Motion of packets of an intense wave field in smoothly inhomogeneous media with nonlocal nonlinearity

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The motion of quasilocalized packets of an intense wave field in inhomogeneous media with nonlocal nonlinearity is studied both analytically and numerically. The approach takes into account the reaction of the radiation emitted by the packets. The paths of packet motion are constructed, and the lifetime and the depth of packet penetration in the shadow region are calculated. Finally the possibility is demonstrated of particle-like packets penetrating regions inaccessible to classical light fluxes. © *1995 American Institute of Physics.*

1. INTRODUCTION

Interest in the motion of packets of an intense wave field in inhomogeneous media stems both from the fundamental problems of the theory of nonlinear waves and from the applications in which packets of strong electromagnetic fields penetrate dense plasma layers. The most thoroughly studied problems in this area of research are spatially localized high-frequency field packets whose paths are described by the equations of geometrical optics in smoothly inhomogeneous media with local[1–3] and nonlocal[4] striction nonlinearities. the effect of the emission of wave fields by packets moving in smoothly inhomogeneous media have been analyzed only by perturbation theory techniques.[5]

In this paper we analyze the motion of packets of an intense one-dimensional wave field in smoothly inhomogeneous plane-layered media with nonlocal nonlinearity, where a packet consists of a kernel and the high- and low-frequency radiations emitted by the kernel. We show that the low-frequency radiation caused by the nonlocal nonlinearity changes the effective potential profile and forces the path of packet motion to deviate from the classical path. In particular, for wave packets emitting waves in the direction of rarefied plasma layers and moving in the direction of the density gradient the depth of their penetration of dense plasma layers increases in comparison to that for localized wave packets.

2. STATEMENT OF THE PROBLEM

We take a one-dimensional field $\Psi(z,t)$, the envelope of the wave process $\Psi(z,t)\exp\{-it\}$ in a smoothly inhomogeneous plane-layered medium with a linear inhomogeneity profile βz and a nonlocal nonlinearity:

$$2i\frac{\partial\Psi}{\partial t}+\frac{\partial^2\Psi}{\partial z^2}-n\Psi-\beta z\Psi=0, \tag{1}$$

$$\frac{\partial^2 n}{\partial t^2}-\frac{\partial^2 n}{\partial z^2}=\frac{\partial^2(|\Psi|^2)}{\partial z^2}. \tag{2}$$

Here n is the concentration perturbation. This system of equations describes packets of, say, a Langmuir or electromagnetic wave field, in the approximation of a "parabolic" law of dispersion for these waves and serves as a generalization of a well-known system of equations[6] to the case of a smoothly inhomogeneous medium. Assuming the wave packet velocity equal to $a(t)$, where $a(t)$ is interpreted as the velocity of motion of the center of "gravity" of field Ψ, we go over in Eqs. (1) and (2) to the comoving reference frame:

$$\xi=z-z_0-\int_0^t a(\tilde{t})d\tilde{t},\quad t'=t$$

(in what follows the prime on t is dropped). Substituting the sought field function in the form

$$\Psi(z,t)=\Phi(\xi,t)\exp\left\{i\left[a(t)\xi-\frac{\beta z_0}{2}t+\frac{1}{2}\int_0^t\left(a^2-\beta\int_0^{\tilde{t}}a d\hat{t}\right)d\tilde{t}\right]\right\} \tag{3}$$

into Eqs (1) and (2) reduces these equations to

$$2i\frac{\partial\Phi}{\partial t}+\frac{\partial^2\Phi}{\partial\xi^2}+\left[-n-\xi\left(2\frac{da}{dt}+\beta\right)\right]\Phi=0, \tag{4}$$

$$(a^2-1)\frac{\partial^2 n}{\partial\xi^2}+\frac{\partial^2 n}{\partial t^2}-2a\frac{\partial^2 n}{\partial t\partial\xi}-\frac{da\partial n}{dt\partial\xi}=\frac{\partial^2(|\Phi|^2)}{\partial\xi^2}. \tag{5}$$

