Dynamics of rotating superfluid systems with pinning

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Equations describing the dynamics of motion of superfluid systems with pinning are derived, and analytical solutions of these equations are established for the case where the difference between the angular velocities of the superfluid and normal components is small. The solutions can be used to explain the time-dependent behavior of the angular velocity of the Vela pulsar. It is shown that vortex pinning in the period between two consecutive jumps in the pulsar angular velocity can redistribute the vortex number density so as to produce both the observed jump and the after-jump relaxation of the pulsar. For one thing, the formulas obtained are shown to provide an explanation of the 1988 Christmas discontinuity in the angular velocity of the Vela pulsar. © *1995 American Institute of Physics.*

1. INTRODUCTION

Studies of rotation of superfluid systems with pinning are of interest in connection with the problem of explaining the irregular behavior of the rotation of pulsars and a vessel with superfluid He II. These irregularities were discovered in observations of the behavior of the angular velocity of various pulsars and in experiments with a rotating vessel filled with superfluid He II (see Refs. 1–4). The angular velocities of these two systems behave similarly because in both cases the motion of the Feynman–Onsager quantum vortices is responsible for the interaction between the superfluid and normal components. At the same time, the observed difference in their behavior is due to the difference in the friction between the vortices and the normal component of the system. While in neutron stars (pulsars) the friction between quantum vortices and the normal component depends on the coordinates and can vary by several orders of magnitude (as a function of the distance from the center of the star), in a vessel with He II the friction coefficient is constant.[5,6]

The problem of the dynamics of a rotating vessel with He II was solved by Krasnov,[6] and the solution provides a good description of Tsakadze's experiments.[4]

The case of a rotating neutron star (pulsar) was studied in Ref. 5, where equations describing the dynamics of superfluid systems without pinning of quantum vortices were obtained. In solving these equations it was assumed that the relative moment of inertia of the superfluid liquid is small compared to the moment of inertia of the normal component. Solutions were found to within the term quadratic in the small parameter $P_0=I_s/I_c$, where I_s is the moment of inertia of the superfluid component, and I_c is the moment of inertia of the normal liquid.

The omission of pinning from the dynamic equations and the approximations in the solution of these equations limit the possibilities of explaining time-dependent dynamic phenomena observed in pulsars. Indeed, calculations show that linear relaxation originates in the layers of a star whose relative moment of inertia is of order 0.5. As for pinning, we must take it into account so as to explain the discontinuity in the angular velocity of a pulsar and to understand the multifaceted behavior of relaxation for different jumps. We also note that it is possible to describe relaxation if we know how the angular velocity of the superfluid component depends on the distance to the star's rotation axis prior to the jump. This dependence can be found by solving the dynamic equations with pinning for the "period of preparation" of the discontinuity in the angular velocity of the pulsar.

The goal of the present investigation is to obtain equations describing the dynamics of a rotating two-component system that allow for pinning on the assumption that the external braking torque is time-independent. In this approximation we find the solutions of these equations for any value of P_0 and for an arbitrarily fixed dependence of the coefficient of friction $\eta(r)$ between the quantum vortices and the normal liquid.

Before deriving the equations of motion we note the following. The considerable variations of the coefficient of friction $\eta(r)$ in neutron stars occur over macroscopic distances much greater than intervortex distances. This makes it possible to use hydrodynamic equations to describe the behavior of the superfluid liquid. We assume, in addition, that the normal component rotates like a rigid body. This condition is met thanks to the presence of ultrahigh magnetic fields $B\sim 10^{12}$ G inside the star, which couples the normal component in the superfluid layers with the solid crust of the star. We also assume that during motion the vortices remain parallel to the rotation axis. Our investigation was conducted on the assumption that cylindrical symmetry is present.

The equations of motion for a two-component liquid with pinning are derived in Sec. 2. In Sec. 3 we find an analytical solution for the problem on the assumption that the difference between the angular velocities of the normal and superfluid components is small. In Sec. 4 the solutions are used to explain the jumps and the relaxation of the pulsar angular velocities after jumps.

