Model of the initial stage of the transition from laminar to turbulent flow in a current-carrying plasmalike medium

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(Submitted 2 December 1994)
Zh. Éksp. Teor. Fiz. **107**, 1868–1876 (June 1995)

The present work treats the initial stage of the transition from laminar to turbulent flow in a current-carrying plasmalike medium, regarded as an incompressible fluid with constant transport coefficients. A model similar to that of Lorenz for the initiation of large-scale hydrodynamic and current eddy structures is proposed, based on a small number of modes. It is shown that this model describes a nonequilibrium phase transition as a result of which the initial highly conducting state is destroyed. The types of bifurcation and the dependence on the system parameters are analyzed. © *1995 American Institute of Physics.*

1. INTRODUCTION

The aim of the present work is to develop a model of the initial stage in the transition from laminar to turbulent flow in current-carrying plasmalike media, based on a small number of modes. For this we use the analogy that has been established[1] between the initial stages in the nucleation of turbulence in an incompressible fluid and the electrical explosion of conductors,[2,3] which is a typical example of a nonequilibrium phase transition (NPT) in current-carrying media.

Below, following Refs. 1 and 4, we assume that the plasmalike medium is an incompressible conducting fluid with constant transport coefficients: electrical conductivity σ and shear viscosity η (this permits us to emphasize the dynamic character of the NPT we are considering). Note that in the derivation of the NPT in Refs. 1 and 4 the unperturbed solution of the magnetic diffusion equation was taken to be the solution corresponding to a uniform current density over the cross section of the conductor, and terms quadratic in the perturbation were dropped. Below in Sec. 2, in constructing our model of the initial stage of the transition from laminar to turbulent flow using the spectral Galerkin method[5] we refrain from making these approximations. As a result, as will be shown in Sec. 3, depending on the value of the control parameter, the loss of stability of the unperturbed solution in the present model may be either soft or hard (in the model of Ref. 4 only hard excitation was possible, corresponding to subcritical bifurcation).

2. DERIVATION OF THE NONEQUILIBRIUM PHASE TRANSITION MODEL

As an example of the starting model we will use the equations of magnetohydrodynamics (MHD). Here, as in Refs. 1 and 4, we consider an incompressible liquid-metal conductor with radius r_0 and electrical current $i(t)$. The MHD equations of an incompressible fluid with constant kinetic coefficients take the form

$$\frac{\partial \mathbf{v}}{\partial t}+(\mathbf{v},\nabla)\mathbf{v}=-\frac{1}{\rho}\nabla p+\frac{1}{4\pi\rho}[\mathrm{rot}\,\mathbf{H},\mathbf{H}]+\nu\Delta\mathbf{v}, \tag{1}$$

$$\frac{\partial \mathbf{H}}{\partial t}+(\mathbf{v},\nabla)\mathbf{H}=(\mathbf{H},\nabla)\mathbf{v}+\nu_m\Delta\mathbf{H}, \tag{2}$$

$$\mathrm{div}\,\mathbf{v}=0,\quad \mathrm{div}\,\mathbf{H}=0, \tag{3}$$

where $\mathbf{v}$, $\mathbf{H}$, p are respectively the velocity, magnetic-field strength, and pressure; ρ is the density; $\nu=\eta/\rho$ is the kinematic viscosity, and $\nu_m=c^2(4\pi\sigma)^{-1}$ is the magnetic viscosity.

From the first of Eqs. (3) it follows that the velocity satisfies $\mathbf{v}=\mathrm{curl}\,\mathbf{A}$ (here $\mathbf{A}$ is the vector potential of the velocity, for which we use the Coulomb gauge div $\mathbf{A}=0$). If we direct the z axis along the axis of the conductor and use azimuthal symmetry we can set $\mathbf{v}=\{u_r(r,z,t),0,u_z(r,z,t)\}$, $\mathbf{H}=\{0,H(r,z,t),0\}$, and $\mathbf{A}=\{0,\psi(r,z,t),0\}$. Then we have $u_r=-\partial\psi/\partial z$ and $u_z=\partial\psi/\partial r+\psi/r$. As a result of applying the curl operator to Eq. (1) and using the assumptions given above we find for the functions ψ and H a system of equations