We are interested in nonlinear wave packets moving in a smoothly inhomogeneous medium with a time-independent acceleration

$$\dot{a}=(\lambda-\beta)/2, \tag{6}$$

where λ is a constant characterizing the deviation of $\dot{a}$ from the "classical" acceleration $-\frac{1}{2}\beta$. In what follows we limit ourselves to examining wave packets whose velocity is sufficiently low in comparison to the velocity of low-frequency perturbations. In this case, neglecting in Eq. (5) the terms of orders a and a^2 and the term with the second time derivative and integrating the resulting relationship with respect to ξ from ξ to infinity as $n(\xi\to+\infty,t)\to 0$ and $\Phi(\xi\to+\infty,t)\to 0$, from Eqs. (4) and (5) with (6) we arrive at the following system of equations:

$$2i\frac{\partial\Phi}{\partial t}+\frac{\partial^2\Phi}{\partial\xi^2}-(n+\lambda\xi)\Phi=0, \tag{7}$$

$$\dot{a}n+\frac{\partial n}{\partial\xi}=-\frac{\partial(|\Phi|^2)}{\partial\xi}. \tag{8}$$

The equations obtained describe the evolution of wave packets of an intense wave field moving with a constant acceleration in a smoothly inhomogeneous medium with a nonlocal nonlinearity.

3. PHASE-CONJUGATE SOLUTIONS

Equations (7) and (8) are invariant under the simultaneous substitutions

$$t\to -t,\quad \Phi\to\Phi^*,\quad n\to n, \tag{9}$$

where Φ^* is the complex-conjugate of Φ. This implies that for packets $\Phi(\xi,t)$ described at the moment $t=0$ at which the packet center $\xi=0$ passes the reversal point by the real functions

$$\Phi(\xi,0)=\Phi^*(\xi,0),$$

solutions are possible that relate the packet parameters before and after reflection via phase-conjugate relations:

$$\Phi(\xi,t)=\Phi^*(\xi,-t);\quad n(\xi,t)=n(\xi,-t). \tag{10}$$

In this case the packet parameters before and after reflection differ only in the sign of the phase (phase-conjugate solutions).

4. METASTABLE PACKETS OF AN INTENSE WAVE FIELD

For further analysis we take Eq. (8) and express the concentration perturbation n in terms of the field's intensity $|\Phi|^2$ explicitly. To this end we multiply the equation by $\exp(\dot{a}\xi)$ and integrate the product with respect to ξ from ξ to infinity as $n(\xi\to+\infty,t)\to 0$ and $\Phi(\xi\to+\infty,t)\to 0$. The result is

$$n(\xi,t)=-|\Phi|^2-\dot{a}\exp(-\dot{a}\xi)\int_\xi^\infty|\Phi|^2\exp(\dot{a}\xi)d\xi. \tag{11}$$

For wave packets whose acceleration is so low that $L\sim(\dot{a})^{-1}\gg L_\Phi$, where L_Φ is the scale of nonuniformity of the field Φ, the exponential factors in (11) can be neglected. As a result for Φ we have the equation

$$2i\frac{\partial\Phi}{\partial t}+\frac{\partial^2\Phi}{\partial\xi^2}-U_{\text{eff}}(\xi,|\Phi|^2)\Phi=0 \tag{12}$$

with an effective potential

$$U_{\text{eff}}(\xi,|\Phi|^2)=\lambda\xi-|\Phi|^2-\dot{a}\int_\xi^\infty|\Phi|^2d\xi. \tag{13}$$

Equation (12) describes quasilocalized packets of an intense wave field whose decay is due to emission of low-frequency and high-frequency waves.

4.1. Emission of low-frequency radiation

This process is caused by the nonuniformity of packet motion in a medium with a nonlocal nonlinearity (the last term on the right-hand side of Eq. (13)) modifies the profile of the effective potential U_{eff}. Equation (13) shows that when the packet is retarded, ($\dot{a}<0$,) emission of low-frequency radiation leads to an increase in the effective potential behind the packet's kernel and in its vicinity, thereby lowering the value of the rolling-down force with which the inhomogeneous medium forces the high-frequency kernel to the region of small values of the unperturbed potential. At the center of the packet's kernel and at $\xi=0$ and

$$\left.\frac{\partial\Phi(t,\xi)}{\partial\xi}\right|_{\xi=0}=0,$$

the slope of the effective potential, which is proportional to the rolling-down force f, is

$$f\sim\left.\frac{\partial U(t,\xi)}{\partial\xi}\right|_{\xi=0}=\lambda+\dot{a}|\Phi(t,\xi=0)|^2. \tag{14}$$

This implies that the critical amplitude of the packet, $|\Phi_c(t,\xi=0)|^2$, at which the force acting on the packet's center is zero, is given by the following formula:

$$|\Phi_c(t,\xi=0)|=\sqrt{\lambda/-\dot{a}}. \tag{15}$$

When $|\Phi(t,\xi=0)|<|\Phi_c(t,\xi=0)|$, the inhomogeneous medium moves the packet into the region of small values of the effective potential. In the opposite case, when $|\Phi(t,\xi=0)|>|\Phi_c(t,\xi=0)|$, the low-frequency radiation shifts the packet into the region of high values of the effective potential. Note that condition (15) is met if $\lambda(-\dot{a})\equiv\frac{1}{2}\lambda(\beta-\lambda)>0$, from which, allowing for the fact that $\beta>0$, we get $\beta>\lambda>0$. The packet's acceleration $\dot{a}$ varies from $-\beta$ to zero.

