Nonlocal hydrodynamics of a quantum relativistic many-particle system

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The transition to the hydrodynamic limit is carried out for a certain class of external forces in a quantum relativistic many-particle system with N local conservation laws. It is shown that the hydrodynamic equations are nonlocal in space and time and that the hydrodynamic model is equivalent to the original quantum-statistical model. The kernels that enter into the constitutive relations are expressed in terms of current Green's functions. The hydrodynamic model satisfies the conditions of causality, dissipativity, and covariance. When the quantum field model is T-invariant the kernels are related by reciprocity conditions (analogous to the Onsager relations). The algebraic structure of the nonlocal hydrodynamics is analyzed in detail for a one-component quantum relativistic fluid. The relation between the classical transport coefficients and the hydrodynamic kernels is established.

1. INTRODUCTION

In many-particle theory the following three levels of description are employed for physical systems: 1) statistical mechanics, a description in terms of a many-particle distribution function or density matrix; 2) kinetic theory, a description in terms of a single-particle distribution function; 3) hydrodynamics, a description in terms of macroscopic fields of certain quantities. One proceeds from a higher to a lower level of description by reducing the number of degrees of freedom of the system.

In deriving the equations of dissipative hydrodynamics from statistical mechanics[1,2] or kinetic theory[3–5] one customarily makes the following assumptions: A) no external forces act on the system in question; B) some sort of expansion in small gradients is applicable. Condition A implies that explicitly or implicitly we are studying a Cauchy problem. Specifically, for a dissipative hydrodynamic system without sources only the state of rest is defined along the entire temporal axis. Condition B implies that the model has local constitutive relations.

However, the description of dissipative hydrodynamic processes in a systematic relativistic theory is associated with dropping locality (or at least dropping locality in time[6–10]). In a system which is nonlocal in time a Cauchy problem cannot be posed, and so it is natural to relax condition A and study a system with sources. Then condition B loses its meaning.

Dinariev[11] derived nonlocal hydrodynamic equations using the nonequilibrium statistical operator of Zubarev,[1] generalized to the case in which sources are present. In the present work the nonlocal relativistic hydrodynamics of a quantum many-particle system is constructed by explicitly imposing a source in the equation for the density matrix, describing an external force acting on the system. The nonlocal kernels appearing in the constitutive relations are expressed in terms of current Green's functions. The properties of the kernels that follow from those of the Green's functions are analyzed. Thus, the kernels satisfy the usual conditions of dissipativity,[12] which hold in mechanical models with inheritance. Because of nonlocality the presence of dissipative processes (in particular, viscosity and thermal conductivity) does not violate the theory's causality. If the original quantum field model is T invariant, then the kernels are related by the reciprocity conditions, an analog of the Onsager relations. The algebraic structure of the constitutive relations is analyzed in detail for a one-component relativistic quantum fluid.

Previously it was shown that in a certain class of sources the transition from relativistic kinetic theory to nonlocal hydrodynamics is equivalent.[9] Below an analogous assertion is also proved for the transition from quantum statistical mechanics to nonlocal hydrodynamics.

We use a system of units in which Planck's constant $\hbar$, the speed of light in vacuum c, and the Boltzmann constant k are equal to unity. The Greek subscripts run over the values 0, 1, 2, 3, corresponding to an inertial system of spatial and temporal coordinates x^α, where $x^0=t$ is time; the Roman subscripts a, b, c run over the values 1, 2, 3, corresponding to spatial Cartesian coordinates x^a; here $\partial_\alpha=\partial/\partial x^\alpha$. The space—time subscripts are raised and lowered using the Minkowsky metric $(\eta_{\alpha\beta})=\mathrm{diag}\,(1,-1,-1,-1)$. The Roman subscripts A, B, C run over the values $0,\ldots,(N-1)$, where N is the number of local conservation laws (1). Summation is understood over repeating indices unless otherwise specified.

2. DERIVATION OF THE NONLOCAL RELATIVISTIC HYDRODYNAMIC EQUATIONS

We will use the following notation: the function $f=f(x^\alpha)$ has Fourier transform $f_F(k_\alpha)$ defined by

$$f_F=f_F(k_\alpha)=\int \exp(-ik_\alpha x^\alpha)f(x^\alpha)dx^\alpha,$$

where $k_0=\omega$ is the frequency.

Consider a quantum many-particle system, which in the absence of interaction with the surrounding medium has a Hamiltonian H_0 and possesses N local conservation laws (energy, momentum, charge, etc.). If there is interaction with the surrounding medium, then the Hamiltonian assumes the form

$$H=H_0+H_1,\quad H_1=H_1(t),$$

where the operator $H_1=H_1(t)$ describes the influence of the surrounding medium and sources appear in the conservation equations:

$$\partial_\alpha J^\alpha_{AH}=S_A, \tag{1}$$

where $J^\alpha_{AH}=J^\alpha_{AH}(x^\beta)$ are 4-current operators in the Heisenberg representation.