$$\frac{\partial\hat{D}\psi}{\partial t}=\frac{\partial\psi}{\partial z}\left(\frac{\partial\hat{D}\psi}{\partial r}-\frac{\hat{D}\psi}{r}\right)-\frac{\partial\hat{D}\psi}{\partial z}\left(\frac{\partial\psi}{\partial r}+\frac{\psi}{r}\right)+\frac{1}{4\pi\rho r}\frac{\partial H^2}{\partial z}+\nu\hat{D}^2\psi, \tag{4}$$

$$\frac{\partial H}{\partial t}=\frac{\partial\psi}{\partial z}\left(\frac{\partial H}{\partial r}-\frac{H}{r}\right)-\frac{\partial H}{\partial z}\left(\frac{\partial\psi}{\partial r}+\frac{\psi}{r}\right)+\nu_m\hat{D}H, \tag{5}$$

where $\hat{D}A=\partial^2A/\partial r^2+(1/r)(\partial A/\partial r)-A/r^2+\partial^2A/\partial z^2$.

Note that in contrast to Refs. 1 and 4, where terms of the form $(\partial\psi/\partial z)(\hat{D}\psi/r)$ and $(\partial\psi/\partial z)(H/r)$ were dropped, here we include all terms of Eqs. (4) and (5).

We look for a solution of Eqs. (4) and (5) in the form $\psi=\psi(r,z,t)$ and $H=H_1(r,t)+h(r,z,t)$, where $H_1(r,t)$ is the unperturbed solution of Eq. (5) satisfying the boundary conditions $H_1(0,t)=0$ and $H_0(t)\equiv H_1(r_0,t)=2i(t)/cr_0$, ψ and h are perturbations with zero boundary conditions, $\psi|_{r=0}=\psi|_{r=r_0}=0$ and $h|_{r=0}=h|_{r=r_0}=0$.

The unperturbed solution $H_1(r,t)$ is easily found from Eq. (5), applied to a stationary medium:

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$$H_1(r,t)=\frac{2}{cr_0}\left[i\left(\frac{t\nu_m}{r_0^2}\right)R\left(\frac{r}{r_0},0\right)+\int_0^{t\nu_m/r_0^2} i(r)\dot{R}\left(\frac{t\nu_m}{r_0^2}-\tau,\frac{r}{r_0}\right)d\tau\right], \quad (6)$$

where $\dot{R}(\tau,x)=\partial R(\tau,x)/\partial\tau$; here the function $R(\tau,x)$ is defined by

$$R(\tau,x)=x-2\sum_{n=1}^{\infty}\exp(-\mu_m\tau)\frac{J_1(\mu_n x)}{\mu_n J_2(\mu_n)}; \quad (7)$$

μ_n is the nth root of the Bessel function $J_1(x)$. The simplest special case is that in which the current $i(t)$ and consequently $H_0(t)$ depend on time as $i(t)=i(0)\exp(\nu_m r_0^{-2}\eta t)$, where $\eta>0$ is a dimensionless parameter. Then by using (6) and (7) we can represent the unperturbed solution in the form

$$H_1(r,t)=H_0(t)\frac{I_1\left(\frac{r}{r_0}\sqrt{\eta}\right)}{I_1(\sqrt{\eta})}\equiv\frac{2}{cr_0}i(t)\frac{I_1\left(\frac{r}{r_0}\sqrt{\eta}\right)}{I_1(\sqrt{\eta})}, \quad (8)$$

where $I_1(x)=iJ_1(ix)$ is a modified Bessel function of the first kind (the main results of the present work are derived using this expression for H_1).

The use of the unperturbed solution in the form (6) and (7) or (8) permits us to relax one additional assumption made in Refs. 1 and 4, specifically, the assumption that the unperturbed current density profile is uniform over the cross section of the conductor. Then we can construct a model of the NPT which applies for large di/dt.

From Eqs. (4) and (5) we find a system of equations for the perturbations:

$$\frac{\partial\hat{D}\psi}{\partial t}=\frac{\partial(\hat{D}\psi/r,\psi r)}{\partial(r,z)}+\frac{1}{2\pi\rho r}\frac{\partial h}{\partial z}(H_1+h)+\nu\hat{D}^2\psi, \quad (9)$$

$$\frac{\partial h}{\partial t}=\frac{\partial(h/r,\psi r)}{\partial(r,z)}+\frac{\partial\psi}{\partial z}\left(\frac{\partial H_1}{\partial r}-\frac{H_1}{r}\right)+\nu_m\hat{D}h, \quad (10)$$

where $\partial(A,B)/\partial(r,z)=(\partial A/\partial r)(\partial B/\partial z)-(\partial A/\partial z)(\partial B/\partial r)$.