To analyze the given effect numerically, we introduce the following variables into Eqs. (7) and (8):

$$\eta=-\dot{a}\xi,\quad \tau=(-\dot{a})^2t,\quad n=-\tilde{n}\frac{\lambda}{\dot{a}},\quad \Phi=\varphi\sqrt{\frac{\lambda}{-\dot{a}}}.$$

As a result we have

$$2i\frac{\partial\varphi}{\partial\tau}+\frac{\partial^2\varphi}{\partial\eta^2}-q\left(\eta-|\varphi|^2+\int_\eta^\infty|\varphi|^2d\eta\right)\varphi=0, \tag{16}$$

where $q=\lambda/(-\dot{a})^3$. For the initial wave field in (16) we take the one-soliton solution of Eq. (16) for a homogeneous medium with a local nonlinearity:

$$\varphi(\eta,\tau=0)=\frac{\varphi_0}{\cosh(\varphi_0\eta\sqrt{q/2})}. \tag{17}$$

Equation (16) was analyzed for a value of the dimensionless parameter q equal to 10^3 and different values of the initial amplitude φ_0. Figure 1 depicts the distributions of the amplitude $|\varphi(\tau,\eta)|$ and the concentration n at $\varphi_0=0.5$, 1.2247, and 1.5 for different moments of time τ. Numerical calculations reveal that there is a critical value of the packet amplitude,

$$\varphi_{0c}\simeq 1.2247, \tag{18}$$

at which the packet is immobile in the selected reference frame. In the old variables Eq. (18) has the form

$$\Phi_c\simeq 1.2247\sqrt{\lambda/(\dot{a})} \tag{19}$$

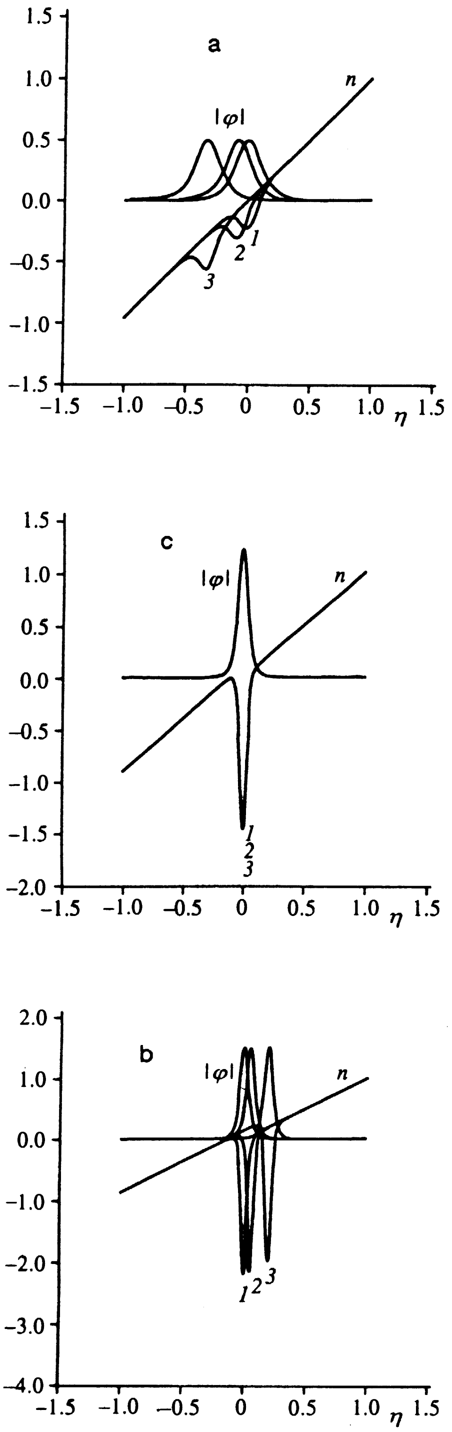

FIG. 1. Distributions of the high-frequency field amplitude $|\varphi(\tau,\eta)|$ and the concentration n of Eq. (16) at (a) $\varphi_0=0.5$, (b) $\varphi_0=1.2247$, and (c) $\varphi_0=1.5$ at different moments of time: curves *1*, $\tau=0$; curves *2*, $\tau=2\times10^{-2}$; and curves *3*, $\tau=4\times10^{-2}$.