Assume that the currents J^α_{AH} constitute a complete set in the sense that the equilibrium state is completely characterized by the average values of the densities J^0_{AH}. Furthermore, assume that the macroscopic fields generated by these densities (such as gravitation, electromagnetism, and so on) are negligibly small.

In addition to the Heisenberg representation we will use the Schrödinger representation $J^\alpha_{AS}=J^\alpha_{AS}(x^a)$ and the interaction representation $J^\alpha_{AI}=J^\alpha_{AI}(x^\beta)$ for the current operators. These representations are related by

$$J^\alpha_{AS}(x^a)=J^\alpha_{AH}(x^\beta)|_{x^0=0}=J^\alpha_{AI}(x^\beta)|_{x^0=0}.$$

From the definition of the interaction representation we have the local conservation laws

$$\partial_\alpha J^\alpha_{AI}=0. \tag{2}$$

Since the 4-current operators in the Schrödinger representation are defined without regard to the interaction with the surrounding medium, in connection with Eq. (2) we have the equations

$$i[H_0,J^0_{AS}]+\partial_a J^a_{AS}=0. \tag{3}$$

Here, in order to avoid problems connected with Haag's theorem,[13,14] it is convenient to set $H_1(t)=0$ for $t\leqslant a$, $a>0$. Otherwise relation (3) does not hold, e.g., when the operator H_1 is nonzero and time-independent.

We define the evolution operator for a quantum system as the solution of the Cauchy problem

$$\frac{d}{dt}U^t_{t_0}=-iHU^t_{t_0},\quad U^{t_0}_{t_0}=1. \tag{4}$$

The current operators in the Heisenberg representation and the interaction representation are expressed in terms of the current operators in the Schrödinger representation:

$$\begin{aligned}&J^\alpha_{AH}(x^\beta)=\overset{*}{U}{}^{x^0}_0 J^\alpha_{AS}(x^b)U^{x^0}_0,\\ &J^\alpha_{AI}(x^\beta)=Z(-x^0)J^\alpha_{AS}(x^b)Z(x^0),\\ &Z(t)=\exp(-itH_0).\end{aligned} \tag{5}$$

From Eqs. (1), (3)–(5) we find the representation for the sources

$$S_A(x^\alpha)=i\overset{*}{U}{}^{x^0}_0[H_1(x^0),J^0_{AS}(x^b)]U^{x^0}_0, \tag{6}$$

We assume that for $t\leqslant a$ there is an equilibrium state with density matrix

$$\rho_0=Q_0^{-1}E_0,\quad E_0=\exp(-\beta H_0),\quad Q_0=\mathrm{Tr}\ E_0, \tag{7}$$

where $T=\beta^{-1}$ is the temperature in the equilibrium state. In the Heisenberg representation the state of the system (7) does not vary. In the Schrödinger representation the evolution of the system is described by the quantum Liouville equation

$$i\frac{d}{dt}\rho=[H,\rho],\quad \rho|_{t=0}=\rho_0. \tag{8}$$

For an arbitrary operator B we define

$$\langle B\rangle_0=\mathrm{Tr}(B\rho_0),\quad \langle B\rangle=\mathrm{Tr}(B\rho).$$

We will consider states close to the equilibrium state (7) and restrict ourselves to the first-order perturbation with respect to the external force $H_1(t)$. Let us recall some standard formulas for calculating averages over states close to the equilibrium.[1]

We introduce the definitions

$$\begin{aligned}&A_1=\beta H_0+V,\quad E_1=\exp(-A_1),\quad Q_1=\mathrm{Tr}\,E_1,\\ &\rho_1=Q_1^{-1}E_1,\\ &\langle B\rangle_1=\mathrm{Tr}(B\rho_1),\end{aligned} \tag{9}$$

where V is a small operator and B is an arbitrary operator. We define $W(t)=Z(-t)VZ(t)$.

