Transformation of second sound into surface waves in superfluid helium

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The Hamiltonian theory of superfluid liquid with a free boundary is developed. Nonlinear amplitudes of parametric Cherenkov radiation of a surface wave by second sound and the inner decay of second sound waves are found. Threshold amplitudes of second sound waves for these two processes are determined. © *1995 American Institute of Physics.*

1. INTRODUCTION

In this paper we consider nonlinear processes in the hydrodynamics of a superfluid with a free boundary. Such a system has conventional bulk wave modes (first and second sound) as well as surface waves propagating along the boundary and attenuating in the bulk.

The most proper method to describe nonlinear processes is the Hamiltonian formalism. The two-component hydrodynamics equations have been shown to have the Hamiltonian form with the energy of the liquid playing the role of the Hamiltonian.[1]

We introduce the following system of coordinates. Axis z is directed vertically, perpendicular to the surface of the liquid and to the rigid boundary bounding the system from below (the latter has zero z-coordinate). The unperturbed depth of the liquid will be denoted by h. The equation of the free boundary is

$$z_s(\mathbf{r},t)=h+\zeta(\mathbf{r},t),$$

where $\mathbf{r}$ is a 2D-vector with coordinates (x,y) (to distinguish it from a 3D radius vector $\mathbf{R}$) and ζ is the deviation of the z-coordinate of the surface from its equilibrium value $z=h$.

In this coordinate system the Hamiltonian has the following form:

$$H=\int d^2\mathbf{r}\int_0^{h+\zeta} dz\left[\frac{\rho_n}{2}\,\mathbf{v}_n^2+\frac{\rho_s}{2}\,\mathbf{v}_s^2+\varepsilon(\rho,s,\mathbf{P})\right]+\int d^2\mathbf{r}\tilde{\alpha}\sqrt{1+(\nabla\zeta)^2}. \quad (1)$$

Here and henceforth ρ is the density of the liquid, ρ_s and ρ_n are its superfluid and normal components respectively, $\mathbf{v}_s$ is the superfluid velocity, $\mathbf{v}_n$ is the normal velocity, $\tilde{\alpha}$ is the surface tension, ε is the internal energy per unit volume, which depends on the entropy per unit mass s and momentum per unit volume $\mathbf{P}$, measured in a frame of reference moving with velocity $\mathbf{v}_s$:

$$\mathbf{P}=\rho_n(\mathbf{v}_n-\mathbf{v}_s).$$

It was shown[1] that an infinite superfluid can be described with three pairs of canonically conjugate variables (ρ,α), (S,β), and (ψ,γ), where α is the superfluid velocity potential, S is the entropy density, β is a phase variable conjugate to S, and ψ and γ are Clebsch variables, responsible for the vorticity.

The application of the canonical formalism with local variables to a liquid with a free boundary meets with difficulty. Indeed, the coordinates play the role of the indices in the Lagrangian description of the liquid. Since the free boundary is time-dependent, some variables appear and others disappear.[2] To avoid this difficulty nonlocal variables should be introduced. For the simple case of a one-component ideal compressible liquid with a free boundary, appropriate variables were introduced in.[2] These are the amplitudes of the bulk and surface waves. In this paper we will determine corresponding nonlocal variables for a two-component liquid.

Let us discuss the role of gravitation. Since the compressibility is small, one can neglect the influence of the gravitational force upon the propagation of bulk sound waves. The variation of the liquid density $\Delta\rho/\rho$ in the gravitational field on the scale of the wavelength is small, $g\lambda/c_1^2\ll 1$, where c_1 is the velocity of first sound. In fact, in a real experiment a much stronger condition is fulfilled, $gh/c_1^2\ll 1$, where h is the depth of the liquid. This means that the unperturbed density does not depend on the z-coordinate.