Equations (9) and (10) are analogous to the Saltzman system for the perturbed temperature and velocity fields in the theory of the Benard effect[6] except that the Cartesian coordinate system must be replaced by cylindrical coordinates and the scalar temperature must be replaced by the vector magnetic field. This permits us to use a substitution that resembles that employed by Lorenz:[7]

$$\psi(r,z,t)=X(t)J_1(k_1 r)\sin(kz), \quad (11)$$

$$h(r,z,t)=Y(t)J_1(k_1 r)\cos(kz)+Z(t)J_1(k_2 r), \quad (12)$$

where we have introduced the notation $k_i=\mu_i/r_0$. Here k is the wave number of the structures in the z direction. The justification for restricting ourselves to three perturbation modes in Eqs. (11) and (12) (one for ψ and two for h) may be found in the well-known Ruelle–Takens hypothesis,[8] which says that the transition from laminar to turbulent flow in an incompressible fluid is determined by a finite number of unstable modes, as well as the hypothesis of Volkov and Iskol'dskii[4] regarding the discrete nature of the reduction in the spatial scale (see also Ref. 9, where a model of the initial stage of the refinement of the spatial scale in a conductor with a current was constructed using a small number of modes, and it was shown that it is a model for a first-order NPT).

Substituting Eqs. (11) and (12) in (6) and (7) and using the spectral Galerkin method[5] to reduce (6) and (7) to a system of three nonlinear ordinary differential equations for the amplitudes X, Y, and Z, we find

$$\dot{X}=\frac{k}{k_1^2+k^2}\frac{1}{2\pi\rho r_0}Y[\alpha(\eta)H_0+c_1Z]-\nu(k_1^2+k^2)X, \quad (13)$$

$$\dot{Y}=\frac{k}{r_0}X[\beta(\eta)H_0+c_2Z]-\nu_m(k_1^2+k^2)Y, \quad (14)$$

$$\dot{Z}=\frac{k}{2r_0}c_3XY-\nu_m k_2^2 Z, \quad (15)$$

where $c_1=0.29$, $c_2=-1.73$, and $c_3=2.34$, and also

$$\alpha(\eta)=2\int_0^1\frac{I_1(x\sqrt{\eta})}{I_1(\sqrt{\eta})}\frac{J_1^2(\mu_1 x)}{J_2(\mu_1)}dx,$$

$$\beta(\eta)=2\int_0^1 x\sqrt{\eta}\frac{I_2(x\sqrt{\eta})}{I_1(\sqrt{\eta})}\frac{J_1^2(\mu_1 x)}{J_2(\mu_1)}dx.$$

It is not difficult to show that in a rectilinear coordinate system we would have $c_1=0$, $c_2=-c_3$, which corresponds to the Lorenz model.[7] Note that for the equations of the model of Refs. 1 and 4 the values of these coefficients correspond to the conditions $c_1=0$, $c_2<0$, and $c_3<0$.

The current $i(t)$ in the conductor is determined by the external electric circuit and its effective resistance, which is given by

$$i^2R_{\text{eff}}=\oint \mathbf{S}\mathbf{n}ds,$$

where $\mathbf{S}=(c/4\pi)[\mathbf{E},\mathbf{H}]$ is the Poynting vector; $\mathbf{E}=\mathbf{j}/\sigma-[\mathbf{v},\mathbf{H}]/c$ is the electric field; $\mathbf{j}$ is the electric current density; ds is an element of the conductor surfaces; and $\mathbf{n}$ is the outward normal to it. Using the assumptions specified above we find

$$R_{\text{eff}}=R_0+R_1=\frac{l}{\sigma H_0}\frac{1}{2\pi r_0^2}\frac{\partial rH_1}{\partial r}\bigg|_{r=r_0}+J_0(\mu_2)\frac{l\mu_2}{2\pi r_0^2\sigma}\frac{Z}{H_0}, \quad (16)$$

where R_0 is the original resistance of the conductors, corresponding to the case $\psi=h=0$, and R_1 is the amount by which it changes (an increase, as will be shown below), associated with the nucleation of vortical current structures.

