Ionization losses by multiply charged relativistic ions

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We examine the energy losses in collisions between heavy multiply charged relativistic ions and light atoms for ion charges $Z \gg 1$ and relative collision velocities $v \gg 1$ such that $Z \sim v \leqslant c$, where c is the speed of light (atomic units), together with the polarization losses associated with the motion of heavy relativistic ions in matter. Strictly speaking, in this range of parameters the Born approximation does not work. We derive simple formulas for the effective retardation. Finally, we compare the results with the experimental data and the results of calculations done by other researchers. © *1995 American Institute of Physics.*

1. INTRODUCTION

The common approach to calculating the ionization losses in collisions of charged relativistic particles with atoms is to employ the Born approximation (see Ref. 1, § 82), which requires $Z/v \ll 1$, where Z is the charge of the incoming particle, and v is the relative collision velocity (here and in what follows we use the atomic system of units). Lately, however, much experimental work has been done in the study of inelastic collisions of atoms with ions whose charges are so great that the Born approximation range for such ions is not reached even for $v \approx c$ (c is the speed of light), so that $Z/v \sim 1$ often holds (see, e.g., Refs. 2–4 and the references listed there). On the other hand, the use of approximations applicable for $Z/v \sim 1$, the eikonal approximation and its modifications,[5,6] the method of sudden perturbations,[7–9] and the classical path method[10] to calculate the cross sections of inelastic processes, entail considerable computation even when the collision velocities are nonrelativistic. The calculations become even more complicated when we move into the relativistic region.[11]

This paper uses the approach developed in Ref. 12 and its generalization to the case of relativistic collisions[13] to examine the energy losses by heavy multiply charged relativistic ions in collisions with light (nonrelativistic) atoms for $Z \sim v \leqslant c$ ($Z \gg 1$, $v \gg 1$, and $c \approx 137$), together with the polarization losses associated with the motion of heavy multiply charged relativistic ions in matter. We derive simple formulas that describe effective retardation. We also compare our results with the experimental data and the results of other researchers.

2. CROSS SECTIONS OF INELASTIC PROCESSES

The most systematic method of calculating the cross sections of inelastic processes in the collisions of fast ($v \gg 1$) multiply charged ($Z \gg 1$) ions with atoms is to use the Glauber approximation, applicable when $Z/v \sim 1$. It is based on the old eikonal approximation, which is close to the quasiclassical approach. The eikonal approximation is usually employed in potential scattering problems in nonrelativistic quantum mechanics when the energy E of the scattered particle is much higher than the potential energy U (see Ref. 14, p. 200). The eikonal approximation is generalized in Ref. 15a, Sec. 1.7.4, to the case of potential scattering of high-energy relativistic particles. In this case, after summation over the polarizations of the scattered particles, the differential cross section of scattering into the solid angle $d\Omega$ encompassing the direction of the unit vector $\mathbf{n}$ has the following form (see Ref. 15a, p. 76):

$$d\sigma = |a(\mathbf{n})|^2 \, d\Omega \tag{1}$$

with a scattering amplitude independent of the spin structure:

$$a(\mathbf{n}) = \frac{ik}{2\pi} \int e^{i\mathbf{qb}} (1 - e^{i\chi(\mathbf{b})}) \, d^2b, \tag{2}$$

where $\mathbf{q} = \mathbf{k}_f - \mathbf{k}_i$, $\mathbf{k}_i$ and $\mathbf{k}_f$ are the particle momenta before and after the collision, with $|\mathbf{k}_i| = |\mathbf{k}_f| = k$,

$$\chi(\mathbf{b}) = -\frac{1}{v} \int_{-\infty}^{\infty} U(x,\mathbf{b}) \, dx,$$

where in the laboratory reference frame the potential $U(\mathbf{R})$ is the temporal component of the 4-potential, $A^{\mu} = (U,0,0,0)$, and depends on the coordinates $\mathbf{R} = (x,\mathbf{b})$ of the scattered particle; here $\mathbf{b} \perp \mathbf{v}$, and the x axis is directed along $\mathbf{v}$, so that $\mathbf{b}$ is usually interpreted as the impact parameter. If the particle velocity is so great that $|U|a/v \ll 1$ holds, Eq. (2) becomes the Born approximation. The eikonal approximation (2) for potential scattering can be generalized (following Ref. 14, § 152) to the case of the collision of a fast particle with a system of particles, with the additional condition that the incoming particle velocity is much higher than the characteristic velocities within the scattering system. More precisely, we examine the collision between an ion moving with a relativistic velocity v and a light (nonrelativistic) atom. We denote the characteristic velocity of the atomic electrons by v_a. Note that when particles with a moderate charge $Z \sim Z_a$ (here Z_a is the effective charge of the atomic nucleus) are scattered by atoms, the condition $v_a \ll v$ coincides (see Ref. 14, p. 741) with the condition for the applicability of the

