. The family of solutions (38) and (39) therefore also includes the "initial" solutions (34) and (35).

It was noted in Ref. 6 that the solutions (39) describe peaks which are moving with a velocity $V<V_c=2\sqrt{qD}$ and which are growing (or damping) with time at a rate $Q=q-V^2/4D-w$. For the solutions (38) we have $V_c=0$ and, hence, this family of solutions shrinks to the fixed solution (34).

It follows from (4), (5), (10), and (13) that the probability for the realization (existence) of the optimal trajectory after a given time T, apart from a pre-exponential factor, will be

$$\mu(T)=P[n_{\text{opt}}]\propto\exp\left[-\sigma\int dr\,dt(ph)^2\right]$$
$$=\exp\left[-\sigma\int dr\,dt(\rho n_1)^2\right]. \tag{40}$$

We then have for the solutions (39)

$$\mu(T)\propto\exp\left[-\frac{4\sqrt{D}}{3\sigma}\left(Q+\frac{V^2}{4D}+w\right)^{3/2}T\right], \tag{41}$$

and for the solutions (38)

$$\mu(T)|_{L\to\infty}\to 0, \tag{42}$$

where L is the characteristic size of the system.

For a given growth rate Q the fixed solutions (35) are thus more probable.

Considering these solutions as a whole we can conclude that as the initial perturbation approaches the attractor of the system (in the $Q<0$ case) the effect of the noise becomes more significant: the solutions are described by the optimal trajectory in the form of the peaks (39) which decrease slowly with time and the fast decay of the unstable [by virtue of (42)] states (34) serves as the source of their formation. After the passage of a certain time interval one will observe the characteristic effect of the critical delay of the approach to equilibrium.[23]

However, in the $Q>0$ case peaks which grow with time can form for the solutions (35). It is just those solutions which determine another kind of behavior, which we shall now consider.

5. DESCRIPTION OF NOISE-INDUCED INTERMITTENCY

It was noted in Ref. 24 that a field distribution $n(r,t)$ in that has structures accompanying high peaks with large concentrations of matter and short life times or a short extension in space is typical for systems described by the stochastic differential equation (31). The spaces between them are characterized by a low concentration of matter and a large extension. In general one calls such a situation intermittency. In Ref. 6 one found for the statistical moments $M_k(t,r_1,\ldots,r_k)$,

$$M_k(t,r_1,\ldots,r_k)=\langle n(r_1,t)\ldots n(r_k,t)\rangle \tag{43}$$

the following asymptotic behavior:

$$M_k\propto\exp\left[(\sigma-w)kt+\frac{\sigma^2}{12D}k(k+1)(k-1)t\right]. \tag{44}$$

This unusual (compared to Gaussian) behavior of the moments is explained[24] by the fact that the main contribution to each moment comes from the peaks (39).

Indeed, the density $n=M_1$ of (39), averaged over an ensemble of realizations, has a temporal behavior which exactly corresponds to the behavior of the solutions of (39) for

$$\sigma=Q-w=q-V^2/4D. \tag{45}$$

If we now leave the most probable solutions (for a given growth rate Q and, in fact, for a given value of the noise level) with $V=0$ we get an important result: for any stationary state n_0 of the system there exists a well defined critical value of the noise level,

$$\sigma_c=w, \tag{46}$$

above which this state loses its stability. In that case the noise-induced deviations grow with time and lead to the formation of fixed concentration peaks (39). For the higher moments the term which is quadratic in σ in Eq. (40) is the joint probability for the vanishing of one peak ($\propto k-1$) and the appearance of yet another peak ($\propto k+1$) in the situation where there are already k pieces.