3. INVESTIGATION OF THE MODEL; DISCUSSION

We study the time-independent solutions of Eqs. (13)–(15). To first order we note that the trivial steady solution $(X=Y=Z=0)$ of Eqs. (13)–(15) becomes unstable for

$$\frac{k^2}{(k_1^2+k^2)^3}\,\frac{\alpha(\eta)\beta(\eta)}{2\pi\rho r_0^2\nu\nu_m}\,H_0^2\gg 1.$$

Introducing the magnetic Rayleigh number $\mathfrak{R}\equiv H_0^2 r_0^2/2\pi\rho\nu\nu_m$ (Refs. 1 and 4) we find that its critical value at which the trivial solution ($\psi=h=0$) of the original equations (9) and (10) becomes unstable corresponds to a minimum of the function

$$\mathfrak{R}(k)=\frac{r_0^4}{\alpha(\eta)\beta(\eta)}\,\frac{(k_1^2+k^2)^3}{k^2}.$$

For this function we have $\mathfrak{R}(k)\to\infty$ in the limits $k\to+0$ and $k\to\infty$. Its minimum value is $\mathfrak{R}_{\min}(k)=\mathfrak{R}_c$, corresponding to a value of k given by $k^*\equiv k_1/\sqrt{2}$. This means that we have $\mathfrak{R}_c=27\mu_1^4[4\alpha(\eta)\beta(\eta)]^{-1}$. Thus, modes with $k=k^*$ becomes unstable in first order, which enables us to set $k=k^*$ in the region close to the critical value in what follows.

As regards the dependence of the critical Rayleigh number on the parameter η, we have $\mathfrak{R}_c\to\infty$ for $\eta\to+0$, whereas in Refs. 1 and 4 the case $i=\text{const}$ corresponded to the finite value $\mathfrak{R}_c=978$. The reason for this is that a current density which is uniform over the cross section of the conductor is stable. In the limit $\eta\to\infty$ we have $\mathfrak{R}_c\to\infty$, because for large values of η the current is concentrated in a narrow surface layer of the conductor, which reduces its effect on the growth of large-scale perturbations that span the entire volume of the conductor. For $\eta=\eta^*=8.73$ we have $\mathfrak{R}_c=\mathfrak{R}_{c\,\min}=7.8\cdot 10^3$, from which it follows that the minimum threshold value of the electric current is $i^*=c\sqrt{\pi\rho\nu\nu_m\mathfrak{R}_{c\,\min}/2}$.

The characteristic size of structures in the z direction will be $\lambda=2\pi/k^*=r_0 2^{3/2}\pi/\mu_1=2.32r_0$. As shown in Ref. 9, this corresponds to a pair of vortex rings (Benard cells closed into a torus). Figure 1 shows the hydrodynamic velocity field in the conductor. It is clear that slices (strata) can develop between vortex rings with oppositely directed rotation (i.e., with oppositely directed vectors curl **v**). In Ref. 11 it is shown that this distance correlates well with the experimentally observed separations between strata.[3]

Note that so far we have assumed that a constant current corresponds to a current density profile which is uniform over the conductor cross section. It is clear, however, that the current density profile will be nonuniform. This may be caused by conductor geometry that differs from cylindrical, by nonuniform heating, or by some other process (see, e.g., Ref. 12) that also gives rise to a U-shaped current density profile. In order to treat a nonuniform unperturbed current profile we introduce corrections to the coefficients α and β of Eqs. (13)–(15), i.e., we replace them by the coefficients $\alpha(\eta)+\alpha^*$ and $\beta(\eta)+\beta^*$, where $\beta^*>0$.