2. EQUATIONS OF MOTION. HARMONIC APPROXIMATION

For the reader's convenience, we reproduce here the well-known equations of hydrodynamics for a superfluid liquid without dissipation (see, e.g. Ref. 3).

$$\frac{\partial\rho}{\partial t}+\operatorname{div}\mathbf{j}=0, \quad (2)$$

$$\frac{\partial\rho s}{\partial t}+\operatorname{div}(\rho s\mathbf{v}_n)=0, \quad (3)$$

$$\frac{\partial\mathbf{v}_s}{\partial t}+\nabla\left(\mu+\frac{1}{2}\,\mathbf{v}_s^2\right)=0, \quad (4)$$

$$\frac{\partial j_i}{\partial t}+\frac{\partial\Pi_{ik}}{\partial x_k}=0. \quad (5)$$

Here $\mathbf{j}$ is the mass flux,

$$\mathbf{j}=\rho_s\mathbf{v}_s+\rho_n\mathbf{v}_n, \quad (6)$$

and Π_{ik} is the momentum flux tensor

$$\Pi_{ik}=p\,\delta_{ik}+\rho_s v_{si}v_{sk}+\rho_n v_{ni}v_{nk}. \quad (7)$$

The chemical potential per unit mass is denoted μ.

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$$\mu=\frac{\partial\varepsilon}{\partial\rho}. \tag{8}$$

For a nondissipative liquid, the boundary conditions at a rigid bottom are

$$v_{nz}|_{z=0}=v_{sz}|_{z=0}=0. \tag{9}$$

The conditions at the free surface are

$$\dot{\zeta}=[v_{sz}-\zeta_\nu v_{s\nu}]_{z=h+\zeta}=[v_{nz}-\zeta_\nu v_{n\nu}]_{z=h+\zeta}, \tag{10}$$

where $\zeta_\nu=\partial\zeta/\partial x_\nu$, $\nu=1,2$, and

$$p|_{z=h+\zeta}+\nabla\left(\frac{\tilde{\alpha}\nabla\zeta}{\sqrt{1+(\nabla\zeta)^2}}\right)=0. \tag{11}$$

Equation (10) means that the component of the velocity perpendicular to the surface at the free boundary coincides with the same component of the velocity of the liquid near the surface. We assume that the velocities $\mathbf{v}_n$ and $\mathbf{v}_s$ have equal components perpendicular to the boundary. Equation (11) is equivalent to the assumption of zero external pressure above the surface.

The total mass flux $\mathbf{j}$ may be written as

$$\mathbf{j}=\rho\mathbf{v}_s+\rho_n(\mathbf{v}_n-\mathbf{v}_s)=\rho\nabla\alpha+\rho s\nabla\beta,$$

where potentials α and β depend on the coordinates and time. The velocities $\mathbf{v}_s$ and $\mathbf{v}_n$ can be expressed in terms of α and β:

$$\mathbf{v}_s=\nabla\alpha, \tag{12}$$

$$\mathbf{v}_n=\nabla\alpha+\frac{\rho s}{\rho_n}\nabla\beta. \tag{13}$$

We introduce the dimensionless variables $\tilde{\rho}$ and σ characterizing the deviations of the density and entropy from their unperturbed values, which are denoted by the subscript "0":

$$\delta\rho=\rho_0\tilde{\rho}, \tag{14}$$

$$\delta s=s_0\left(\frac{\rho_{s0}}{\rho_{n0}}\right)^{1/2}\sigma. \tag{15}$$

First we consider a simplified version of the theory with the coefficient of thermal expansion $\kappa_0=(\partial\ln\rho/\partial\ln T)$ assumed to be zero. This coefficient is indeed small (10^{-2} in order of magnitude) over the full range of temperature from $T=0$ to the λ-point. At $T=1.15$ K, the coefficient κ_0 changes sign. Nevertheless for one parametric process (decay of second sound into two surface waves), this coefficient is crucial and should be taken into account. In this paper we restrict our consideration to other processes for which thermal expansion is not significant. Neglecting thermal expansion, the velocities of first and second sound are given by well-known formulas:

$$c_1^2=\left(\frac{\partial p}{\partial\rho}\right)_s, \tag{16}$$

$$c_2^2=s_0^2\left(\frac{\partial T}{\partial s}\right)\frac{\rho_{s0}}{\rho_{n0}}. \tag{17}$$