and differs from Eq. (15), which was obtained earlier from the requirement that the forces acting on the packet kernel in an inhomogeneous medium with a nonlocal nonlinearity are balanced. This discrepancy is due to the finite size of a packet in the presence of an inhomogeneous effective potential. For instance, for the packet's acceleration

$$\ddot{\bar{\eta}}=\frac{1}{N_0}\frac{d^2}{dt^2}\int_{-\infty}^{+\infty}\eta|\varphi|^2 d\eta,$$

with $\bar{\eta}=(1/N_0)\int_{-\infty}^{\infty}\eta|\varphi|^2 d\eta$ the packet's center of "gravity" and $N_0=\int_{-\infty}^{\infty}|\varphi|^2 d\eta$ the number of wave-field quanta in the packet, we obtain, with allowance for (16), the following relationship valid in a reference frame moving with velocity a:

$$\ddot{\bar{\eta}}=-\frac{1}{2N_0}\int_{-\infty}^{\infty}\frac{\partial U}{\partial\eta}|\varphi|^2 d\eta=-\frac{q}{2}\left(1-\frac{N_1}{N_0}\right), \tag{20}$$

where $N_1=\int_{-\infty}^{\infty}|\varphi|^4 d\eta$. For wave packets corresponding to the initial distribution (17) we have

$$\ddot{\bar{\eta}}=-\frac{q}{2}\left(1-\frac{2}{3}\varphi_0^2\right). \tag{21}$$

This implies that a packet remains immobile in a reference frame moving with velocity a if its amplitude is $\varphi_{0c}=\sqrt{3/2}\simeq 1.2247$, which agrees with the result of numerical calculations (18). With allowance for (18) the packet's acceleration in the laboratory reference frame is

$$\dot{a}=-\frac{1}{2}\frac{\beta}{(1+\Phi_0^2/3)}. \tag{22}$$

In this case the parameter q is given by the following relationship:

$$q=\frac{8}{3}\frac{\Phi_0^2}{\beta^2}\left(1+\frac{1}{3}\Phi_0^2\right)^2. \tag{23}$$

4.2. Emission of high-frequency radiation

Emission of high-frequency radiation from a packet's kernel region is due to the slope of the effective potential $U_{\text{eff}}\propto\lambda\xi$ related to the deviation of the path of the wave packet from the classical path, on which $\lambda=0$. For instance, for packets that are immobile in an accelerated reference frame with allowance for the condition $\beta>\lambda>0$, which corresponds to a retardation of the packet that is slower than the classical deceleration ($\dot{a}=-\frac{1}{2}\beta$), the emission of high-frequency radiation is in the direction of rarefied plasma: $\xi\to-\infty$ (Fig. 2a). For $\lambda<0$, which corresponds to the case where the retardation is faster than in the classical case, emission is in the direction of dense plasma: $\xi\to\infty$ (Fig. 2c). The value $\lambda=0$ corresponds to packets that do not emit high- frequency radiation (Fig. 2b). If emission of low-frequency radiation is ignored, the lifetime of the given kernels of wave packets is infinite.

Finite values of λ correspond to metastable kernels of wave packets whose lifetime is finite and increases as $|\lambda|$ drops. At the same time, as λ grows, so does the depth of penetration of dense plasma layers by the high-frequency kernel of the wave packet. Obviously, transport of the energy of the high-frequency field to dense plasma layers requires that the packet's lifetime t_0 be longer than the time it takes the packet to get to its reversal point, t_r, i.e., $t_0>t_r$.