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Born approximation: the atomic electrons usually have a velocity $v_a \sim Z_a$, with the result that at $Z_a \sim Z$ the condition $v_a \ll v$ yields $Z/v \ll 1$. But when atoms collide with an ion whose charge Z is much greater than Z_a, the condition $v_a \ll v$ does not automatically imply $Z/v \ll 1$. However, the assumption that the velocity of the incident particle is much higher than the characteristic velocities within the scattering system (the nonrelativistic atom) makes it possible to consider ion motion with the atomic electrons held fixed. Hence the natural generalization of Eq. (2) to the case of elastic scattering of a relativistic ion by a nonrelativistic atom has the form

$$a_{fi}=\frac{ik_i}{2\pi}\int\left\{1-\exp\left[-\frac{i}{v}\int U(x,\mathbf{b};\{\mathbf{r}_a\})\,dx\right]\right\} \times|\Psi_0(\{\mathbf{r}_a\})|^2\prod_{n=1}^{N} d^3r_n\, e^{-i\mathbf{qb}}\, d^2b, \tag{3}$$

where the scattering potential is a function not only of the ion coordinates $\mathbf{R}=(x,\mathbf{b})$ but also of the instantaneous positions of the atomic electron, whose set of coordinates is designated by $\{\mathbf{r}_a\}$, i.e., $U=U(x,\mathbf{b};\{\mathbf{r}_a\})$. Integrating over the coordinates of the atomic electrons (whose total number is N) corresponds to averaging over the internal (ground) state of the atom described by the wave function $\Psi_0(\{\mathbf{r}_a\})$. The general expression for the inelastic scattering amplitude for the case where the atom goes from state $|\Psi_i\rangle$ to state $|\Psi_f\rangle$ has the form (cf. Ref. 14, § 152)

$$a_{if}(\mathbf{q})=\frac{ik_i}{2\pi}\int e^{-i\mathbf{qb}}\langle\Psi_f|1 -\exp\left[-\frac{i}{v}\int U(x,\mathbf{b};\{\mathbf{r}_a\})\,dx\right]|\Psi_i\rangle\, d^2b, \tag{4}$$

where, as in Eq. (3), the ion momentum transfer $\mathbf{q}$ is $\mathbf{k}_f-\mathbf{k}_i$. When $Z/v \ll 1$ holds, Eqs. (3) and (4) become the Born approximation.

The inelastic scattering amplitude can be obtained via the well-known formula

$$\sigma=\int\frac{k_i}{k_f}|a_{fi}|^2\,d\Omega, \tag{5}$$

where Ω is the scattering solid angle of the ion.

However, even in the case of a nonrelativistic charged particle colliding with such a simple system as the hydrogen atom, calculations of the excitation and ionization cross sections employing the Glauber approximation lead to cumbersome expressions.[5,6] Certain simplifications can be achieved by proceeding as follows: for small scattering angles we have $d\Omega \approx d^2q/k_fk_i \approx d^2q/k^2$; we then represent $|a_{fi}|^2$ from (4) in the form of a double integral with respect to d^2b and d^2b' and integrate with respect to d^2q via the integral representation of the δ-function; after this the latter is eliminated by integrating with respect to d^2b'. As a result for Eq. (5) we come to

$$\sigma=\int d^2b\left|\langle\Psi_f|1 -\exp\left[-\frac{i}{v}\int_{-\infty}^{\infty}dx\, U(x,\mathbf{b};\{\mathbf{r}_a\})\right]|\Psi_i\rangle\right|^2. \tag{6}$$

Next, using $dt=dx/v$ and the mutual orthogonality of the functions Ψ_i and Ψ_f, we arrive at an expression for the inelastic scattering cross section that can be directly obtained from the sudden perturbation approximation:[7,13]

$$\sigma=\int d^2b\left|\langle\Psi_f|\exp\left[-i\int_{-\infty}^{\infty}dt\, U(x,\mathbf{b};\{\mathbf{r}_a\})\right]|\Psi_i\rangle\right|^2. \tag{7}$$

Accordingly, the integrand is interpreted as the probability for the atom to go from the state Ψ_i to the state Ψ_f in a collision with impact parameter $\mathbf{b}$.