When we exceed the critical value $\sigma_c=w$ of the noise level there are no longer stable equilibrium states in the system (31) and it starts to wander about in phase space, appearing as intermittency. The intermittency in the problem (31) looks as follows:[24] for $\sigma>\sigma_c$ there occurs an exponential growth of the optimal realization of the field $n(r,t)$ [for $\sigma<\sigma_c$ the statistical moments of second and higher orders may also grow when the diffusion is sufficiently small—see (40)]. As the control parameter σ increases the formation of moving solutions becomes more probable in the system. By a fixed time the statistical and spatial averages are exactly the same up to a certain value of k_0 (the order of the moment) while for larger k the statistical moments show the largest growth. This is because the collection of peaks corresponding to each statistical moment is such a rare event that it is as a rule not realized in the system.

6. NOISE-INDUCED ESCAPE FROM A POTENTIAL AND POPULATION OF THE MEDIUM

We established in §4 the equivalence in the mathematical statement of the problem for describing intermittency between the problem with monomolecular reactions described by the stochastic differential equation (31) and the problem of the escape from a potential described by Eq. (1). This is because our method makes it possible to split the motions in the phase space of the system (1) into motions in two mutually orthogonal subspaces where the noise-induced subspace is the same as the phase space of the system (31). We showed that as a result of the action of multiplicative noise with an intensity $\sigma>\sigma_c$ the state n_0 loses its stability and breaks up into concentration troughs and peaks:

$$n=n_0\pm|\varepsilon|\frac{e^{(\sigma-w)t}}{\cosh(r\sqrt{\sigma/D})}. \tag{47}$$

The probability that they are realized is independent of the value n_0 of the concentration:

$$\mu(T) \propto \exp\left[-\frac{4}{3} T\sqrt{\sigma D}\right]. \tag{48}$$

It follows from Eq. (47) that the peaks are most stable for a value of the control parameter σ close to its critical value σ_c of (46).

It follows from the analysis just given that the system (1) evolves on the characteristic slow time scale $\tau_m = \varepsilon^{1/m} t$:

$$n = n_0 \pm |\varepsilon| \frac{e^{\sigma \tau_m}}{\cosh(r\sqrt{\sigma/D})}, \tag{49}$$

where ε and the index m connected with it are defined in (12). Now dividing the space occupied by the peaks (35) into small cells with a constant concentration of the reactant R one can consider the system (1) approximately as being in a state of quasi-equilibrium inside each of them. In such an adiabatic approximation each of these cells corresponds to its own value of the quasi-equilibrium concentration $n_0(\tau_m)$ and thus with its own potential barrier (11), $w(\tau_m)$. By applying our method to describe the loss of stability of the (quasi) equilibrium state we can describe the transition of the system from one stable state into another and also the formation of new, noise-induced states.

We note that the medium is populated[4] through the loss of stability of the attractor in the attraction region in which the system finds itself initially and the transition of the system into two neighboring (for a given noise level) stable deterministic attractors with a higher and a lower population concentration n_i corresponding to the plus and minus signs in (12): $n_1 < n_0 < n_2$.

When the noise level σ is insufficiently high to take the system completely out of the attraction region n_0 of the attractor (or when one of the attractors n_1 or n_2 is not present), it is possible to form in the system (1) a new, noise-induced attractor $\hat{n}_0$ (in contrast to problem (31) with a linear form of the reactive term $g(n)$). From (11) and (42) we get the conditions for the stabilization of this new stable state through noise with a strength σ:

$$-g'(\hat{n}_0) = \sigma. \tag{50}$$

Varying the value of the control parameter—the noise level σ—we may observe a hysteresis effect: depending on the initial conditions the system will go to neighboring attractors for a given value of the control parameter.

7. CONCLUSION

We have shown that when the noise level reaches a certain critical value a state which is an attractor for the given initial conditions loses stability. When one goes above this critical value of the noise level the behavior of the system becomes intermittent. Such a behavior is characterized by the presence of self-localized structures which are accompanied by high peaks with a large concentration of matter and a small spatial extension. Knowledge of the critical values of the noise level for each attractor of the system enables us to control the behavior of the system by varying the noise level.

One may expect that by virtue of its visualizability and the many possible sources of external noise the proposed theoretical approach will often be applied to describe effects in multiplicative systems, especially in applications to systems with biomolecular kinetics.[25]

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