Then we have[1]

$$\Delta\rho=\rho_1-\rho_0=-T\int_0^\beta(W(is)-\langle V\rangle_0)ds\rho_0, \tag{10}$$

$$\langle B\rangle_1=\langle B\rangle_0-T\int_0^\beta\langle(B-\langle B\rangle_0)W(is)\rangle_0 ds. \tag{11}$$

Then, if we assume the condition

$$\langle(B-\langle B\rangle_0)W(t+is)\rangle_0\to 0 \tag{12}$$

for $t\to-\infty$, Eq. (11) can be transformed to[1]

$$\langle B\rangle_1=\langle B\rangle_0-Ti^{-1}\int_{-\infty}^0\langle[W(t),B]\rangle_0 dt. \tag{13}$$

We will assume that the original quantum field model satisfies the following two conditions related to the local nature of the theory:

a) for a spacelike 4-vector $(x^\alpha-y^\alpha)$

$$[J^\alpha_{AI}(x^\beta),J^\gamma_{BI}(y^\beta)]=0, \tag{14}$$

b) for $\rho_1=\rho_1[f]$ of the form (9), where

$$V=-\int f^A(x^a)J^0_{AS}(x^a)dx^a, \tag{15}$$

a nonzero 3-vector (x^a-y^a) satisfies

$$\mathrm{Tr}\left(J^\alpha_{AS}(x^a)\frac{\delta\rho_1[f]}{\delta f(y^a)}\bigg|_{f=0}\right)=0. \tag{16}$$

Let us direct our attention to the Cauchy problem (8). If we linearize it with respect to the external perturbations we find

$$\frac{d}{dt}\rho=i[\rho,H_0]+\sigma(t),\quad \sigma(t)=i[\rho_0,H_1(t)]. \tag{17}$$

We assume that the source term $\sigma(t)$ in Eq. (17) has the functional form (10) with

$$V=V(t)=\int h^A(t,x^a)J^0_{AS}(x^a)dx^a. \tag{18}$$

Here $h^A(t,x^a)$ are given functions which characterize the external forces. This assumption is central in the present derivation of the hydrodynamic equations. In order that the quantum field system be described by hydrodynamics the external forces should belong to a certain class. If this condition is not satisfied, then the processes that occur in the system are not describable by hydrodynamics. A similar situation also obtains in kinetic theory.[9]

The problem (17) together with (18) has the simple solution

$$\rho(t)=\rho_0+\kappa(t),$$

$$\kappa(t)=\int_0^t Z(t-t_1)\sigma(t_1)Z(t_1-t)dt_1$$

$$=-T\int_0^t dt_1\int_0^\beta ds\int dx^a h^A(s,x^a)$$

$$\times(J^0_{AI}(t_1-t+is,x^a)-\langle J^0_{AS}\rangle_0)\rho_0. \tag{19}$$

We introduce the hydrodynamic 4-currents:

$$g^\beta_A(x^\alpha)=\langle J^\beta_{AS}(x^\alpha)-\langle J^\beta_{AS}\rangle_0\rangle. \tag{20}$$

Knowing the density matrix (19) we can express them in terms of the external forces. To obtain an expression in compact form it is convenient to introduce the reduced 4-currents $j^\beta_{AI}(x^\alpha)=J^\beta_{AI}(x^\alpha)-\langle J^\beta_{AS}\rangle_0$; the spectral functions $S^{\alpha\beta}_{AB}(k_\gamma)$:

$$S^{\alpha\beta}_{AB}(k_\gamma)=\Sigma^{\alpha\beta}_{ABF}(k_\gamma),$$

$$\Sigma^{\alpha\beta}_{AB}(x^\gamma)=\langle j^\alpha_{AI}(x^\gamma)j^\beta_{BI}(0)\rangle_0,$$

and the Green's functions for the currents,

$$D^{\alpha\beta}_{AB}(x^\gamma)=i^{-1}\theta(x^0)\langle[j^\alpha_{AI}(x^\gamma),j^\beta_{BI}(0)]\rangle_0,$$

$$L^{\alpha\beta}_{AB}(x^\gamma)=\theta(x^0)\int_{x^0}^{+\infty}dsD^{\alpha\beta}_{AB}(s,x^c). \tag{21}$$

The properties of these functions needed in the exposition which follows, as well as the properties of the corresponding spectral functions, are given in the Appendix.

Substituting expressions (18) in (19) and using Eq. (13), we arrive at a representation of the hydrodynamic currents in terms of the external forces:

$$g^\beta_A(x^\alpha)=T\int dy^\alpha L^{\beta 0}_{AB}(x^\alpha-y^\alpha)h^B(y^\alpha). \tag{22}$$

In this case in the limit $t\to-\infty$ condition (12) takes the form (the principle of diminishing correlations)

$$\langle(J^\alpha_{AS}(x^a)-\langle J^\alpha_{AS}\rangle_0)J^0_{BI}(t+is,y^b)\rangle_0\to 0.$$

In the Fourier representation Eqs. (22) can be written in the form

$$g^\beta_{AF}(k_\alpha)=TL^{\beta 0}_{ABF}(k_\alpha)h^B_F(k_\alpha). \tag{23}$$

At first glance it appears from the definitions (21) and Eqs. (22) that causality is violated in the theory (superluminal action of the external forces on the hydrodynamic variables is possible). However, more careful analysis reveals that the kernels $L^{\alpha 0}_{AB}$ satisfy causality [see Eq. (A9) in Sec. 2 of the Appendix].