4.3. The lifetime of metastable packets

To find the packet's lifetime determined by emission of high-frequency radiation we employ the method of a "frozen" nonlinearity. For the emitting wave kernel we take the one-soliton solution

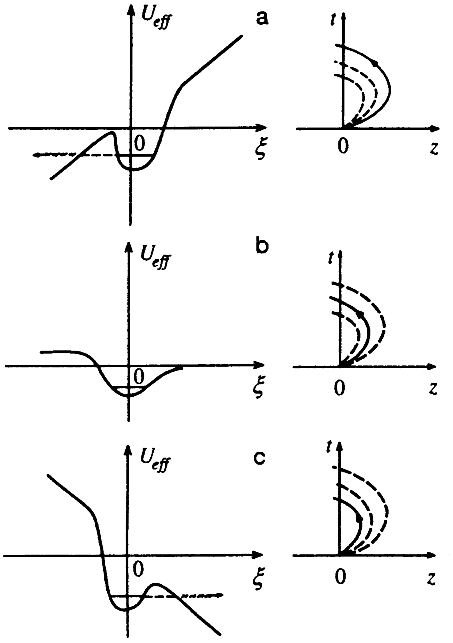

FIG. 2. Profiles of the effective potential $U_{\text{eff}}(\xi,|\varphi|^2)$ of Eq. (12) (the left diagrams) and the corresponding paths of motion of high- frequency wave packets (the solid curves in the right diagrams) for different values of λ: (a) $\lambda>0$, (b) $\lambda=0$, and (c) $\lambda<0$.

$$\varphi(\eta,\tau)=\frac{\varphi_0}{\cosh(\varphi_0\eta\sqrt{q/2})}\exp\left\{\frac{iq\varphi_0^2\tau}{4}\right\}\tag{24}$$

of Eq. (12) for a homogeneous medium with a local nonlinearity and at the packet's critical amplitude $\varphi_0=1.2247$. Approximating the smooth profile of the effective potential of Eq. (16),

$$U_{\text{eff}}(\eta,|\varphi|^2)=q\left(\eta-|\varphi|^2+\int_\eta^\infty|\varphi|^2d\eta\right)$$

with a given field distribution $|\varphi|^2$ (the dashed curve in Fig. 3) by a broken line (the solid curve in Fig. 3), we find the coefficient of transmission of high-frequency waves through a triangular potential barrier into the rarefied-plasma region:[7]

$$T=\exp\left[-\frac{4}{3q}\left(2\varphi_0\sqrt{\frac{2}{q}}-\mu+\frac{1}{2}\varphi_0^2q\right)^{3/2}\right],\tag{25}$$

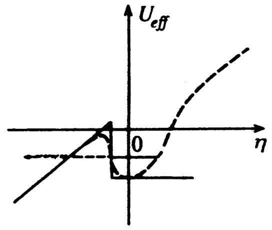

FIG. 3. Approximation of the smooth profile of the effective potential $U_{\text{eff}}(\eta,|\varphi|^2)$ of Eq. (16) (the dashed curve) by the broken solid line used in calculating the coefficient of transmission of a high-frequency field through a supercritical barrier in the frozen nonlinearity approximation.

where $\mu=\cosh^{-1}\sqrt{2}$. We find the time τ_0 that it takes photons to travel along a closed path in the potential well created by the soliton (24) in a homogeneous medium by ignoring the fact of radiation emission into the region of small values of the effective potential $U_{\text{eff}}(\eta,|\varphi|^2)$. In this case we have

$$\tau_0=\frac{2\pi}{\Delta^2}=\frac{4\pi}{q\varphi_0^2}.$$

As a result we arrive at the following expression for the lifetime of metastable wave packets, defined as $\tau_L=\tau_0/T$, with the packet's critical amplitude $\varphi_0^2=\frac{3}{2}$:

$$\tau_L=\frac{8\pi}{3q}\exp\left[\frac{4}{3q}\left(2\sqrt{\frac{3}{q}}-\mu+\frac{3}{4}q\right)^{3/2}\right].\tag{26}$$

The lifetime τ_L is commensurate with the time that it takes the packet center to reach the reversal point, τ_r, which according to (6) for $\dot{a}<0$ is equal to $a_0/(-\dot{a})$, where $a_0=a(\tau=0)$ is the velocity of the packet's motion at the initial moment. The ratio of these two times is

$$K\equiv\frac{\tau_L}{\tau_r}=\frac{8\pi(-\dot{a})}{3qa_0}\exp\left[\frac{4}{3q}\left(2\sqrt{\frac{3}{q}}-\mu+\frac{3}{4}q\right)^{3/2}\right].\tag{27}$$

When $q\gg1$,

$$K=\frac{8\pi(-\dot{a})}{3qa_0}\exp\left\{\frac{\sqrt{3q}}{2}\right\}.\tag{28}$$

Substituting the expressions for Eq. (28) $\dot{a}$ and q with $\Phi_0^2\ll1$,

$$\dot{a}\simeq-\frac{\beta}{2},\quad q\simeq\frac{8\Phi_0^2}{3\beta^2},$$

we get

$$K\simeq\frac{\pi\beta^3}{2\Phi_0^2a_0}\exp\left\{\sqrt{2}\frac{\Phi_0}{\beta}\right\}.\tag{29}$$

This implies that for $\Phi_0\leqslant\beta$ the packet lifetime in Eq. (29) is much longer than the time that it takes the packet's center to reach the reversal point: $K\gg1$.