First and second sound show up as waves propagating along the surface with wave vector $\mathbf{p}$, and a standing wave in the vertical direction. A general form of potential for such waves is:

$$\phi=\bar{\phi}_{\mathbf{k}}\cos q_1z\exp i(\mathbf{pr}-\omega t), \tag{18}$$

$$\tilde{\rho}=\bar{\rho}_{\mathbf{k}}\cos q_1z\exp i(\mathbf{pr}-\omega t), \tag{19}$$

$$\psi=\bar{\psi}_{\mathbf{k}}\cos q_2z\exp i(\mathbf{pr}-\omega t), \tag{20}$$

$$\sigma=\bar{\sigma}_{\mathbf{k}}\cos q_2z\exp i(\mathbf{pr}-\omega t), \tag{21}$$

where

$$\phi=\alpha+s_0\beta, \tag{22}$$

$$\psi=s_0\left(\frac{\rho_{s0}}{\rho_{n0}}\right)^{1/2}\beta, \tag{23}$$

and $q_{1,2}$ obey the following conditions:

$$k_i^2=q_i^2+\mathbf{p}^2, \tag{24}$$

$$k_i=\omega/c_i\quad(i=1,2), \tag{25}$$

$$\cot q_1h=-\frac{\Lambda(\mathbf{p})q_1}{\bar{\omega}^2}, \tag{26}$$

$$\sin q_2h=0, \tag{27}$$

and

$$\Lambda(\mathbf{p})=g+\frac{\tilde{\alpha}}{\rho_0}p^2.$$

The relations between amplitudes can be found from the linearized equations of motion:

$$\bar{\rho}=\frac{i\omega}{c_1^2}\bar{\phi}, \tag{28}$$

$$\bar{\zeta}=\frac{q_1}{i\omega}\sin q_1h\bar{\phi}=\frac{i\omega}{\Lambda(\mathbf{p})}\cos q_1h\bar{\phi}, \tag{29}$$

$$\bar{\sigma}=\frac{i\omega}{c_2^2}\bar{\psi}. \tag{30}$$

Equations (24)–(27) define the dispersion $\omega(\mathbf{k})$ and the discrete spectrum of q. Assuming the coefficient of thermal expansion to be zero, first sound can be treated as an adiabatic oscillation of density (pressure), while second sound comprises isobaric oscillations of temperature (entropy). Formally it means that $\sigma=\psi=0$ for first sound, while $\tilde{\rho}=\phi=0$ for second sound. Oscillations of the density can occur for purely real or purely imaginary q. The first case holds for first sound, and the second corresponds to surface waves.

Putting $q=i\kappa$ for the surface wave, one can find a recognizable relationship:

$$\coth\kappa h=\frac{\Lambda(\mathbf{p})\kappa}{\omega^2}, \tag{31}$$

$$\mathbf{k}_1^2=\mathbf{p}^2-\kappa^2. \tag{32}$$

Usually the velocity of the surface waves is much lower than the velocity of first sound. From this, we find approximately $\kappa=|\mathbf{p}|$. The spectrum of surface waves in a two-component liquid coincides with that in a one-component liquid. In the case of deep water, $h\to\infty$, the spectrum is greatly simplified:

$$\kappa=p,\quad \omega^2=p\Lambda(p). \tag{33}$$

3. THE HAMILTONIAN. ORTHOGONALITY RELATIONS

It is easy to see that the linearized equations of motion can be described by the Hamiltonian

$$H=\int d^2\mathbf{r}\int_0^h dz(\rho_0(\nabla\phi^*)(\nabla\phi)+\rho_0(\nabla\psi^*)(\nabla\psi)+\rho_0 c_1^2\tilde{\rho}^*\tilde{\rho}+\rho_0 c_2^2\sigma^*\sigma)+\rho_0\int d^2\mathbf{r}\zeta^*\hat{\Lambda}\zeta, \tag{34}$$

where

$$\hat{\Lambda}=g-\frac{\tilde{\alpha}}{\rho_0}\Delta_2,$$

Δ_2 is the two-dimensional Laplacian:

$$\Delta_2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2},$$

and the Poisson brackets are

$$\{\phi^*(\mathbf{R},t),\rho_0\tilde{\rho}(\mathbf{R}',t)\}=\delta(\mathbf{R}-\mathbf{R}'), \tag{35}$$

$$\{\psi^*(\mathbf{R},t),\rho_0\sigma(\mathbf{R}',t)\}=\delta(\mathbf{R}-\mathbf{R}'), \tag{36}$$

$$\{\phi_\Gamma^*(\mathbf{r},t),\rho_0\zeta(\mathbf{r}',t)\}=\delta(\mathbf{r}-\mathbf{r}'). \tag{37}$$