These procedures for obtaining Eqs. (6) and (7) from the Glauber approximation, which include changing the order of integration, are proper only for potentials with a limited range. For long-range potentials these procedures make the integrals in (6) and (7) divergent for large impact parameters. This divergence, however, is unimportant[7,8] because at large impact parameters the Born approximation becomes applicable, and the ranges of applicability of the Born approximation and sudden perturbations approximation overlap, which makes it possible to do proper matching in the impact parameter. But for a nonrelativistic hydrogen atom computations with Eq. (7) result in considerable time expenditure.[7,8] For this reason Refs. 12 and 13 suggested an approach based on the fact that at $v \sim Z \gg 1$ the inelastic cross sections are substantial and the range of large impact parameters provides the main contribution. The corresponding cross section has the form[12,13]

$$\sigma=2\pi\int_{b_1}^{b_0}b\,db\left|\langle\Psi_f|\exp\left(-i\mathbf{q}\sum_{a=1}^{N}\mathbf{r}_a\right)|\Psi_i\rangle\right|^2, \tag{8}$$

where b_0 and b_1 are estimated from the conditions for the applicability of the approach. Equation (8) can also be obtained directly from (7), provided the expansion of the potential $U(x,\mathbf{b};\{\mathbf{r}_a\})$ in (7) in powers of the small parameter $|\mathbf{r}_a|/b$ stops after the dipole terms.

3. COLLISIONS WITH SEPARATE ATOMS

According to Ref. 14, § 49, the mean energy losses in collisions are characterized by a quantity known as the effective retardation:

$$\kappa=\sum_n(\epsilon_n-\epsilon_0)\sigma_n, \tag{9}$$

where ϵ_n and ϵ_0 are the energies of the excited $|n\rangle$ and ground $|0\rangle$ states of the atom, and σ_n is the cross section for excitation of state $|n\rangle$. For simplicity we start by examining the collision of a multiply charged relativistic ion with the hydrogen atom.

According to Refs. 12 and 13, it is convenient to partition the entire interval $0<b<\infty$ of possible values of the impact parameter b into three regions:

A) $0<b<b_1$, B) $b_1<b<b_0$, C) $b_0<b<\infty$, (10)

which correspond to small, medium, and large impact parameters. From Ref. 13 the values of the boundaries of the regions are

$$b_1 \sim 1, \quad b_0 \sim \frac{v}{\sqrt{1-\beta^2}}, \quad \beta = \frac{v}{c}.$$

We calculate κ in each region specified in (10) and obtain the total effective retardation by adding the contribution of each region. Here the exact values of the boundaries are unimportant, since in each region the dependence of κ on the parameters b_1 and b_0 proves to be logarithmic, leading to a proper matching of the contributions of adjacent regions and to a situation in which the dependence of κ on the matching parameters b_1 and b_0 disappears from the final result.

Small impact parameters corresponds to large momentum transfers, and, following Ref. 1, § 82, we ignore the binding of the electron to the atom and assume that the multiply charged ion is scattered elastically by an electron that initially was at rest. Then the effective retardation is expressed in terms of $\sigma(\epsilon)$, the cross section of transfer of energy ϵ, as follows:

$$\kappa = \int_{\epsilon_{\min}}^{\epsilon_{\max}} \epsilon \sigma(\epsilon)\, d\epsilon. \tag{11}$$