We define the effectiveness with which quasilocalized wave packets penetrate dense plasma layers by the ratio of the displacement Δz_r of the coordinate z_r of the reversal point of the radiation-emitting kernels of wave packets, $z_r=a_0^2(\beta-\lambda)$, from the coordinate of the reversal point, $z_r(\lambda=0)=a_0^2\beta$, of nonradiative packets when the latter move along the classical path $\Delta z_r=z_r^{\text{opt}}-z_r(\lambda=0)$, to the scale of inhomogeneity of the medium, $L\sim\beta^{-1}$:

$$R=\frac{\Delta z_r}{L}=\frac{a_0^2\lambda}{\beta-\lambda}.\tag{30}$$

For one thing, for packets whose amplitude and acceleration are related by (22) we have

$$R=a_0^2\Phi_0^2/3.$$

The depth of penetration grows with the initial velocity a_0 of packet motion and the packet's amplitude Φ_0 and can reach considerable values for $a_0\gg1$.

In conclusion we estimate the time of modulation instability of the nonlinear wave packets considered here. Since the medium's inhomogeneity is smooth ($\beta \ll 1$) and the velocity of motion of the packets is low ($a \ll 1$), we estimate the time that it takes an instability to develop by using the two-dimensional nonlinear Schrödinger equation

$$2i\frac{\partial\varphi}{\partial t}+\frac{\partial^2\varphi}{\partial\xi^2}+\frac{\partial^2\varphi}{\partial y^2}+q|\varphi|^2\varphi=0, \tag{31}$$

where y is the transverse coordinate. For the unperturbed one-dimensional nonlinear solution of (31) we take the Langmuir soliton (17). In this approximation there is no longitudinal instability leading to the partitioning of packets into separate time pulses, while the partitioning into transverse structures takes place with an optimum scale $\Delta_\perp$ of the order of the soliton width $\Delta=\varphi_0\sqrt{2/q}$. The value of the increment on this scale reaches its maximum and amounts to $\Gamma=q\varphi_0^2 4$ (see Ref. 8). In this case, comparing the time it takes transverse perturbations to develop, $\tau_\perp=\Gamma^{-1}$, with the lifetime (26) of nonlinear wave packets for which emission of high-frequency radiation is responsible, we arrive at the following ratio for $q \gg 1$:

$$\frac{\tau_L}{\tau_\perp}=\frac{2\pi\varphi_0^2}{3}\exp\left(\frac{\sqrt{3q}}{2}\right). \tag{32}$$

Equation (32) implies that for $\varphi_0^2 \ll 1$ the lifetime of nonlinear wave packets for which emission of high-frequency radiation is responsible can be shorter than the time of development of transverse perturbations.

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[1] H. H. Chen and C. S. Liu, Phys. Fluids **21**, 377 (1978).
[2] M. R. Gupta, B. K. Som, and B. Dasgupta, Phys. Lett. A **69**, 172 (1978).
[3] E. M. Gromov, Fiz. Plazmy **15**, 631 (1989) [Sov. J. Plasma Phys. **15**, 368 (1989)].
[4] E. M. Gromov, V. M. Nakaryakov, and V. I. Talanov, Zh. Éksp. Teor. Fiz. **100**, 1785 (1991) [Sov. Phys. JETP **73**, 987 (1991)].
[5] K. V. Chukbar and V. V. Yan'kov, Fiz. Plazmy **3**, 1496 (1978) [*sic*].
[6] V. E. Zakharov, Zh. Éksp. Teor. Fiz. **62**, 1745 (1972) [Sov. Phys. JETP **35**, 908 (1972)].
[7] L. D. Landau and E. M. Lifshitz, *Quantum Mechanics: Nonrelativistic Theory*, Nauka, Moscow (1989), p. 220 [3rd English edition, Pergamon Press, Oxford (1977)].
[8] L. M. Degyarev, V. E. Zakharov, and L. I. Rudakov, Zh. Éksp. Teor. Fiz. **68**, 115 (1975) [Sov. Phys. JETP **41**, 57 (1975)].

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