Here $\phi_\Gamma=\phi|_{z=h}$, and other brackets equal to zero. We represent the dynamical variables by two vectors Φ_1 and Φ_2:

$$\Phi_1=\begin{pmatrix}\phi(\mathbf{R},t)\\ \tilde{\rho}(\mathbf{R},t)\\ \zeta(\mathbf{r},t)\end{pmatrix}, \tag{38}$$

$$\Phi_2=\begin{pmatrix}\psi(\mathbf{R},t)\\ \sigma(\mathbf{R},t)\end{pmatrix}. \tag{39}$$

In terms of Φ_1 and Φ_2, the Hamiltonian (34) can be represented as a sum of scalar products:

$$H=(\Phi_1,\Phi_1)+(\Phi_2,\Phi_2), \tag{40}$$

where the two bilinear forms (Φ_1',Φ_1) and (Φ_2',Φ_2) are defined by

$$(\Phi_1',\Phi_1)=\int d^2\mathbf{r}\rho_0\left[\phi\frac{\partial\phi'^*}{\partial z}\right]_{z=h}+\int d^2\rho_0\mathbf{r}\zeta'^*\hat{\Lambda}\zeta+\int d^3\mathbf{R}(-\rho_0(\Delta\phi'^*)\phi+\rho_0 c_2^2\tilde{\rho}'^*\tilde{\rho}), \tag{41}$$

$$(\Phi_2',\Phi_2)=\int d^2\mathbf{r}\rho_0\left[\psi\frac{\partial\psi'^*}{\partial z}\right]_{z=h}+\int d^3\mathbf{R}(-\rho_0(\Delta\psi'^*)\psi+\rho_0 c_2^2\tilde{\sigma}'^*\tilde{\sigma}). \tag{42}$$

It is easy to check that the forms (41) and (42) satisfy all the conditions of a scalar product. The elementary monochromatic solutions (18)–(21) are orthogonal with respect to these products if they belong to different modes or to different $\mathbf{k}$. (Note that the surface integral in Eq. (42) vanishes in the harmonic approximation, but should be accounted for in nonlinear processes.)

Let us normalize the function Φ_1 and Φ_2 by the conditions:

$$(\Phi_{1\mathbf{k}_1},\Phi_{1\mathbf{k}_1})=\omega_{\mathbf{k}_1}, \tag{43}$$

$$(\Phi_{2\mathbf{k}_2},\Phi_{2\mathbf{k}_2})=\omega_{\mathbf{k}_2}, \tag{44}$$

where

$$\Phi_{1\mathbf{k}_1}(\mathbf{R},t)=\begin{pmatrix}\phi_{\mathbf{k}_1}(\mathbf{R},t)\\ \tilde{\rho}_{\mathbf{k}_1}(\mathbf{R},t)\\ \zeta_{\mathbf{p}_1}(\mathbf{r},t)\end{pmatrix}, \tag{45}$$

$$\Phi_{2\mathbf{k}_2}(\mathbf{R},t)=\begin{pmatrix}\psi_{\mathbf{k}_2}(\mathbf{R},t)\\ \sigma_{\mathbf{k}_2}(\mathbf{R},t)\end{pmatrix}, \tag{46}$$

and $\mathbf{k}_1$ and $\mathbf{k}_2$ are wave vectors corresponding to first and second sound respectively.