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2. THE EQUATIONS OF MOTION

Suppose that the system rotates with an angular velocity $\omega(t) > \omega_{c1}$, where ω_{c1} is the critical angular velocity of formation of a vortex lattice. The circulation of the average velocity of the superfluid component is quantized and has the form

$$\text{curl}\, \mathbf{v}_s = \nu_0 n(r,t), \tag{1}$$

where $n(r,t)$ is the density of the vortex lattice, $\nu_0 = 2\pi\hbar/m$ is the circulation quantum, $\boldsymbol{\nu}_0/\nu_0$ is the unit vector parallel to the vortex direction, and m is the mass of a helium atom or the mass of a Cooper pair of neutrons in the pulsar.

If instead of the linear velocity $\mathbf{v}_s$ we introduce the angular velocity $\omega_s(r,t)$ by the formula $\mathbf{v}_s = [\boldsymbol{\omega}_s \mathbf{r}]$ and integrate Eq. (1) with respect to r, we arrive at a relationship linking $\omega_s(r,t)$ and $n(r,t)$:

$$\omega_s(r,t) = \frac{\nu_0}{r^2} \int_0^r n(r',t) r' \, dr'. \tag{2}$$

From Eqs. (1) and (2) we can easily derive another form of the same equations to be used in what follows:

$$\frac{\partial}{\partial r} [r^2 \omega_s(r,t)] = \nu_0 r n(r,t). \tag{3}$$

Now we discuss the equation of continuity of the moving vortex system. If we denote the velocity of free vortex motion by $\mathbf{v}_L$ and the number density of the pinned vortices at point r and time t by $n_p(r,t)$, the equation of continuity assumes the form

$$\frac{\partial}{\partial t} n(r,t) = -\, \text{div}[n(r,t) - n_p(r,t)] \mathbf{v}_L, \tag{4}$$

where we have allowed for the fact that pinned vortices do not participate in the motion of the vortex system. In the absence of pinning, i.e., when $n_p(r,t) = 0$, the equation of continuity (4) assumes the usual form.[5] Integrating Eq. (4), we get

$$\frac{\partial}{\partial t} \int_0^r n(r',t) r' \, dr' = -[n(r,t) - n_p(r,t)] r v_{Lr}. \tag{5}$$

Allowing for (2), we can write Eq. (5) as

$$\frac{\partial}{\partial t} [r^2 \omega_s(r,t)] = -\nu_0 [n(r,t) - n_p(r,t)] r v_{Lr}. \tag{6}$$

Any variation in the angular velocity of the vessel or the crust of the neutron star causes the free vortices to move and leads to a new quasiequilibrium distribution of vortices. Hence the vortex distribution depends on the velocity field $\mathbf{v}_L$. This field can be found from the equation of vortex motion, i.e., the requirement that the sum of forces acting on each element of a vortex vanish:[6,7]

$$\rho_s [\mathbf{v}_s - \mathbf{v}_L \boldsymbol{\nu}_0] - \eta(r)(\mathbf{v}_L - \mathbf{v}_n) - \beta(r)[\mathbf{v}_L - \mathbf{v}_n \boldsymbol{\nu}_0] = 0. \tag{7}$$

Here the first term is the Magnus force, and the second and third terms are the forces of friction between the vortices and the normal liquid, ρ_s is the mass density of the superfluid component, $\mathbf{v}_n$ is the velocity of the normal component, and $\eta(r)$ and $\beta(r)$ are the coordinate-dependent longitudinal and tangential (in relation to $\mathbf{v}_L - \mathbf{v}_n$) friction coefficients. Equation (7) was solved in Refs. 5 and 6 and the following expressions for the velocity components v_{Lr} and $v_{L\varphi}$ were obtained:

$$v_{Lr} = k[\omega_s(r,t) - \omega_c(t)] r, \qquad v_{L\varphi} = \frac{\nu_0 \rho_s - \beta(r)}{\eta(r)} v_{Lr}, \tag{8}$$

where

$$k = \frac{\nu_0 \rho_s / \eta(r)}{1 + \left(\dfrac{\nu_0 \rho_s - \beta(r)}{\eta(r)} \right)^2}.$$

Here $\omega_c(t)$ is the angular velocity of the normal component of the liquid, and since we assume that the normal component rotates like a rigid body, $\omega_c(t)$ depends only on time.