The only difference from Ref. 1, where it is assumed that $Z/v \ll 1$, is that we cannot substitute $\sigma(\epsilon)$ into the Born approximation, since in our case Z/v could be of order unity. As in Ref. 1, we assume that the incoming ion is an infinitely heavy particle, meaning that it does not change its motion in the course of the collision. Then the cross section for the scattering of an electron that initially was at rest by an ion moving with a constant velocity can be obtained by going over to the reference frame in which the moving electron is scattered by the immobile ion. We denote the corresponding scattering angle by θ. Following Ref. 1, we assume (excluding the ultrahigh energy region) that the energy loss ϵ is θ-dependent,

$$\epsilon(\theta) = \frac{2v^2}{1-\beta^2} \sin^2 \frac{\theta}{2}, \tag{12}$$

and the values $\epsilon_{\max}$ and $\epsilon_{\min}$ are attained at $\theta = \pi$ and $\theta = \theta_{\min}$, respectively. As a result we can write Eq. (11) as

$$\kappa = 2\pi \frac{Z^2}{v^2} \int_{\theta_{\min}}^{\pi} \frac{\sigma(\theta)}{\sigma_R(\theta)} \cot \frac{\theta}{2}\, d\theta, \tag{13}$$

where $\sigma(\theta)$ is the cross section of electron scattering on an immobile ion of charge Z for an arbitrary electron velocity v (it was obtained in Ref. 15b, § 11.5, by solving the scattering problem for the Dirac equation), and $\sigma_R(\theta)$ is the Rutherford cross section:

$$\sigma_R = \frac{Z^2(1-\beta^2)}{c^4\beta^4(1-\cos\theta)^2}. \tag{14}$$

According to Refs. 15b and 16, the ratio of the exact cross section to the Rutherford cross section, $\sigma(\theta)/\sigma_R(\theta)$, tends to unity as $\theta \to 0$. Consequently, to determine the scattering angle θ we can use the quasiclassical relationship (Ref. 17, Problem 2 to § 39) that links the scattering angle with the impact parameter (see also the graphic qualitative picture of a collision suggested in Ref. 12):

$$\theta_{\min} = \frac{2Z}{v^2 b_1}\sqrt{1-\beta^2}. \tag{15}$$

Clearly, for small $\theta_{\min}$ the integral in Eq. (13) depends on $\theta_{\min}$ logarithmically, with the result that Eq. (13) can be written as

$$\kappa = 4\pi \frac{Z^2}{v^2} \ln \frac{v^2 b_1}{Z\sqrt{1-\beta^2}\, a(Z,v)}, \tag{16}$$

where the function $a(Z,v)$ is determined by comparing (16) with the results of numerical calculations by Eq. (13), which were done using the values of $\sigma(\theta)$ of Ref. 16. As a result $a(Z,v)$ can be approximated by the following formula:

$$a(Z,v) = [-0.23016\alpha - (1.00832\alpha - 0.32388)\beta^2 + 1]^2, \tag{17}$$

where $\alpha = Z/c$. To validate the approximation (17), in Tables I and II we list the values of the effective retardation calculated (at $b_1 = 1$) by Eq. (13) (Table I) and by Eq. (16) (Table II) with the use of $a(Z,v)$ from (17): the first column gives the energy of the multiply charged ion (in units of MeV per nucleon) corresponding to the relative velocities for the data of Ref. 16; the other columns list values of κ (in atomic units) for ion charges 6, 13, 29, 50, 82, and 92, respectively. As Tables I and II demonstrate, the suggested approximation

TABLE I. Effective retardations (atomic units) obtained through numerical integration of Eq. (13) with the use of σ/σ_R from Ref. 16.

Ion energy, in MeV/nucleon	Ion charges 6	13	29	50	82	92
91.8	0.894	3.714	16.247	44.368	110.067	135.497
183.6	0.554	2.333	10.411	29.185	75.125	93.469
367.2	0.372	1.580	7.160	20.452	53.402	68.540
734.4	0.280	1.200	5.495	15.893	43.418	55.109
1285.2	0.245	1.056	4.872	14.181	39.220	49.993
1836	0.235	1.015	4.697	13.705	38.063	48.618
3672	0.231	1.003	4.670	13.671	38.079	48.701
7344	0.239	1.043	4.879	14.310	40.246	50.941
18 360	0.257	1.129	5.311	15.595	43.308	55.302

TABLE II. Effective retardations (atomic units) calculated by Eq. (13) with the use of the function $a(Z,v)$ defined in Eq. (17).