In Ref. 10 it was shown that for metals like copper we have $s\equiv\nu/\nu_m\sim10^{-3}$, i.e., $s\ll1$. In this case the amplitudes Y and Z will be adjusted adiabatically in response to the slowly changing amplitude X, whose time variation is therefore given by a single first-order ordinary differential equation. For the case $H_0=\text{const}$ (or what is the same thing, $\eta=0$) this equation can be written in terms of the dimensionless amplitude $X_1=X(\nu_m\mu_2)^{-1}\sqrt{-c_2c_3/6}$ and time $\tau=3t\nu_m k_1^2/2$, in the form

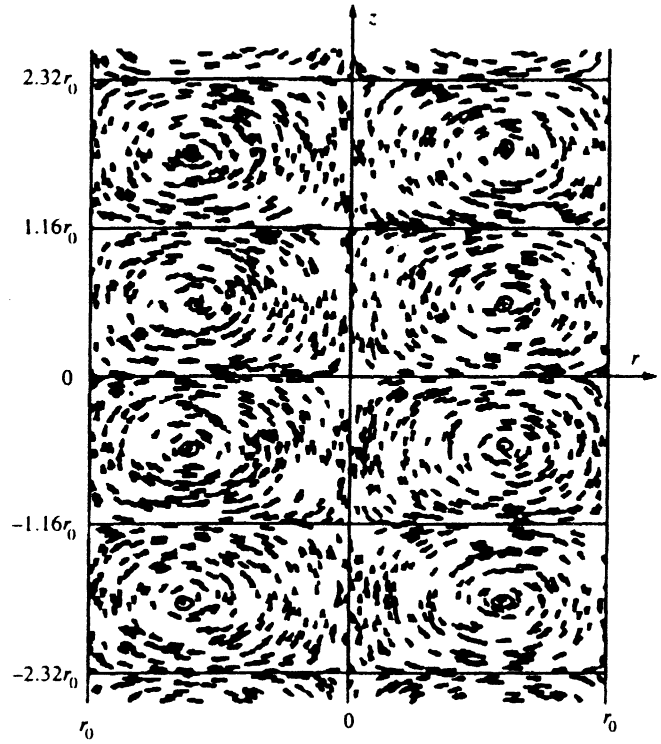

FIG. 1. Distribution of the hydrodynamic velocity **v** in the conductor.

$$\frac{d}{d\tau}X_1=sX_1\left[\frac{p}{1+X_1^2}\left(1+\xi\,\frac{X_1^2}{1+X_1^2}\right)-1\right], \tag{17}$$

where $p=\mathfrak{R}/\mathfrak{R}_c$ and $\xi=-c_1\beta^*/c_2(\alpha(0)+\alpha^*)>0$.

The bifurcation curves of this equation for two different values of the parameter ξ are depicted in Fig. 2. The trivial solution of the equation ($X_1=0$) corresponds to the absence of vortex structures in the conductor. It is stable for $0<p<1$ and unstable for $p>1$.

But in the case of the nontrivial time-independent solutions, as can be seen from the figure, for $p>1$ (if $0<\xi<1$ holds) or for $p>p_0\equiv4\xi/(\xi+1)^2$ (if $\xi>1$ holds) a pair of

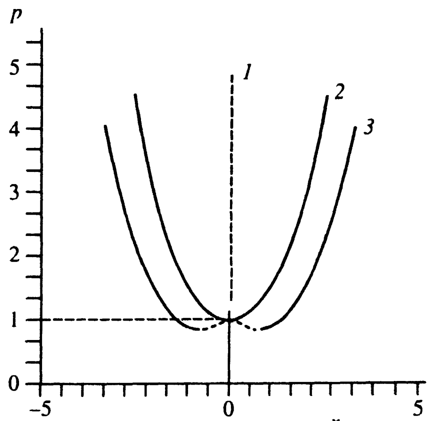

FIG. 2. Bifurcation curves of Eq. (17). Here *1* is the trivial time-independent solution. Trace *2* corresponds to $\xi=0.75<1$ and trace *3* to $\xi=1.75>1$. Unstable solutions are represented by the broken traces.

stable stationary points develops, corresponding to a steady vortex motion of the conducting fluid. But for $p_0<p<1$ (if $\xi<0$ holds) a second pair of stationary points develops, which in contrast to the first pair is unstable. Thus, for $0<\xi<1$ the bifurcation is supercritical as in the Lorenz model, while for $\xi>1$ it is subcritical, as in the model of Refs. 1 and 4.