The next relationship of interest is the equation determining the number density of the pinned vortices, $n_p(r,t)$. We assume that the vortices are captured by pinning centers, i.e., the "roughness of the inner surface" of the boundary region between the core and the crust of the star, and are freed only after a discontinuity in the angular velocity of the pulsar. Then the rate of variation of the number density of the pinned vortices is proportional to the number density of free vortices, i.e.,

$$\frac{\partial}{\partial t} n_p(r,t) = \frac{n(r,t) - n_p(r,t)}{\tau_p(r,t)}, \tag{9}$$

where $\tau_p(r,t)$ is the characteristic time describing the pinning process. The law (9) assumes not only the presence of a certain distribution of pinning centers but also that all the events in which moving vortices are captured are equally probable. Knowing these two factors makes it possible to find the function $\tau_p(r,t)$. However, in this paper we assume the function to be given, and in the period between two jumps in the pulsar angular velocity it depends only on r.

Finally, to complete the system of equations determining the dependent variables of the problem we must write the equation of motion of the normal component in the system:

$$I_c \frac{d}{dt} \omega_c(t) = K_{\text{int}} + K_{\text{ext}}. \tag{10}$$

Here I_c is the moment of inertia of the normal component of the system, K_{int} is the moment of the force acting between the superfluid and normal components of the system, and K_{ext} is the external braking torque, $K_{\text{ext}} = c\omega_c^k$, where $k=3$ for pulsars and $k = 1$ for a vessel with He II if the latter is rotating in a viscous medium.

Now we obtain an expression for the moment of forces of internal friction. As is well known,

$$\mathbf{K}_{\text{int}} = \int (\mathbf{F}(r)\mathbf{r})[n(r,t) - n_p(r,t)] \, dV, \tag{11}$$

where $\mathbf{F}$ is the force of friction between the vortices and the normal component of the liquid. If we allow for the fact that

$$\mathbf{F}\boldsymbol{\nu}_0 = \eta(r) v_{L\varphi} = \nu_0 \rho_s v_{Lr},$$

then

$$K_{\mathrm{int}}=2\pi\nu_0\int \rho_s[n(r,t)-n_p(r,t)]v_{Lr}lr^2\,dr,$$

where l is the length of a vortex filament. Using Eq. (6), we can eliminate $n(r,t)-n_p(r,t)$ from this expression. After fairly simple transformations we finally obtain

$$K_{\mathrm{int}}=-\frac{d}{dt}\int \omega_s(r,t)\,dI_s, \tag{12}$$

where $dI_s=r^2\,dm=2\pi\rho_s lr^3\,dr$. Here I_s is the moment of inertia of the superfluid component in the volume of a cylinder of radius r and length l. Substituting (12) into (10) we finally get

$$I_c\frac{d}{dt}\,\omega_c(t)+\frac{d}{dt}\int \omega_s(r,t)\,dI_s=K_{\mathrm{ext}}. \tag{13}$$

If we substitute v_{Lr} specified in (8) into Eq. (6), then Eqs. (3), (6), (9), and (13) form a closed system of equations for finding the unknown functions of our problem: $\omega_s(r,t)$, $n(r,t),n_p(r,t)$, and $\omega_c(t)$. Here it is assumed that the functions $\eta(r)$, $\beta(r)$, and $\tau_p(r,t)$ are given.