Ion energy, in MeV/nucleon	Ion charges 6	13	29	50	82	92
91.8	0.891	3.697	16.056	43.666	110.134	137.246
183.6	0.552	2.319	10.268	28.454	73.567	92.369
367.2	0.370	1.570	7.066	19.922	52.838	66.916
734.4	0.278	1.192	5.439	15.559	42.255	53.985
1285.2	0.244	1.051	4.834	13.951	38.430	49.379
1836	0.234	1.010	4.666	13.520	37.462	48.252
3672	0.230	0.999	4.647	13.529	37.685	48.636
7344	0.238	1.040	4.860	14.185	39.523	50.991
18 360	0.256	1.125	5.291	15.475	43.018	55.405

is fairly good, at least for the ranges of Z and v for which the data of Ref. 16 are given. We also note that in the nonrelativistic limit ($\beta\to 0$) and as $\alpha\to 0$, according to (17), $a(Z,v)\to 1$, since $\sigma/\sigma_R\to 1$ as $\beta\to 0$ and $\alpha\to 0$.

The range of moderate impact parameters, $b_1<b<b_0$, corresponds to the range of moderate momentum transfers, which provides[12,13] the main contribution to the inelastic cross section. In this range the transferred energy ϵ is less than unity, i.e., the ionization potential of the atom, and the contribution of this region cannot be taken into account by perturbation theory techniques.[7-9,12,13] We also note that in this range the electron of the hydrogen atom is nonrelativistic both before and after the collision.[13] The contribution of this range to the effective retardation can be obtained from (9) by substituting the inelastic cross section (8),

$$\sigma_n=\int_{b_1}^{b_0}2\pi b|\langle n|e^{-i\mathbf{qr}}|0\rangle|^2\,db, \tag{18}$$

and repeating the calculations (see Ref. 14, § 152). The only simplification is that the upper limit of integration in (18) is independent of the final state of the atom:

$$\kappa=\sum_n(\epsilon_n-\epsilon_0)\sigma_n=4\pi\frac{Z^2}{v^2}\ln\frac{q_1}{q_0}, \tag{19}$$

where $q_0=2Z/vb_0$, and $q_1=2Z/vb_1$.

In the range of large impact parameters, $b_0<b<\infty$ the interaction of the incident ion with the atom can be taken into account perturbatively.[7-9,12,13] The corresponding amplitude of the atom's transition from state $|0\rangle$ to state $|n\rangle$ can be obtained following the reasoning of Ref. 18:

$$A_{0n}=\frac{2iZ}{v^2}\Omega_n\mathbf{r}_{0n}\left[i\frac{\mathbf{v}}{v}(1-\beta^2)K_0(\xi)+\frac{\mathbf{b}}{b}\sqrt{1-\beta^2}K_1(\xi)\right], \tag{20}$$

where $\Omega_n=\epsilon_n-\epsilon_0$, $\xi=\Omega_n b\sqrt{1-\beta^2}/v$, $K_0(\xi)$ and $K_1(\xi)$ are modified Bessel functions, and $\mathbf{r}_{0n}=\langle 0|\mathbf{r}|n\rangle$.

The cross section corresponding to (20),

$$\sigma_n(b_0<b<\infty)=\int d^2b\,|A_{0n}|^2,$$

is obtained by integrating with respect to the impact parameter $\mathbf{b}$ within the following limits: the angle of $\mathbf{b}$ varies from 0 to 2π, and $b_0<b<\infty$. As a result we get[1)]

$$\sigma_n=4\pi\frac{Z^2}{v^2}|x_{0n}|^2\left[\ln\frac{4v^2}{\eta^2b_0^2\Omega_n^2(1-\beta^2)}-\beta^2\right], \tag{21}$$

where $\eta=e^B=1.781$ ($B=0.5772$ is Euler's constant), and $x_{0n}=\langle n|x|0\rangle$. The contribution of this range to the effective retardation can be obtained by substituting (21) into (9):

$$\kappa=4\pi\frac{Z^2}{v^2}\left(\ln\frac{2v}{\eta Ib_0\sqrt{1-\beta^2}}-\frac{\beta^2}{2}\right), \tag{22}$$

where, following Ref. 1, § 82, we have introduced the "average atomic energy" I such that