These conditions fix normalized amplitudes of first and second sounds and the surface wave:

$$|\phi_{\mathbf{k}_1}|^{-2}=\frac{2\rho_0 A}{\omega}\left(h\mathbf{k}_1^2+\frac{\mathbf{p}^2-q_1^2}{2q_1}\sin 2q_1h\right), \tag{47}$$

$$|\psi_{\mathbf{k}_2}|^{-2}=\frac{2\rho_0 A\mathbf{k}_2^2 h}{\omega}, \tag{48}$$

$$|\phi_{\mathbf{p}}|^{-2}=\frac{2\rho_0 Ap}{\omega_s}. \tag{49}$$

Here A is the area of the free surface and ω_s is the frequency of the surface wave. The vector functions Φ_1, Φ_2 can be represented as series:

$$\Phi_1(\mathbf{R},t)=\sum_{\mathbf{k}_1}a_{\mathbf{k}_1}\Phi_{1\mathbf{k}_1}(\mathbf{R},t)+\sum_{\mathbf{p}}c_{\mathbf{p}}\Phi_{1\mathbf{p}}(\mathbf{R},t), \tag{50}$$

$$\Phi_2(\mathbf{R},t)=\sum_{\mathbf{k}_2}b_{\mathbf{k}_2}\Phi_{2\mathbf{k}_2}(\mathbf{R},t), \tag{51}$$

where $a_{\mathbf{k}}$, $b_{\mathbf{k}}$ and $c_{\mathbf{p}}$ are the canonical amplitudes of first and second sound and surface waves respectively. In such a representation, the Hamiltonian H acquires a standard diagonal form,

$$H=\sum_{\mathbf{k}_1}a_{\mathbf{k}_1}^*a_{\mathbf{k}_1}\omega_{\mathbf{k}_1}+\sum_{\mathbf{k}_2}b_{\mathbf{k}_2}^*b_{\mathbf{k}_2}\omega_{\mathbf{k}_2}+\sum_{\mathbf{p}}c_{\mathbf{p}}^*c_{\mathbf{p}}\omega_{\mathbf{p}}. \tag{52}$$

The equation of motion derived from this Hamiltonian and the canonical Poisson brackets:

$$\{a_{\mathbf{k}_1}^*,a_{\mathbf{k}_1'}\}=-i\delta_{\mathbf{k}_1\mathbf{k}_1'}, \tag{53}$$

$$\{b_{\mathbf{k}_2}^*,b_{\mathbf{k}_2'}\}=-i\delta_{\mathbf{k}_2\mathbf{k}_2'}, \tag{54}$$

$$\{c_{\mathbf{p}}^*,c_{\mathbf{p}'}\}=-i\delta_{\mathbf{p}\mathbf{p}'}, \tag{55}$$

have the standard form:

$$\dot{a}_{\mathbf{k}_1} = -i\omega_{\mathbf{k}_1} a_{\mathbf{k}_1},$$

$$\dot{b}_{\mathbf{k}_2} = -i\omega_{\mathbf{k}_2} b_{\mathbf{k}_2}, \quad \dot{c}_{\mathbf{p}} = -i\omega_{\mathbf{p}} c_{\mathbf{p}}. \tag{56}$$

4. NONLINEAR RESONANT PROCESSES

Nonlinear effects such as parametric generation and decay of sound waves correspond to anharmonic terms in the Hamiltonian. The coefficients of this expansion are proportional to the amplitudes of these processes.

We consider only resonant processes that conserve energy and momentum simultaneously:

$$\omega_{\mathbf{k}} = \omega_{\mathbf{k}'} + \omega_{\mathbf{k}''}, \tag{57}$$

$$\mathbf{p} = \mathbf{p}' + \mathbf{p}''. \tag{58}$$

In a restricted geometry, only the components of momentum parallel to the surface are conserved. For infinite depth, interaction amplitudes have been found by direct series expansion of the Hamiltonian in the amplitudes. However, for a system with a free boundary, we prefer to reconstruct the Hamiltonian step by step from the hydrodynamic equations and boundary conditions.

