# A surface barrier for a vortex loop in type-II superconductors

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The magnetic field distribution, the magnetic flux, and the free energy of an Abrikosov vortex loop near the flat surface of a type-II superconductor are calculated in the London approximation. The vortex line of such a vortex is a half-circumference of arbitrary radius. The interaction between the vortex half-ring and an external uniform magnetic field applied along the surface is studied and the value of the energy barrier that hinders vortex expansion into the superconductor is found. Also discussed are mechanisms by which expanding vortex loops near the surface of the type-II superconductor form an equilibrium vortex line with a shape determined by the distribution of the applied magnetic field. 

## 1. INTRODUCTION

In the mixed state, magnetic flux penetrates a type-II superconductor in the form of separate quanta of magnetic flux $\phi_0=\pi\hbar c/e$ or Abrikosov vortices, which in the equilibrium state constitute a regular lattice.[1] The presence of such vortices in the interior of the superconductor and their interaction with inhomogeneities and defects of the material (pinning) determine the magnetic properties of the superconductor[2] and the possibility of dissipation-free current flow.[3] On the whole the equilibrium shape of an Abrikosov vortex resembles the distribution of the lines of force of the external magnetic field. If the external magnetic field is uniform, an Abrikosov vortex with a straight vortex line parallel to the magnetic field correctly represents the distribution of the magnetic field. Penetration of such vortices into the superconductor is blocked by the Bean–Livingston energy barrier, which exists near the surface and has a strong effect on the motion of the vortices in the direction perpendicular to the surface.[4,5] As the external magnetic field $H_0$ gets stronger, the barrier lowers and finally disappears completely in fields on the order of the thermodynamic critical field $H_{cm}$. If the external field $H_0$ is lower than $H_{cm}$, the vortices penetrate the superconductor because of fluctuations and a nucleation center in the form of a vortex loop appears near the surface.[6] There exists a minimum critical size of the loop at which the vortex does not collapse at the surface but continues to expand into the bulk of the superconductor.

In this paper we obtain solutions of the London equation that describe the behavior of Abrikosov vortices in the shape of a half-ring (a loop) near the flat surface of a type-II superconductor. In Sec. 2 we calculate the distribution of the magnetic field, the magnetic flux, and the free energy of an isolated Abrikosov vortex with a vortex line in the form of a half-ring of arbitrary radius. Section 3 studies the interaction of a vortex half-ring and an external uniform magnetic field and calculates the energy barrier that prevents the formation of a vortex expanding into the bulk of the superconductor. In Sec. 4 we discuss the mechanism of formation of the equilibrium shape of a vortex line, determined by the distribution of the external magnetic field $\mathbf{H}_0$, from expanding vortex loops near the surface of a type-II superconductor.

## 2. A VORTEX HALF-RING NEAR THE FLAT SURFACE OF A SUPERCONDUCTOR

Let us assume that a superconductor with a magnetic-field penetration depth $\lambda$ and a coherence length $\xi$ occupies the space $z\leqslant 0$ and that the superconductor surface $S$ coincides with the $(x,y)$ plane. Suppose that the vortex line forms a semicircle of arbitrary radius $\rho_v$ and lies in the $(y,z)$ plane parallel to the external magnetic field $\mathbf{H}_0$ (Fig. 1). Then in view of the continuity of magnetic flux the ends of the vortex must lie on the surface $S$ (Ref. 6). In the London approximation, valid for superconductors with large Ginzburg–Landau parameters, $\kappa=\lambda/\xi\gg 1$, the distribution of the magnetic field $\mathbf{H}$ in the superconductor ($z\leqslant 0$) is described by the London equation

$$\mathbf{H}+\operatorname{curl}\operatorname{curl}\mathbf{H}=\mathbf{e}_v\delta(\mathbf{r}-\mathbf{r}_v). \tag{1}$$