From Eqs. (1) it follows that the hydrodynamic equations with sources are

$$\partial_\alpha g^\alpha_A=s_A,\quad s_A=\langle S_A\rangle_0. \tag{24}$$

The sources s_A can be expressed in terms of the external forces by means of Eq. (6):

$$s_A(x^\alpha)=T\int\Gamma_{AB}(x^a-y^a)h^B(x^0,y^a)dy^a,$$

$$\Gamma_{AB}(x^a)=\dot{L}^{00}_{AB}(x^\alpha)|_{x^0=+0}. \tag{25}$$

If we rewrite Eq. (25) in terms of Fourier transforms, then the definitions (21) and Eq. (A2) yield

$$s_{AF}(k_\alpha)=-T\Lambda_{AB}(k_a)h^B_F(k_\alpha),$$

$$\Lambda_{AB}=(2\pi)^{-1}\int\omega^{-1}(e^{\beta\omega}-1)S^{00}_{AB}(\omega,k_a)d\omega.$$

From Eq. (A9) we have $\Gamma_{AB}(x^a)=0$ for a nonvanishing 3-vector x^a. Hence the hydrodynamic sources s_A are related to the external forces h^A by a local dependence. Consequently, the hydrodynamic variables g^α_A are expressed causally both in terms of the forces h^A and in terms of the sources s_A.

Equations (24) can be used constructively to describe the dynamics of a system if we express the spatial currents g^a_A in terms of the charge g^0_A, i.e., find the constitutive relations. Eliminating the quantity h^A in Eqs. (23) and using the definitions

$$(\Delta_{AB}(k_\alpha))=(L^{00}_{ABF}(k_\alpha))^{-1},$$

$$C^b_{AC}(k_\alpha)=L^{b0}_{AB}(k_\alpha)\Delta_{BC}(k_\alpha),$$

we find the desired constitutive relations:

$$g^b_{AF}(k_\alpha)=C^b_{AB}(k_\alpha)g^0_{BF}(k_\alpha). \tag{26}$$

It is evident that in the coordinate representation the constitutive relations, which represent a combination of transformations satisfying causality, satisfy causality themselves.

Thus we have shown that from the equations of quantum statistical mechanics for a certain class of sources we can derive the equations (24), (26) of nonlocal hydrodynamics. Conversely, in this class of sources the hydrodynamic description is complete. Specifically, if we know the hydrodynamic variables we can reconstruct the parameter h^A [cf. Eq. (23)] and the density matrix (10).

3. PROPERTIES OF THE HYDRODYNAMIC KERNELS OF NONLOCALITY

In this section we study the properties of the coefficients in (26), which in the coordinate representation go over to kernels that characterize the nonlocal nature of the theory.

Where possible we will omit the subscripts A, B,..., associated with the labels of the conserved currents and use the matrix formalism.

From Eqs. (A5) and (A7) we have

$$\Lambda=\Lambda^{+}\geqslant 0. \tag{27}$$

From relations (A3) and (A6) we have identically

$$ik_\alpha L^{\alpha 0}(k_\beta)=-\Lambda.$$

Hence it follows that the coefficients in the constitutive relations are related by

$$ik_a C^a=-\Lambda\Delta-ik_0. \tag{28}$$

We write $A^a=C^a\Lambda$, $B=ik_aA^a$. It turns out to be technically simpler to study the coefficients that enter into the operators A^a. These operators allow us to express the 3-currents g_A^a in terms of external forces such as $f^A=f^A(x^\alpha)$, which for a state $\rho_1=\rho_1[f]$ of the form (9), (15) lead at each instant of time to the same values of the quantity g_A^0:

$$g_{AF}^0(k_\alpha)=T\Lambda_{AB}(k_a)f_F^A(k_\alpha).$$

The transition from the operators A^a to the operator C^a can be carried out if the operator Λ is known.

Using Eqs. (27), (28), (A5), and (A8), we obtain the matrix inequality

$$B+B^{+}\geqslant 0. \tag{29}$$

The inequality (29) is the usual form of the dissipativity conditions for mechanical models with inheritance.[12]

If the original quantum field model is invariant under time inversion (T-invariance, Ref. 14), then we can choose the currents so that

$$\overset{*}{U}_T J_{AI}^0(x^0,x^a)U_T=\varepsilon_A J_{AI}^0(-x^0,x^a),\quad \varepsilon_A=\pm 1, \tag{30}$$

where U_T is the anti-unitary time inversion operator.