To demonstrate our program, we find first the amplitude of Cherenkov radiation of the surface wave by second sound. In order to find it, we must write down the equations of motion and boundary conditions up to the next order in the amplitudes:

$$\Delta\psi + \dot{\sigma} + \frac{\partial}{\partial t}(\tilde{\rho}\sigma) + (1-v)\nabla(\tilde{\rho}\nabla\psi) + \nabla(\sigma\nabla\psi) = 0, \tag{59}$$

$$\dot{\psi} + c_2^2\sigma + (\nabla\phi\nabla\psi) = 0, \tag{60}$$

$$\left.\frac{\partial\psi}{\partial z}\right|_{z=h} + \zeta\left.\frac{\partial^2\psi}{\partial z^2}\right|_{z=h} - (\nabla\psi\nabla\zeta)|_{z=h} = 0, \tag{61}$$

where

$$v = \frac{\rho_0}{\rho_{s0}}\left(\frac{\partial}{\partial \ln\rho}\ln\left(\frac{\rho_n}{\rho}\right)\right)_T.$$

We have omitted terms of the same order that do not contribute to this process. The functions ψ and σ contain the contribution of the incident and reflected waves and corrections of the next order:

$$\psi = b_1\psi_1 + b_2\psi_2 + \text{c.c.} + \psi^{(2)}, \tag{62}$$

$$\sigma = b_1\sigma_1 + b_2\sigma_2 + \text{c.c.} + \sigma^{(2)}, \tag{63}$$

$$\phi = c_3\phi_3 + \text{c.c.} + \phi^{(2)}, \tag{64}$$

$$\tilde{\rho} = c_3\tilde{\rho}_3 + \text{c.c.} + \tilde{\rho}^{(2)}. \tag{65}$$

For the sake of brevity, we write $b_1 = b_{\mathbf{k}_1}$, $\psi_1 = \psi_{\mathbf{k}_1}$, etc., and $\psi^{(2)}$, $\sigma^{(2)}$, etc. are the second-order corrections. Substituting (62)–(65) into the equations of motion, a number of products of the amplitudes $b_{\mathbf{k}_i}$ and $c_{\mathbf{p}}$ appear. We omit all such terms that contribute to the amplitude of the Cherenkov process except b_2c_3, (We mean that the incident wave has wave vector $\mathbf{k}_1$):

$$\dot{b}_1\sigma_1 + b_2c_3\left(\frac{\partial}{\partial t}(\tilde{\rho}_3\sigma_2) + (1-v)\nabla(\tilde{\rho}_3\nabla\psi_2) + \nabla(\sigma_2\nabla\phi_3)\right) = -(-i\omega_1\sigma_1^{(2)} + \Delta\psi^{(2)}), \tag{66}$$

$$\dot{b}_1\psi_1 + b_2c_3(\nabla\psi_2\nabla\phi_3) = -(-i\omega_1\psi_1^{(2)} + c_2^2\Delta\psi^{(2)}), \tag{67}$$

$$b_2c_3\left[\frac{\partial^2\psi_2}{\partial z^2}\zeta_3 - \nabla\psi_2\nabla\zeta_3\right]_{z=h} = -\left.\frac{\partial\psi^{(2)}}{\partial z}\right|_{z=h}. \tag{68}$$

Using the conditions of orthogonality and normalization, we find the amplitude of the Cherenkov radiation process:

$$\Gamma^{\text{Cher}}_{\mathbf{k}_1,\mathbf{k}_2\mathbf{p}_s} = \frac{2\rho_0}{\omega_1}\left[\int d^3\mathbf{R}\left(\mathbf{k}_1^2\psi_1(\nabla\psi_2\nabla\phi_3) + c_2^2\sigma_1\left(\frac{\partial}{\partial t}(\tilde{\rho}_3\sigma_2) + (1-v)\nabla(\tilde{\rho}_3\nabla\psi_2) + \nabla(\sigma_2\nabla\phi_3)\right)\right) - i\omega_1\int d^2\mathbf{r}\psi_1\left[\frac{\partial^2\psi_2}{\partial z^2}\zeta_3 - \nabla\psi_2\nabla\zeta_3\right]_{z=h}\right]. \tag{69}$$

The conservation laws for this elementary process are

$$\omega_1 = \omega_2 + \omega_s, \quad \mathbf{p}_1 = \mathbf{p}_2 + \mathbf{p}_s, \tag{70}$$

where ω_s and $\mathbf{p}_s$ are the frequency and wave vector of the surface wave. We assume $p_s \sim p_{1,2}$, so $\omega_s \ll \omega_{1,2}$. This implies that ω_1 and ω_2 and hence $\mathbf{k}_1$ and $\mathbf{k}_2$ are almost equal to one another, i.e., the wave vectors of second sound waves lie on some sphere in $\mathbf{k}$-space; the difference between their (x,y) components defines $\mathbf{p}_s$. Furthermore, we neglect the difference between ω_1 and ω_2 and denote them simply by ω (similarly for moduli the vector $k_1 = k_2 = k$).