We turn our attention now to the analogy between the behavior of the effective resistance (16) of the conductor close to the stability boundary and that of the specific heat C_p near the critical point in an equilibrium phase transition.[13] Specifically, if the characteristic times τ are much larger than $1/s$, then for $0<\xi<1$, depending on how the effective resistance varies with the external control parameter p, we have an inflection point:

$$\left.\frac{\partial R_{\mathrm{eff}}}{\partial p}\right|_{p=1-0}=0, \quad \left.\frac{\partial R_{\mathrm{eff}}}{\partial p}\right|_{p=1+0}=\frac{\gamma}{1-\xi},$$

where we have written $\gamma=-J_0(\mu_2)l\mu_2\beta^*(2\pi r_0^2\sigma c_2)^{-1}j>0$. This type of nonanalytical behavior of the function R_{eff} is analogous to that of the thermodynamic functions in a second-order phase transition.

In the case $\xi>1$, depending on how the function R_{eff} varies as the external control parameter p increases slowly (i.e., over times much larger than $1/s$), we have a discontinuity:

$$R_{\mathrm{eff}}|_{p=1-0}=R_0, \quad R_{\mathrm{eff}}|_{p=1+0}=R_0+\gamma\frac{\xi-1}{\xi}.$$

But for a slowly decreasing parameter p we have the discontinuity

$$R_{\mathrm{eff}}|_{p=p_0-0}=R_0, \quad R_{\mathrm{eff}}|_{p=p_0+0}=R_0+\gamma\frac{\xi-1}{2\xi}.$$

This behavior of the function R_{eff} is similar to that of thermodynamic functions in a first-order phase transition.

In any case it is clear that the value of the effective resistance R_{eff} of the conductor when vortex structures develop in it exceeds the original resistance R_0 even in the case of constant local kinetic coefficients (see also Refs. 4, 10, and 11). Thus, e.g., in the limit we have $\lim_{p\to\infty} R_{\mathrm{eff}}=R_0+\gamma$. Note also that in plasma physics the behavior of the so-called anomalous resistivity associated with the flow of an electric current through the plasma has been related to the onset of turbulence.[14] The increase in the effective resistance of the conductor which we have derived is also related to the turbulence in a plasmalike medium (in our case, a liquid metal). However, the resulting vortex structures have a spatial scale comparable with the conductor radius. This gives us some reason to expect that the model proposed here is the simplest model of the initial stage of the transition from laminar to turbulent flow in current-carrying plasmalike media.

4. CONCLUSION

Thus, we have derived a model for the initial stage of the transition from laminar to turbulent flow in a current-carrying plasmalike medium, represented as an incompressible conducting fluid with constant transport coefficients. We have shown that a nonuniform electric current density profile is unstable. The instability develops when the magnetic Rayleigh number $\mathfrak{R}$ reaches a critical value and the convective inflow of magnetic flux into the conductor is comparable with that due to diffusion.

The study of how different original current profiles affect stability reveals that the critical Rayleigh number $\mathfrak{R}_c$ depends on the current parameter $\eta\sim d\ln i/dt$, and for $\eta=8.73$ it attains its minimum value, $7.8\cdot10^3$. It should be noted that in this work $\mathfrak{R}_c$ turns out to be much larger than its value 978 found in Refs. 1 and 4. The characteristic size λ of the periodic structures in the conductor turns out to be the same as in Refs. 1 and 4 and is equal to $2.32r_0$.

We have studied the simplest model with a small number of modes for the initial stage in the transition from laminar to turbulent flow in a current-carrying medium, which turns out to be different from that proposed in Refs. 1 and 4, and is generally reminiscent of the Lorenz model.[7] By analyzing the model we find that the appearance of vortex structures in the conductor increases its effective resistance even when the local transport coefficients are constant (see also Refs. 4, 10, and 11). We have shown that, depending on how the function R_{eff} varies with the control parameter $p\sim\mathfrak{R}$ near its critical value, there is a discontinuity or an inflection point, which is analogous to the behavior of thermodynamic functions close to the critical point for an equilibrium phase transition.

It is a pleasure to thank A. M. Iskol'dskii for his unflagging interest in this work and fruitful discussions. This work was performed under the "Theoretical study of the dynamics of large-scale spatial structures in current-carrying conducting media" program, supported by the Russian Fund for Fundamental Research (Project No. 94-02-06654a).

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Translated by David L. Book