Assume that the equilibrium state is T-invariant. Then Eq. (30) yields a relation for the spectral function:

$$S_{AB}^{00}(k_0,k_a)=\varepsilon_A\varepsilon_B S_{AB}^{00}(k_0,-k_a)^*.$$

In turn it follows from this relation using Eqs. (A1)–(A7) that the reciprocity conditions for the components of the Green's functions are

$$L_{ABF}^{\alpha\beta}(k_0,k_c)=\varepsilon_A\varepsilon_B L_{BAF}^{\beta\alpha}(k_0,-k_c). \tag{31}$$

In addition, when (30) is satisfied we have a relation similar to (31):

$$\Lambda_{AB}(k_c)=\varepsilon_A\varepsilon_B\Lambda_{BA}(-k_c). \tag{32}$$

Now from (31) and (32) it is easy to find reciprocity relations for the coefficients that appear in the constitutive relations (26):

$$B_{AB}(k_0,k_a)=\varepsilon_A\varepsilon_B B_{AB}(k_0,-k_a). \tag{33}$$

Assume that the equilibrium state ρ_0 is invariant under the group of spatial rotations $SO(3)$. The requirement of invariance with respect to the action of this group imposes additional restrictions on the functional form of the matrix A^a:

$$(R_gA)_{AB}^a(k_0,gk_b)=A_{AB}^a(k_0,k_b) \tag{34}$$

for arbitrary $g\in SO(3)$. Here R_g is a representation of the group of rotations in the linear space Φ of quantities of the form A_{AB}^a. We recall that the subscripts A and B can also be transformed under the group $SO(3)$. In the space Φ we distinguish a maximal set of linearly independent invariants with respect to the subgroup of rotations that conserve the vector $k_a:I^n=I_{AB}^{na}(k_b)$. These invariants can be chosen in the form of polynomials in k_a such that

$$(I_{AB}^{na}(k_b))^*=I_{AB}^{na}(-k_b). \tag{35}$$

Then we can find the most general functional form of the coefficient matrix A_{AB}^a that satisfies Eq. (33):

$$A_{AB}^a=I_{AB}^{na}X_n. \tag{36}$$

Here $X_n=X_n(k_\alpha)$ are scalar quantities which are invariant under the action of the group of rotations. By virtue of Eqs. (A1) and (35) these functions satisfy the relation $(X_n(k_\alpha))^*=X_n(-k_\alpha)$. This means that the functions X_n are Fourier transforms of some real kernels $Y_n=Y_n(x^\alpha):Y_{nF}=X_n$. Since the theory is causal, the supports of these kernels lie on the future cone

$$x^0\geqslant(-\eta_{ab}x^ax^b)^{1/2}.$$

Substituting expression (36) in (29) or (33) permits us to obtain restrictions on the kernels that follow from the dissipativity or T-invariance respectively.

In this theory the requirements of causality, dissipativity, T-invariance, and $SO(3)$ covariance lead to the same restrictions as in a purely phenomenological approach. It is of fundamental importance, of course, that here these restrictions follow from first principles. However, one condition arises in the theory which would be less than obvious in a purely phenomenological approach. From the identity (28) and Eq. (A3) we have the sum rule

$$\int(ik_0+ik_aC^a)^{-1}dk_0=2\pi. \tag{37}$$

In contrast to the other conditions, for the verification of which it is necessary to know the local values of the hydrodynamic kernels in the space of 4-momenta, the sum rule (37) presupposes a knowledge of the functional form of the kernels globally.

4. HYDRODYNAMICS OF A ONE-COMPONENT QUANTUM FLUID

Consider a model in which energy, momentum, and also some charge (e.g., baryon for a system of nucleons or electrical for an electron gas) are conserved. Let $T^{\alpha\beta}$ be the symmetric energy-momentum tensor and Q^α the 4-current. In the notation of Sec. 2 we set

$$J_A^\alpha=T_A^\alpha,\quad A=0,1,2,3,\quad J_4^\alpha=Q^\alpha.$$

The Hamiltonian for the equilibrium state (7) in this case takes the form

$$H_0=\int (T_S^{0\alpha}(x^b)v_\alpha-\mu Q_S^0(x^b))dx^b. \quad (38)$$

Here μ is a chemical potential, v^α is a 4-vector satisfying the conditions

$$v_\alpha v^\alpha=1, \quad v^0>0, \quad (39)$$

which can be interpreted as the 4-velocity of the medium, and μ is the chemical potential. It is not difficult to verify that expression (38) in the theory without sources is invariant under the orthochronic Poincaré group. For this it suffices to rewrite expression (38) in the interaction representation and use the conservation laws (2).