After substituting (18) and (19) into (69), we find the expression for the nonlinear amplitude Γ of the Cherenkov process. In deep water, the leading term of this expression is

$$\Gamma^{\text{Cher}}_{\mathbf{k}_1,\mathbf{k}_2\mathbf{p}_s} = \frac{\omega}{2h(\rho_0 A)^{1/2}}\left(\frac{p_s}{\omega_s}\right)^{1/2}\left[1 - \frac{\mathbf{p}_1\mathbf{p}_2}{k^2}\right]. \tag{71}$$

In (71) we neglect all terms of order c_2^2/c_1^2, because this quantity is small over the temperature range between 0.8K and the λ-point. Analogous calculations for a thin film yield an expression for Γ that differs from (71) by the presence of an additional factor $(ph)^{1/2}$ on the right-hand side.

Note that in both cases, the main contribution to the nonlinear amplitude comes from the surface integral in (69).

As in our previous discussion, one can determine the nonlinear amplitude of other parametric decay processes of a second sound wave into two second sound waves (so called "inner" decay). The terms in the equations of motion that contribute to this process are proportional to ψ^2, σ^2 and $\psi\sigma$. The equations of motion, to the required precision, read

$$\left(\frac{\rho_{s0}}{\rho_{n0}}\right)^{1/2}\dot{\sigma}+\frac{\rho_0}{2\rho_{s0}}\frac{\gamma}{s_0}\frac{\partial}{\partial t}(\nabla\psi)^2+\left(\frac{\rho_{s0}}{\rho_{n0}}\right)^{1/2}\Delta\psi$$
$$+\frac{1}{\rho_{n0}}\left(2\rho_{s0}+\rho_0\gamma\frac{\partial T}{\partial s}s_0\right)\nabla(\sigma\nabla\psi)=0, \quad (72)$$

$$\left(\frac{\rho_{s0}}{\rho_{n0}}\right)^{1/2}\dot{\psi}_\lambda+s_0^2\frac{\partial T}{\partial s}\left(\frac{\rho_{s0}}{\rho_{n0}}\right)^{3/2}\sigma_\lambda+\frac{1}{2}\left(\frac{\rho_{s0}}{\rho_{n0}}\right)^2 s_0^2\left(\frac{\partial T}{\partial s}\right.$$
$$\left.+s_0\frac{\partial^2 T}{\partial s^2}\right)(\sigma^2)_\lambda+\frac{\rho_{s0}}{\rho_{n0}}\left(\frac{1}{2}(1+v)(\nabla\psi)^2_\lambda\right.$$
$$\left.+(\psi_\mu\psi_\lambda)_\mu\right)=0. \quad (73)$$

Here $\psi_\lambda=\partial\psi/\partial x_\lambda$, $\gamma=(\partial\ln\rho_n/\partial\ln T)_p$, v is the same as in (59), and we sum over repeated Greek indices. The functions ϕ and σ can be represented as a sum of one incoming and two outgoing waves

$$\psi=b_1\psi_1+b_2\psi_2+b_3\psi_3+\text{c.c.}+\psi^{(2)}, \quad (74)$$

$$\sigma=b_1\sigma_1+b_2\sigma_2+b_3\sigma_3+\text{c.c.}+\sigma^{(2)}. \quad (75)$$