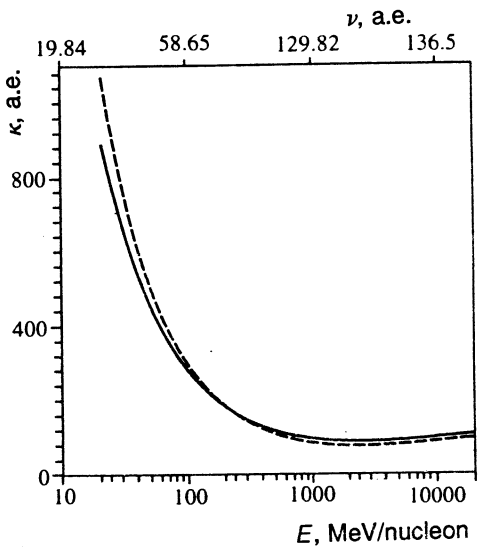

FIG. 1. Effective retardation κ as a function of E, the energy of the incident ion. The solid curve represents the results of our calculation [Eq. (24)], and the dashed curve is the Born approximation (Eq. (25)). For convenience we list some values of the velocity on the upper horizontal axis.

$$\ln I=\frac{\Sigma_n(\epsilon_n-\epsilon_0)|x_{0n}|^2\ln(\epsilon_n-\epsilon_0)}{\Sigma_n(\epsilon_n-\epsilon_0)|x_{0n}|^2}. \tag{23}$$

We obtain the total effective retardation of a multiply charged relativistic ion on the hydrogen atom by adding (22), (19), and (16):

$$\kappa=4\pi\frac{Z^2}{v^2}\left(\ln\frac{2v^3}{\eta IZ(1-\beta^2)a(Z,v)}-\frac{\beta^2}{2}\right). \tag{24}$$

For comparison we give the value of κ listed in Ref. 1, § 82, which was calculated in the Born approximation:

$$\kappa=4\pi\frac{Z^2}{v^2}\left(\ln\frac{2v^2}{I(1-\beta^2)}-\beta^2\right). \tag{25}$$

Figure 1 illustrates the behavior of the effective retardation at $Z=92$: the solid curve represents the results of our calculation [Eq. (24)], and the dashed curve is the Born approximation [Eq. (25)]. As Fig. 1 shows, the Born approximation systematically overestimates the values of κ in the low-energy region, while at high energies the same approximation yields a value of κ smaller than the one given by Eq. (24).

Note that the nonrelativistic limit for the ionization losses (24) (allowing for the fact that $a(Z,v)\to 1$ as $\beta\to 0$ and $\alpha\to 0$),

$$\kappa=4\pi\frac{Z^2}{v^2}\ln\frac{2v^3}{I\eta Z}, \tag{26}$$

has the appearance of the well-known Bohr formula (see Ref. 19, Article 4), which was derived from classical ideas. We also note that our results [Eq. (24)] obtained by the approach developed in Refs. 12 and 13 and valid only for $Z\sim v\gg 1$ does not allow for a transition to the Born approximation, which is valid for $Z/v\ll 1$ (a similar situation occurs with the Bohr formula).

As our derivation of Eq. (24) shows, the generalization of Eq. (24) and the generalization of Born losses (25) to the case of collisions between multiply charged relativistic ions and complex but nonrelativistic atoms (for which the characteristic velocity of the atomic electrons, v_a, is much less than the relative collision velocity v) can be obtained by multiplying the right-hand side of (24) by Z_a, the number of atomic electrons, and by replacing I with the average atomic potential I_a, which is still determined by Eq. (23) with ϵ_n, ϵ_0, and $|x_{0n}|^2$ calculated for the complex atom:

$$\kappa=4\pi\frac{Z^2}{v^2}Z_a\left(\ln\frac{2v^3}{\eta I_a Z(1-\beta^2)a(Z,v)}-\frac{\beta^2}{2}\right). \tag{27}$$

4. ENERGY LOSSES IN MATTER

Now let us examine the question of energy losses by a multiply charged relativistic ion moving in a medium. The losses are represented by the sum of macroscopic (polarization) losses and the energy lost in collisions with separate atoms. Here we assume that the ion velocity is much higher than the characteristic velocities of the atomic electrons (or at least most of these velocities). According to Fermi (see Ref. 20, Article 85), polarization losses in the motion of a charged particle in a medium are determined by the flux of the Poynting vector through a cylinder of radius b_0' constructed around the particle trajectory. The energy loss per unit path length, or the effective retardation, is obtained by dividing the energy flux by the particle velocity:

$$\kappa=\frac{Z^2b_0'}{\pi v^2}\int_{-\infty}^{+\infty}d\omega\, K_0(b_0'\xi)K_1(b_0'\xi^*)\times\left(\frac{1}{\epsilon(\omega)}-\beta^2\right)i\omega\xi^*, \tag{28}$$

where we have written $\xi^2=\omega^2[v^{-2}-c^{-2}\epsilon(\omega)]$ and $\epsilon(\omega)$ is the dielectric constant. When the cylinder radius b_0' is small, i.e.,

$$|b_0'\xi|\ll 1, \tag{29}$$

we have

$$\kappa=\frac{iZ^2}{\pi v^2}\int_{-\infty}^{+\infty}\omega\, d\omega\left(\frac{1}{\epsilon(\omega)}-\beta^2\right)\ln\frac{2}{\eta b_0'\xi}, \tag{30}$$

where, as in (21), $\eta=1.781$. On the other hand, according to Ref. 21, energy losses can be interpreted as the work performed by the field over a path one unit long:

$$\kappa=\frac{iZ^2}{\pi}\int_0^{q_0}\int_{-\infty}^{+\infty}\omega d\omega q dq\,\frac{[v^{-2}-c^{-2}\epsilon(\omega)]}{\epsilon(\omega)(q^2+\xi^2)}. \tag{31}$$

Equations (30) and (31) coincide if condition (29) is met and if $q_0=2/\eta b_0'$. Indeed,

$$\ln\frac{2}{\eta\xi b_0'}\approx\frac{1}{2}\ln\left(\frac{2^2}{\eta^2\xi^2 b_0'^2}+1\right)=\int_0^{q_0}\frac{qdq}{q^2+\xi^2}. \tag{32}$$

Usually the next step is to examine two cases: $v^2<c^2/\epsilon_0$ [where $\epsilon_0=\epsilon(0)$ is the dielectric constant in a static field], and $v^2>c^2/\epsilon_0$. In the first case we have[21]

$$\kappa=4\pi\frac{NZ^2}{v^2}\left(\ln\frac{q_0 v}{\bar\omega\sqrt{1-\beta^2}}-\frac{\beta^2}{2}\right),\quad q_0=\frac{2}{\eta b_0'}, \tag{33}$$

where, as in Ref. 21, $\bar\omega$ is the average value of the frequency of motion of the atomic electrons:

$$\ln\bar\omega=\frac{\int_0^\infty\omega\eta''(\omega)\ln\omega\, d\omega}{\int_0^\infty\omega\eta''(\omega)\, d\omega},\quad \eta''=\mathrm{Im}[\epsilon^{-1}(\omega)].$$

In the second case ($v^2>c^2/\epsilon_0$), for particles whose energy is not too high (the energy of motion is lower than the ion's rest energy), Eq. (33) is usually employed. On the other hand, in the ultrarelativistic limit,[21]

$$\kappa=2\pi\frac{NZ^2}{c^2}\ln\frac{q_0^2c^2}{4\pi N}. \tag{34}$$

Next we must "match" the macroscopic losses to the energy losses on separate atoms. To this end we write condition (29) as

$$b_0'\ll\frac{v}{|\omega\sqrt{1-\beta^2\epsilon}\,|}<\frac{v}{\sqrt{1-\beta^2}}\sim b_0,$$

where for estimates we have assumed that $\omega\sim\omega_a\sim 1$, which is the characteristic atomic frequency. The condition at which the macroscopic approach is valid has the form $b_0'\gg b_1\sim 1$, i.e., characteristic atomic dimension. Thus, the lower boundary of b_0' lies between b_1 and b_0:

$$b_1\ll b_0'\ll b_0. \tag{35}$$

Comparing this with the conditions (10), we conclude that to obtain the total energy losses when a multiply charged relativistic ion moves in a medium we must add the contributions of the regions A and B defined in (10) (the upper limit in B is b_0') to the polarization losses. According to Eqs. (16) and (19), the sum of the contributions provided by A and B is

$$\kappa=4\pi\frac{Z^2N}{v^2}\ln\frac{v^2b_0'}{Za(Z,v)\sqrt{1-\beta^2}}, \tag{36}$$

where we have replaced the number Z_a of atomic electrons by the number of electrons per unit volume, N, as is assumed in Ref. 21. Adding (33) and (36), we arrive at an expression for the total energy lost by a multiply charged relativistic medium traveling through matter for the case $v^2<c^2/\epsilon_0$:

$$\kappa=4\pi\frac{Z^2N}{v^2}\left(\ln\frac{2v^3}{Z\eta(1-\beta^2)a(Z,v)\bar\omega}-\frac{\beta^2}{2}\right). \tag{37}$$