Thus, the equilibrium state is characterized by the parameters β, v^α, μ. When Eq. (39) is taken into account this amounts to five degrees of freedom. We set

$$f_e^0=-\beta, \quad f_e^a=-\beta v^a, \quad f_e^4=\beta\mu. \quad (40)$$

A functional relationship holds in the form:

$$\langle J_{AS}^0\rangle_0=F_A(f_e^B). \quad (41)$$

Now we study the constitutive relations (the matrices A^a) for perturbations of the rest state. In the rest state we have $f_e^a=0$. The perturbing forces $f^A(x^\alpha)$ (see Sec. 3) can naturally be interpreted as local variations of the right-hand sides of Eqs. (40).

The constitutive relations are determined by the following coefficients A_{aAB} with different transformation properties under the group $SO(3)$: A_{a00}, A_{a04}, A_{a40}, A_{a44}, A_{a0b}, A_{a4b}, A_{ab0}, A_{ab4}, A_{abc}. Hence, by virtue of the symmetry of the energy-momentum tensor we have, first, $A_{abA}=A_{baA}$, and second, $A_{a0A}=\Lambda_{aA}$ (since $g_a^0=-g_0^a$).

Now it is not hard to write the representation of the form (36):

$$A_{a00}=ik_aX_0, \quad A_{a04}=ik_aX_1,$$

$$A_{a40}=ik_aX_2, \quad A_{a44}=ik_aX_3,$$

$$A_{a0b}=\delta_{ab}X_4+ik_aik_bX_5+i\varepsilon_{abc}k_cX_6,$$

$$A_{a4b}=\delta_{ab}X_7+ik_aik_bX_8+i\varepsilon_{abc}k_cX_9,$$

$$A_{ab0}=\delta_{ab}X_{10}+ik_aik_bX_{11},$$

$$A_{ab4}=\delta_{ab}X_{12}+ik_aik_bX_{13},$$

$$A_{abc}=\delta_{ab}ik_cX_{14}+ik_aik_bik_cX_{15}+(ik_a\delta_{bc}+ik_b\delta_{ac})X_{16} +(ik_aik_d\varepsilon_{bcd}+ik_bik_d\varepsilon_{acd})X_{17}.$$

Let us construct the matrix $B_{AB}=ik^aA_{aAB}$. Here we will set $\lambda=-\eta^{ab}k_ak_b$:

$$B_{00}=\lambda X_0, \quad B_{04}=\lambda X_1, \quad B_{40}=\lambda X_2, \quad B_{44}=\lambda X_3,$$

$$B_{0b}=ik_b(-X_4+\lambda X_5), \quad B_{4b}=ik_b(-X_7+\lambda X_8),$$

$$B_{b0}=ik_b(-X_{10}+\lambda X_{11}), \quad B_{b4}=ik_b(-X_{12}+\lambda X_{13}),$$

$$B_{ab}=ik_aik_b(-X_{14}+\lambda X_{15}-X_{16})+\delta_{ab}\lambda X_{16}.$$

This matrix satisfies the dissipativity conditions (29). In particular, we have the inequalities

$$\mathrm{Re}\, X_{16}\geqslant 0, \quad \mathrm{Re}(2X_{16}+X_{14}-\lambda X_{15})\geqslant 0, \quad \mathrm{Re}\, X_3\geqslant 0. \quad (42)$$

If the quantum field model is T-invariant, then we assume $\varepsilon_0=\varepsilon_4=1$, $\varepsilon_a=-1$. Hence the reciprocity relations lead to the following restrictions on the kernels:

$$X_1-X_2=0, \quad -X_4+\lambda X_5+X_{10}-\lambda X_{11}=0,$$
$$-X_7+\lambda X_8+X_{12}-\lambda X_{13}=0. \quad (43)$$

We can simplify the constitutive relations considerably by assuming that the linear mapping $(T\Lambda)$ is identical with the linearization of the functions (41).

Let us consider this case in more detail. We introduce the notation $\varepsilon=\langle T^{00}\rangle_0$ for the internal energy density, $p=-\langle T^{aa}\rangle_0$ (no summation!) for the pressure, $n=\langle Q^0\rangle_0$ for the particle density, and $h=\varepsilon+p$ is the enthalpy. From thermodynamics we have the well-known differential relations

$$d(\beta p)=\varepsilon d(-\beta)+nd(\mu\beta).$$

From this follows the representation

$$\Lambda_{AB}=\beta\frac{\partial^2\varphi}{\partial f^A\partial f^B}, \quad \varphi=\varphi(f^0,f^4)=\beta p,$$
$$A,B=0,4. \quad (44)$$

It is also easy to verify that

$$\Lambda_{aA}=h\delta_{aA}. \quad (45)$$

From Eq. (44) we immediately find

$$X_0=0, \quad X_1=0, \quad X_4=Th, \quad X_5=0, \quad X_6=0.$$

From this and from the reciprocity relations (43) we have

$$X_2=0, \quad X_{10}-\lambda X_{11}=Th.$$

With this the study of the purely algebraic properties of the hydrodynamics of a one-component relativistic quantum fluid can be regarded as complete. Further progress in this model can be made through the actual evaluation of the Green's function for the currents (on the basis of some specific field theory) or by postulating some properties for it.