Since the velocity of the vortex motion depends on $\delta\omega(r,t)=\omega_s(r,t)-\omega_c(t)$, it is convenient, as will be seen shortly, instead of $\omega_s(r,t)$ to select $\delta\omega(r,t)$ and instead of $n_p(r,t)$ the corresponding angular velocity $\omega_p(r,t)$ determined by the following relationship:

$$\omega_p(r,t)=\frac{\nu_0}{r^2}\int_0^r n_p(r',t)r'\,dr'.$$

We introduce the dimensionless functions

$$\delta\Omega=\frac{\delta\omega(r,t)}{\omega_c(0)},\quad \Omega_c=\frac{\omega_c(t)}{\omega_c(0)},\quad \delta\Omega_p=\frac{\omega_p(r,t)-\omega_c(t)}{\omega_c(0)},$$

and the notation

$$2\omega_c(0)k=\frac{1}{\tau'(r)},\qquad n_0=\frac{2\omega_c(0)}{\nu_0}.$$

Then Eqs. (6), (3), and (13) assume the form

$$\frac{\partial}{\partial t}\,\delta\Omega+\frac{\delta\Omega}{2r\tau'(r)}\frac{\partial}{\partial r}[r^2(\delta\Omega-\delta\Omega_p)]=-\frac{d}{dt}\,\Omega_c, \tag{14}$$

$$\int\frac{\partial}{\partial t}\,\delta\Omega\,dP=-\frac{d}{dt}\,\Omega_c-\gamma', \tag{15}$$

$$\frac{n(r,t)-n_p(r,t)}{n_0}=\frac{1}{2r}\frac{\partial}{\partial r}\,[r^2(\delta\Omega-\delta\Omega_p)], \tag{16}$$

where

$$dP=\frac{dI_s}{I},\quad \gamma'=-\frac{K_{\mathrm{ext}}}{I\omega_c(0)},\quad I=I_s+I_c.$$

If we integrate Eq. (14) with respect to P and use (15) to eliminate $d\Omega_c/dt$, we obtain

$$\int_0^1\Big\{(1-P_0)\frac{\partial}{\partial t}\,\delta\Omega+\frac{\delta\Omega}{2r\tau'(r)}\frac{\partial}{\partial r}[r^2(\delta\Omega-\delta\Omega_p)]-\gamma'\Big\}\,dy=0, \tag{17}$$

where we have introduced the notation $dP=P_0\,dy$, with P_0 the total relative moment of inertia of the superfluid component. Since the condition (17) must be met for any arbitrary functions $\tau'(r)$ and $\tau_p(r,t)$, we finally arrive at the following equation determining $\delta\Omega$:

$$\frac{\partial}{\partial t}\,\delta\Omega+\frac{\delta\Omega}{2r\tau(r)}\frac{\partial}{\partial r}[r^2(\delta\Omega-\delta\Omega_p)]=\gamma. \tag{18}$$

Here we have introduced the notation

$$\tau(r)=(1-P_0)\tau'(r),\quad \gamma=\frac{\gamma'}{1-P_0}.$$

Finally, we can integrate Eq. (15) with respect to t. Since $\gamma=\gamma(t)$, the integral of Eq. (15) with the initial conditions $\Omega_c(0)=1$ and $\delta\Omega(r,0)=\delta\Omega_0$ has the form

$$\Omega_c(t)=1-P_0\int(\delta\Omega-\delta\Omega_0)\,dy-\int\gamma(t)\,dt. \tag{19}$$

As a result Eqs. (9), (16), (18), and (19) can be said to constitute a closed system of dynamic equations for rotating superfluid systems with pinning.