As mentioned earlier, this expression is also often used for the case where $v^2>c^2/\epsilon_0$ and for particles that do not move very fast. Note that (37) differs in one respect from Eq. (27), whose behavior is illustrated in Fig. 1 and which describes losses on separate atoms: the average potential I_a is replaced by $\bar\omega$ (cf. a similar situation in Ref. 21). Acting in the same manner in the ultrarelativistic case, we find the total effective retardation:

$$\kappa=2\pi\frac{Z^2N}{c^2}\ln\frac{4c^6}{Z^2\eta^2(1-\beta^2)^2a^2(Z,v)4\pi N}. \tag{38}$$

TABLE III. Experimental and theoretical values of energy losses in MeV $(mg/cm^2)^{-1}$ for different ion-target combinations.

Projectile	Target	Calculations[3]		Our results	Experiment[3]
$^{86}_{36}Kr$ 900 MeV/nucleon ($\beta=0.861$)	Be	2.346	2.438	2.552 00	2.432±0.037
$^{136}_{54}Xe$ 780 MeV/nucleon ($\beta=0.839$)	Be	5.488	5.812	5.861 87	5.861±0.076
	C	6.014	6.378	6.434 78	6.524±0.084
	Al	5.404	5.755	5.809 85	5.806±0.121
	Cu	4.703	5.036	5.087 41	5.077±0.066
	Pb	3.654	3.942	3.987 36	3.959±0.063

Thus, allowance for polarization losses in the ultrarelativistic case leads to a growth of effective retardation with velocity that is less than that of losses on separate atoms [Eq. (27)]. For convenience we write Eq. (37) in a form that expresses polarization losses explicitly:

$$\kappa=4\pi\frac{Z^2N}{v^2}\left[\ln\frac{2v^3}{Z\eta(1-\beta^2)a(Z,v)I_a}-\frac{\beta^2}{2}-\frac{\delta}{2}\right], \quad (39)$$

where the values of the average ionization potential I_a and the Fermi correction $\delta/2$ for polarization losses can be taken from Refs. 22 and 23.

Table III lists the experimental and theoretical values of the effective retardation [in units of $(mg/cm^2)^{-1}$] for different ions (we limit our investigation to Kr and Xe, whose nuclei carry a fairly large charge) and targets from Ref. 3 and also our results: the first column specifies the ion, the ion's energy (in units of MeV per nucleon), and the value of $\beta=v/c$; the second column specifies the type of target; the third lists the results of calculations in Ref. 3 by Bethe's formula with the Fermi correction for polarization losses; the fourth column lists the results of calculations in Ref. 3 by Bethe's formula with the Fermi, Mott,[24] and Bloch[25] corrections; the fifth column lists our results; and the sixth the experimental values for effective retardation.[3] Our results may be seen to agree fairly well with the experimental data. This corroborates the conclusion of Scheidenberger *et al.*[3] about systematic deviations from Bethe's retardation theory in the case of heavy relativistic ions.

5. CONCLUSION

The present simple approach makes it possible to estimate the effective retardation of multiply charged relativistic ions in their collisions with separate atoms and in passing through matter in many cases that are important from the practical viewpoint, since the formulas allow using the widespread method of introducing phenomenological correction, usually employed in applied calculations. The range in which our formulas can be applied, $Z\sim v\leqslant c$, does not allow for a direct transition to the Born approximation ($Z/v\ll 1$), but we believe this is a minor limitation since for ions whose charge is sufficiently high (e.g., $Z=92$) the range of the Born approximation is not reached even when $v\approx c$.

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[1)]Strictly speaking, Eq. (21) was derived on the assumption that $\xi=\Omega_n bv^{-1}\sqrt{1-\beta^2}\ll 1$, where $\Omega_n\sim 1$. Then $b\ll v(1-\beta^2)^{-1/2}\sim b_0$, so that subsequent matching can be carried out precisely at such values of b.

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