5. CONCLUSION

To conclude this work it is appropriate to make a number of comments about the various stages of the arguments given above.

It is obvious that the physically natural representation (44), (45) significantly narrows the set of unknown coefficient functions necessary to make the models specific. Hence the assumption (15), which is associated with causality, is satisfied automatically. However, the proof of the representation (44), (45) from first principles is difficult in connection with the possibly complicated structure of the Schrödinger terms in the commutators for the currents.

The assumption (18) is fundamental. If we impose external forces, e.g., through the Hamiltonian

$$H_1(t)=\int h^A(t,x^a)J_{AS}^0(x^a)dx^a,$$

then by following arguments analogous to those of Sec. 2 we can derive equations identical in form with those of hydrodynamics. The resulting theory is causal but not dissipative.

In the constitutive relations (26) one customarily distinguishes the so-called equilibrium components l_A^a of the spatial currents.[1] It is evident that the exact definition of the equilibrium components does not change the final equations. In Refs. 1 and 11 the following definition was used (if we combine the appropriate notation with that of the present work):

$$l_{AF}^b(k_\alpha)=Z_{AB}^b(k_\alpha)|_{k_0=0}g_{BF}^0(k_\alpha),$$

$$Z^b(k_\alpha)=D^{b0}(k_\alpha)(\Lambda(k_\alpha))^{-1}.$$

This expression yields a vanishing contribution to the left-hand side of the dissipative inequality (29). The alternative definition

$$l_{AF}^b(k_\alpha)=Z_{AB}^b(k_\alpha)|_{k_\alpha=0}g_{BF}^0(k_\alpha)$$

yields vanishing dissipation only in the case when Λ does not depend on the wave vector k_a.

In relativistic hydromechanics the definitions of the velocity of the medium given by Eckart[15] and by Landau and Lifshitz[16] are used. Intermediate definitions were considered in Ref. 5. In Sec. 3 we implicitly used the definition of the velocity of the medium given by Landau and Lifshitz.

If we disregard nonlocality, nonlocal hydrodynamics of a one-component fluid can be introduced to the standard model of a viscous thermally conducting Navier–Stokes–Fourier fluid with bulk and shear viscosity coefficients η_V, η_S and with thermal conductivity κ. The analysis of the dispersion relation for the homogeneous hydrodynamic equations allows us to establish the identification

$$\eta_S=\beta X_{16}|_{k_\alpha=0},$$

$$\left(\eta_V+\frac{4}{3}\eta_S\right)=\beta(2X_{16}+X_{14})|_{k_\alpha=0},$$

$$\kappa=\Lambda_{00}\beta(\Lambda_{00}\Lambda_{44}-\Lambda_{04}\Lambda_{40})^{-1}X_3|_{k_\alpha=0}.$$

The inequalities (42) cause the quantities η_S, $(\eta_V+(4/3)\eta_S)$, and κ to be nonnegative, but do not guarantee the nonnegativity of the bulk viscosity. This is one of the most distinctive characteristics of nonlocal hydrodynamics. The nonnegativity of the bulk viscosity can be proved by starting from the condition that the entropy produced in a particle of the medium be positive.[8]

The properties of the nonlocal hydrodynamics of a quantum relativistic many-particle system treated in the present work have a different level of significance. Causality, covariance, and dissipativity have a universal character. The reciprocity conditions (the Onsager relations) can be violated if T-invariance breaks down. This situation can occur, e.g., when weak interactions are taken into account.

APPENDIX

1. In the present section we present well-known properties of the current Green's functions taken from the literature (see, e.g., Ref. 1).

The Green's functions for the currents are real:

$$(D_{AB}^{\alpha\beta}(x^\gamma))^*=D_{AB}^{\alpha\beta}(x^\gamma),\quad (L_{AB}^{\alpha\beta}(x^\gamma))^*=L_{AB}^{\alpha\beta}(x^\gamma).$$

This relation in the Fourier representation yields

$$(D_{ABF}^{\alpha\beta}(k_\gamma))^*=D_{ABF}^{\alpha\beta}(-k_\gamma),$$

$$(L_{ABF}^{\alpha\beta}(k_\gamma))^*=L_{ABF}^{\alpha\beta}(-k_\gamma). \tag{A1}$$