3. THE GENERAL SOLUTION OF THE EQUATIONS OF MOTION FOR RELATIVELY SMALL JUMPS IN ANGULAR VELOCITY

Now let us solve the system of dynamic equations in the approximation in which the discontinuities in the angular velocity of the normal component of the system are fairly moderate. In this approximation, both in experiments with a rotating vessel with He II and in pulsars the functions $\delta\Omega$ and δn and all their variations are small compared to Ω_0 and n_0. This makes it possible to linearize Eq. (18). Indeed, if we substitute n_0 for n in Eq. (16), the factor of $\delta\Omega/\tau(r)$ in Eq. (18) becomes equal to $1-n_p(r,t)/n_0$ and no longer depends on $\delta\Omega$. The same substitution can be done in Eq. (9) since on the whole the rate of variation of the number density of pinned vortices is determined by the variation of $n_p(r,t)$. If we assume that between two consecutive jumps in angular velocity the function $\tau_p(r)$ is time-independent, Eq. (9) can be immediately integrated, and its solution has the form

$$n_p(r,t)=n_0(1-e^{-t/\tau_p}). \tag{20}$$

This solution corresponds to the initial condition $n_p(r,0)=0$, which in turn follows from the assumption that after each jump all vortices become free. The moment $t=0$ is the moment of a delta-like discontinuity in the angular velocity of the normal component of the system. Substituting the solution (20) into Eq. (18), we get

$$\frac{\partial}{\partial t}\,\delta\Omega+\frac{\delta\Omega}{\tau(r)}e^{-t/\tau_p}-\gamma=0. \tag{21}$$

If we allow for the fact that γ depend weakly on time, the unknown function $\delta\Omega$ is determined from (21) with the condition that $\gamma=\mathrm{const}$. Knowing $\delta\Omega$ and employing Eq. (19), we can easily obtain the observed angular velocity $\Omega_c(t)$ of the normal component.

The general solution of Eq. (21) with the initial condition $\delta\Omega(r,0)=\delta\Omega_0$ has the form

$$\delta\Omega-\delta\Omega_0=\gamma e^{-x(t)}\int e^{x(t')}\,dt'-\delta\Omega_0(1-e^{-x(t)}), \tag{22}$$

where

$$x(t)=\frac{\tau_p}{\tau}(1-e^{-t/\tau_p})=\frac{n_p}{n_0}\frac{\tau_p}{\tau}.$$

From the general solution (22) it is clear how to obtain a solution in the absence of pinning, i.e., the solution for free vortices. For this it is sufficient to send τ_p to infinity in (22). Under this condition the function $x(t)$ tends to t/τ and hence

$$\delta\Omega-\delta\Omega_0=(\gamma\tau-\delta\Omega_0)(1-e^{-t/\tau}). \tag{23}$$

Substituting this solution into Eq. (19), we finally obtain

$$\Omega_c(t)=1-P_0\int(\gamma\tau-\delta\Omega_0)(1-e^{-t/\tau})\,dy-\gamma t. \tag{24}$$

Finally, we note that the problem of free vortices has a steady-state solution. Indeed, for times t satisfying the condition $t\gg\tau$ the solution becomes time-independent:

$$\delta\Omega=\gamma\tau(r). \tag{25}$$

Such a difference between the angular velocities of the superfluid and normal components of the system ensures an equal rate of decrease of the angular velocities of these components, i.e., $\dot{\Omega}_c=\dot{\Omega}_s=\gamma$. Note that the solution (23) and (24) for $P_0\ll 1$ coincides with the solution obtained in Ref. 5.

4. USING THE SOLUTIONS OBTAINED TO EXPLAIN THE DISCONTINUITIES AND SUBSEQUENT RELAXATION OF THE PULSAR ANGULAR VELOCITY

Theoretically, as will shortly be seen, the equations obtained can be used to describe irregularities in the behavior of the angular velocities of pulsars. Leaving a qualitative comparison of the theory with the specific observational data for later, we first formulate what requirements following from observation the theoretical model must meet.

Observations of the jumps in angular velocity of the Vela pulsar have revealed sudden increases in the spin of this pulsar; for instance, the eighth jump in the angular velocity of this pulsar took less than two minutes.[3] As for the relaxation times, they can be as long as several years: first, exponential relaxation with a growing characteristic time, whose value changes from several days to values of order of a hundred days, and then linear (in time) relaxation with a characteristic time of the order of the interjump time, which for the Vela pulsar is three years on the average.[1] Note that although the characteristics of different jumps in the angular velocity of the Vela pulsar are similar in their general features, both the relative value of the jump in angular velocity and the spectrum of characteristic times describing relaxation change from jump to jump.