To describe inner decay, only the products b_2b_3 should be included among the anharmonic terms in (72)–(73). After simple but lengthy calculations, we have

$$\Gamma^{\text{inn}}_{\mathbf{k}_1,\mathbf{k}_2,\mathbf{p}}=-\left(\frac{\rho_{s0}}{\rho_{n0}}\right)^{1/2}\frac{c_2^3}{2(\rho_0V)^{1/2}(\omega_1^3\omega_2\omega_3)^{1/2}}\Big[[k_1(u+z-1)$$
$$+(k_2+k_3)(u+2)]k_1k_2k_3+(\mathbf{k}_1\mathbf{k}_2)k_3^2+(\mathbf{k}_1\mathbf{k}_3)k_2^2$$
$$+k_1\left[(2+v)k_1+(k_2+k_3)\left(2+\frac{\rho_0}{\rho_{n0}}u\right)\right]$$
$$\times[\mathbf{p}_2\mathbf{p}_2-q_2q_3]\Big]. \quad (76)$$

Here

$$z=\left(\frac{s_0}{c}\frac{\partial\ln c_2^2}{\partial\ln T}\right)_p,\quad u=\frac{\rho_0}{\rho_{s0}}\frac{s_0}{c}\left(\frac{\partial\ln\rho_n}{\partial\ln T}\right)_p=\frac{\rho_0}{\rho_{s0}}\frac{s_0}{c}\gamma,$$

and

$$c=T_0\frac{\partial s}{\partial T}$$

is the heat capacity per unit mass. We have used the conservation laws (57) and (58), and the z-components of the wave vectors are related by

$$|q_1|=|q_2\pm q_3|.$$

This obviously stems from the specular reflection of the wave by the boundary.

The threshold amplitudes for the Cherenkov process and for the inner decay of second sound become

$$|b_{\text{Cher}}|=\frac{[\delta_2(\omega_2)\delta_s(\omega_s)]^{1/2}}{\Gamma_{\text{Cher}}}, \quad (77)$$

$$|b_{\text{inn}}|=\frac{\delta_s\left(\frac{\omega_s}{2}\right)}{\Gamma_{\text{inn}}}, \quad (78)$$

where δ_2, δ_s are the attenuation of constants for second sound and the surface wave respectively. These are related to the viscosity η and thermal conductivity κ (see Ref. 3):

$$\delta_2(\omega_2)=\frac{\omega^2}{2\rho_0c_2^2}\frac{\rho_{s0}}{\rho_{n0}}\left(\frac{4}{3}\eta+\frac{\rho_{n0}}{\rho_{s0}}\frac{\kappa}{T}\frac{\partial T}{\partial s}\right), \quad (79)$$

$$\delta_s(\omega_s)=\frac{2\eta}{\rho_0}p_s^2. \quad (80)$$

Substituting (79) and (80) into (77) and (78), we have

$$\left|\frac{b_{\text{Cher}}}{b_{\text{inn}}}\right|=8\left(\frac{4}{3}-\frac{\rho_{n0}}{\rho_{s0}}\frac{\kappa}{T\eta}\frac{\partial T}{\partial s}\right)^{-1/2}\left(\frac{\omega_s}{\omega}\right)^{1/2}(p_sh)^{1/2}. \quad (81)$$

Equation (81) shows that in the deep water, the threshold amplitude for Cherenkov radiation exceeds that for inner decay if

$$(\omega/\omega_s)^{1/2}\leqslant 8(p_sh)^{1/2}. \quad (82)$$

Otherwise, the Cherenkov process dominates.

In the opposite thin-film limit, the ratio of the threshold amplitudes differs from (81) by the absence of the factor $(p_sh)^{1/2}$. Therefore, Cherenkov radiation always dominates in a thin film.

5. CONCLUSION

We have described the parametric transformation of a second sound wave into two second sound waves (inner decay) and Cherenkov radiation of a surface wave. Our conclusion is that in the temperature range $0.8\text{K}<T<T_\lambda$, the Cherenkov process dominates in a thin film of liquid helium, or more generally, in a layer with thickness h satisfying

$$h<\frac{1}{64p_s}\frac{\omega}{\omega_s},$$

where ω is the frequency of the incident second sound and ω_s,p_s are the frequency and wave vector of the surface wave. Otherwise, inner decay dominates. The amplitudes of these processes are given by equations (71) and (76). All processes described here are accessible to experimental observation. For instance, under the experimental conditions described in Ref. 4, these phenomena can in principle be observed far from resonance frequencies.

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