The Green's functions are related to the spectral density by

$$D_{ABF}^{\alpha\beta}(k_\gamma)=(2\pi)^{-1}\int(e^{\beta\omega}-1)S_{AB}^{\alpha\beta}(\omega,k_c)\times(k_0-\omega-i\varepsilon)^{-1}d\omega, \tag{A2}$$

$$L_{ABF}^{\alpha\beta}(k_\gamma)=(2\pi)^{-1}i\int(e^{\beta\omega}-1)S_{AB}^{\alpha\beta}(\omega,k_c)\times\omega^{-1}(k_0-\omega-i\varepsilon)^{-1}d\omega. \tag{A3}$$

The spectral function satisfies the relations that follow from the definition (17) and the relation (2):

$$S_{AB}^{\alpha\beta}(-k_\gamma)=\exp(\beta k_0)S_{BA}^{\beta\alpha}(k_\gamma), \tag{A4}$$

$$c_A^\alpha c_B^{\beta*}S_{AB}^{\alpha\beta}(k_\gamma)\geq 0,\ \text{for arbitrary } c_A^\alpha, \tag{A5}$$

$$k_\alpha S_{AB}^{\alpha\beta}(k_\gamma)=0, \tag{A6}$$

$$S_{AB}^{\alpha\beta}(k_\gamma)^*=S_{BA}^{\beta\alpha}(k_\gamma). \tag{A7}$$

From (A3) and (A4) we find the relation

$$L_{ABF}^{\alpha\beta}(k_\gamma)+L_{BAF}^{\beta\alpha}(-k_\gamma)=-(k_0)^{-1}(e^{\beta k_0}-1)S_{AB}^{\alpha\beta}(k_\gamma). \tag{A8}$$

2. In this section we give a proof of the assertion that for a spacelike vector x^γ we have

$$L_{AB}^{\alpha 0}(x^\gamma)=0. \tag{A9}$$

Specifically, consider expression (15). Using Eq. (13) and the definitions (21), we see that this relation is equivalent to the condition that for a nonzero 3-vector x^a we have

$$L_{AB}^{\alpha 0}(x^\gamma)|_{x^0=+0}=0. \tag{A10}$$

Then, from condition (14) and definitions (21) it follows that in the range $0<x^0<(-\eta_{ab}x^ax^b)^{1/2}$ the function $L_{AB}^{\alpha 0}(x^\gamma)$ does not depend on x^0. Comparison of this assertion with Eq. (A10) leads to the relation (A9).

[1] D. N. Zubarev, *Nonequilibrium Statistical Thermodynamics*, Consultants Bureau, New York (1974).
[2] A. I. Akhiezer and S. V. Peletminskii, *Methods of Statistical Physics*, Pergamon Press, Oxford (1981).
[3] S. Chapman and T. G. Cowling, *The Mathematical Theory of Nonuniform Gases*, Cambridge University Press (1953).
[4] J. H. Ferziger and H. G. Kaper, *Mathematical Theory of Transport Processes in Gases*, North-Holland, Amsterdam (1972).
[5] S. P. de Groot, W. A. Van Leeuwen, and C. C. Van Weert, *Relativistic Kinetic Theory, Principles and Applications*, North-Holland, Amsterdam (1980).
[6] B. Carter, "Covariant theory of conductivity in ideal fluid or solid media," in *Lecture Notes in Mathematics*, Vol. 1385 (1989).
[7] W. Israel, "Covariant fluid mechanics and thermodynamics: An introduction," in *Lecture Notes in Mathematics*, Vol. 1385 (1989).
[8] O. Yu. Dinariev, Prikl. Mat. Mekh. **56(1)**, 250 (1992).
[9] O. Yu. Dinariev, Zh. Éksp. Teor. Fiz. **106**, 161 (1994) [JETP **79**, 88 (1994)].

[10] O. Yu. Dinariev, Izv. Vuzov. Ser. Fiz. **57**, 13 (1993).
[11] O. Yu. Dinariev, Izv. Vuzov. Ser. Fiz. **58**, 70 (1994).
[12] O. Yu. Dinariev, Dokl. Akad. Nauk SSSR **309**, No. 3, 615 (1989) [Sov. Phys. Dokl. **34**, 206 (1989)].
[13] G. G. Emch, *Algebraic Methods in Statistical Mechanics*, Wiley–Interscience, New York (1972).
[14] N. N. Bogolyubov, A. A. Logunov, A. I. Oksak, and I. T. Todorov, *General Principles of Quantum Field Theory*, Kluwer Academic, Dordrecht, The Netherlands (1990).
[15] C. Eckart, Phys. Rev. **58**, 919 (1940).
[16] L. D. Landau and E. M. Lifshitz, *Fluid Mechanics*, 2nd ed., Pergamon, Oxford (1987).

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