To explain such diversity in the behavior of discontinuities and relaxation of the angular velocity of the Vela pulsar (in a single model of a neutron star), one must assume that the pinning mode changes after each jump in angular velocity. For a given pinning mode, i.e., for a given function $\tau_p(r)$, relaxation is accompanied by a change in vortex structure, which "prepares" the star for the next jump in angular velocity. The solutions that we found make it possible, by specifying $\tau_p(r)$, to find an initial condition $\delta\Omega_0$ that ensures both the observed spontaneous jump in angular velocity and the angular-velocity relaxation in the pulsar. Different initial conditions $\delta\Omega_0$ before each jump ensure the difference in the magnitude and relaxation behavior of the jumps in the angular velocity of the star. On the whole, however, $\tau_p(r)$ and hence $\delta\Omega_0$ change little from jump to jump, with the result that the overall features of the jumps in the angular velocity of the Vela pulsar are the same.

Thus, to solve the problem completely we must be able to construct the function $\tau_p(r)$ from a knowledge of the inner structure of the neutron star and the internal processes in the star. This problem has yet to be examined because it requires selecting a certain pinning mechanism and studying the mechanism in detail. Here we consider the inverse problem and discuss the requirements imposed on the properties of $\tau_p(r)$ that follow from the observation data on $\Omega_c(t)$ and $\dot{\Omega}_c(t)$.

In Ref. 5 we showed that the time of dynamic relaxation of vortices in the n–p–e phase of a neutron star varies from several seconds to several years. This time is short near the boundary between the n–p–e and A–e–n phases and grows as we move closer within the core of the neutron star towards its center. The jumps observed in the angular velocity can be explained by a sudden release of the pinned vortices that are in the region where the dynamic relation time is shorter than one minute. As for the relaxation of the angular velocity, its cause lies in that region of the star region where dynamic relaxation times vary from one day to two thousand days. We will call the first layer the discontinuity zone and the second the relaxation zone. Since the phenomena of the discontinuity in angular velocity and of relaxation are different qualitatively, the function $\tau_p(r)$ must also be different in the two zones. The logic of the problem implies that the time $\tau_p(r)$ is short in the discontinuity zone and long in the relaxation zone. The characteristic time with which we must compare $\tau_p(r)$ must be of the order of the time between two consecutive jumps in angular velocity, t_g. For one thing, we assume that in the discontinuity zone $\tau_p(r)\ll t_g$, while in the relaxation zone the opposite is true: $\tau_p(r)\gg t_g$.

We start with the discontinuity zone and assume $\tau_p\ll t_g$. We will show that if this condition is met, then in a time interval $t\approx t_g$ the discontinuity zone accumulates a sufficient number of vortices to explain the observed jump in the angular velocity of the state. The function $\delta\Omega(r,t)$ in this zone is determined from the general solution (22). Let us examine its asymptotic behavior as $t\to t_g$. If we require $\tau_p\ll t_g$ it follows that $t\gg\tau_p$ holds, and asymptotically the function $x(t)$ tends to the time-independent quantity τ_p/τ. The solution (22) in these conditions assumes the form

$$\delta\Omega-\delta\Omega_0=\gamma t-\delta\Omega_0(1-e^{-\tau_p/\tau}). \tag{26}$$

Irrespective of the value of τ_p/τ the second term in (26) is always smaller than the first. Then we finally have

$$\delta\Omega(r,t_g)=t_g/\tau_0,$$

where $\tau_0 = 1/\gamma$ is the pulsar characteristic lifetime.

When a discontinuity occurs, the Magnus force on the vortices in the discontinuity zone exceeds the pinning force, which leads to a catastrophic release of all vortices accompanied by a rapid decrease in their number density. The rapid jump observed in the angular velocity of the star can be described by the solution (24) if we assume that the relative moment of inertia of this zone is low and that at the center of the zone the dynamic relaxation time is of order of several minutes. Then for $\Omega_c(t)$ we have the following expression:

$$\Omega_c(t) = 1 - \frac{I_g}{I}[\gamma\tau - \delta\Omega(r,t_g)](1-e^{-t/\tau}) - \gamma t, \tag{27}$$

where I_g/I is the relative moment of inertia of this zone. The solution (27) suggests that the angular velocity of the star grows rapidly. Eliminating $\delta\Omega(r,t_g)$ from (27), we arrive at the following expression for the pulsar angular velocity:

$$(\Delta\Omega_c)_0 = \frac{I_g}{I}\frac{t_g}{\tau_0}. \tag{28}$$

As noted earlier, after the jump in the angular velocity relaxation occurs, for which the relaxation zone is responsible. Here the reverse condition is met: $\tau_p(r) \gg t_g$. Since $t \lesssim t_g$, we have $t \ll \tau_p$. In this approximation, according to (22), the function $x(t)$ has the form

$$x(t) = \frac{\tau_p}{\tau}(1-e^{-t/\tau_p})$$

$$\approx \frac{t}{\tau} - \frac{t^2}{2\tau\tau_p}. \tag{29}$$

Substituting $x(t)$ into (22) and replacing the function $x(t')$ by $(t'/\tau)(1-t/4\tau\tau_p)$ in the integrand, we finally get

$$\delta\Omega - \delta\Omega_0 = (\gamma\tau - \delta\Omega_0)(1-e^{-t/\tau})$$

$$+ \gamma\tau(e^{t^2/4\tau\tau_p} - 1), \tag{30}$$

and from Eq. (19) for $\Omega_c(t)$ we obtain

$$\Omega_c(t) = 1 - \int (\gamma\tau - \delta\Omega_0)(1-e^{-t/\tau_p})\, dP$$

$$-\gamma \int \tau(e^{t^2/4\tau\tau_p} - 1)\, dP - \gamma t. \tag{31}$$

Let us show that on the whole the relaxation processes for $\delta\Omega(r,t)$ and $\Omega_c(t)$ are described by the first term in (30) and the second term in (31). Indeed, the exponents in the solutions satisfy the conditions

$$\frac{t^2}{4\tau\tau_p} \lesssim \frac{t}{\tau}\frac{t_g}{4\tau_p} \ll \frac{t}{\tau},$$

since $t_g \ll \tau_p$. Hence in the relaxation process, i.e., when $t \lesssim t_g$ holds, the exponent in the first exponential terms is much greater than the exponents in the subsequent exponential terms. This means that the above statement is true. Neglecting the third term in Eq. (31), we obtain an expression for the relaxation of the star's angular velocity, $\Omega_c(t)$, that coincides with the solution (24) for free vortices.

For the final solution of the problem we must specify the initial condition $\delta\Omega_0$. The value of $\delta\Omega_0$ is determined by two terms: $\delta\Omega(r,t_g)$, a quantity that depends on the distribution of vortices along the radius of the star before the jump in the pulsar angular velocity; and $\Delta\Omega_c$, the size of the jump. While the second term is determined from observations, the first can be found from the solution (30). Substituting into this solution $t = t_g$ and allowing for the fact that $\tau \ll t_g$ holds, we obtain

$$\delta\Omega(r,t_g) = \gamma\tau + \gamma\tau(e^{t_g^2/4\tau\tau_p} - 1). \tag{32}$$

If we introduce the notation $\Delta\Omega = \delta\Omega(r,t_g) - \gamma\tau$, we have

$$\Delta\Omega = \gamma\tau(e^{t_g^2/4\tau\tau_p} - 1),$$

or

$$\frac{t_g^2}{4\tau\tau_p} = \ln\left(1 + \frac{\Delta\Omega}{\gamma\tau}\right). \tag{33}$$

Since in the relaxation zone the value of τ changes from days to several thousand days and it satisfies $10^{-6} \lesssim \gamma\tau \lesssim 10^{-4}$, at $\Delta\Omega \approx 10^{-6}$ the following condition holds: $\Delta\Omega/\gamma\tau \lesssim 1$. Then from (33) we obtain

$$\Delta\Omega \approx \frac{t_g^2}{\tau_0\tau_p}. \tag{34}$$

For the solution (31) to correctly describe the observed relaxation of the Vela pulsar, $\Delta\Omega$ must be of order 10^{-6}. If we allow for the fact that $\tau_0 = 10^4$ yr and $t_g = 3$ yr hold, we obtain $\tau_p \approx 10^2 t_g$. Note that we assumed that in the relaxation zone $t_g \ll \tau_p$, which agrees with this estimate.

Thus, the solution (24) describes the relaxation of the pulsar angular velocity if the integral is taken over the relaxation zone, and the initial condition $\Delta\Omega_0$ is determined by the relationship

$$\delta\Omega_0 = \delta\Omega(r,t_g) - \Delta\Omega_c = \gamma\tau + \Delta\Omega - \Delta\Omega_c, \tag{35}$$

where $\Delta\Omega$ is given by (33). Since the quantity measured in observations is $\dot{\Omega}_0(t)$, by differentiating (24) with respect to t and substituting $\delta\Omega_0$ from (35) we finally obtain

$$\dot{\Omega}_c(t) = -\int (\Delta\Omega_c - \Delta\Omega)\frac{e^{-t/\tau}}{\tau}\, dP - \gamma. \tag{36}$$

It is this solution that must be compared with the observation data. We will now show that the above formulas explain the observed characteristics of discontinuities in the angular velocity of the Vela pulsar.

We apply the formulas to the 1988 Christmas jump in the angular velocity of the Vela pulsar. A distinctive feature of this discontinuity is that $(\Delta\dot{\Omega}_c)_0$ was greater almost by a factor of ten than in the other discontinuities in the angular velocity of the same pulsar; the jump time was less than two minutes, $(\Delta\Omega_c)_0 = 1.8\times10^{-6}$, and $t_g = 907$ days (see Ref. 8). These data are sufficient to determine I_g/I in the region of the jump. It follows from (28) that $I_g/I = 1.4\times10^{-2}$.

For the same jump in the angular velocity of the Vela pulsar one can observe fast relaxation of the time derivitive of the angular velocity with characteristic time $\tau \approx 0.4$ days and $(\Delta\dot{\Omega}_c)_0/(\dot{\Omega}_c)_0 \approx 0.2$ (Ref. 8). Substituting $t = 0$ into (36) we get

$$(\Delta\dot{\Omega}_c)_0 = (\Delta\Omega_c)_0 \tau_d^{-1}, \tag{37}$$

where

$$\tau_d^{-1} = \int \left(1 - \frac{\Delta\Omega}{\Delta\Omega_c}\right) \frac{dP}{\tau}.$$

Since the relative moment of inertia of the relaxation zone is low, according to (37) we have

$$\tau_d^{-1} = \frac{1}{\tau} \frac{\Delta I_s}{I}. \tag{38}$$

Here we have allowed for the fact that in this zone $\Delta\Omega \ll \Delta\Omega_c$. Substituting (38) into (37) yields

$$\frac{\Delta I_s}{I} = \frac{\tau}{\tau_0} \frac{(\Delta\dot{\Omega}_c/\dot{\Omega}_c)_0}{(\Delta\Omega_c/\Omega_c)_0}, \tag{39}$$

and for the particular discontinuity considered $\Delta I_s/I = 5.3 \times 10^{-3}$.

In conclusion we note that these values of I_g/I and $\Delta I_s/I$ agree with the standard model for a neutron star with a mass of $1.4 M_\odot$, where for $\tau \lesssim 2$ min the value of I_g/I is of order 10^{-2}, and for the zone with an average relaxation time $\tau \approx 0.4$ day the value of $\Delta I_s/I$ is of order 5×10^{-3}.

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