Here $\mathbf{r}_v$ is the radius vector determining the position of the vortex line in the superconductor, and $\mathbf{e}_v$ is the unit vector directed along the tangent to the vortex line. Outside the superconductor ($z>0$) the magnetic field $\mathbf{H}_s$ satisfies the Maxwell equation

$$\operatorname{curl}\mathbf{H}_s=0. \tag{2}$$

The boundary conditions at the surface $S$ can be obtained from the condition that all components of the magnetic field are continuous at $z=0$,

$$\mathbf{H}=\mathbf{H}_s. \tag{3}$$

We have used dimensionless variables everywhere, assuming that the magnetic flux is measured in units of the flux quantum $\Phi_0$ and that all units of length are normalized to $\lambda$. In terms of these variables the unit of magnetic field $H$ is $\Phi_0/\lambda^2$ and the current density $j$ is measured in units of $c\Phi_0/\lambda^3$.

We now use the fact that Eq. (1) and the boundary conditions (3) are linear and write the solution for the field $\mathbf{H}$ in the superconductor in the form of a linear combination,

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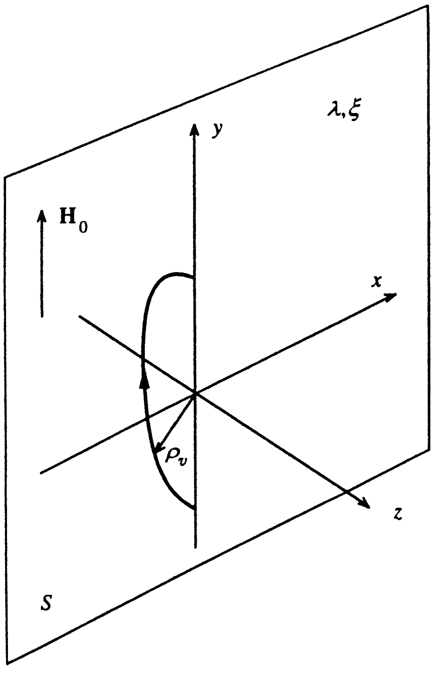


FIG. 1. An Abrikosov vortex in the form of a semicircle radius $\rho_v$ near the flat surface $S$ of a type-II superconductor. The superconductor occupies the half-space $z\leq 0$ and is characterized by the depth λ to which the magnetic field penetrates and the coherence length ξ; the vortex line lies in the $(x,y)$ plane.

$$\mathbf{H}=\mathbf{H}_v+\mathbf{H}_d. \tag{4}$$

The term $\mathbf{H}_v=H_v\mathbf{e}_v$ describes the distribution of the magnetic field of a closed Abrikosov vortex with the vortex line in the form of a circle of radius $\rho_v$ and in cylindrical coordinates ($\rho=\sqrt{y^2+z^2},\varphi,x$) satisfies the following equation

$$\frac{\partial^2 H_v}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial H_v}{\partial\rho}+\frac{\partial^2 H_v}{\partial x^2}-\left(1+\frac{1}{\rho^2}\right)H_v=-\delta(\rho-\rho_v)\delta(x). \tag{5}$$

The solution of Eq. (5) that describes the structure of a toroidal Abrikosov vortex and is valid for a vortex of arbitrary radius $\rho_v$ was obtained in Refs. 7–9 via a Fourier–Bessel transformation. It can be written as

$$H_v(\rho,x)=\frac{\rho_v}{2\pi}\int_{-\infty}^{+\infty}dq\ \exp\{iqx\}\times\begin{cases}I_1(\rho\sqrt{1+q^2})K_1(\rho_v\sqrt{1+q^2}) & \text{for } \rho\leq\rho_v,\\ I_1(\rho_v\sqrt{1+q^2})K_1(\rho\sqrt{1+q^2}) & \text{for } \rho\geq\rho_v,\end{cases} \tag{6}$$

where $I_1(z)$ and $K_1(z)$ are modified Bessel functions. The term $\mathbf{H}_d$ is the solution of the homogeneous London equation

$$\mathbf{H}_d+\text{curl curl }\mathbf{H}_d=0 \tag{7}$$

with the boundary conditions (3) in the region $z\leq 0$. To solve Eq. (7) we write the projections $\mathbf{H}_d=(H_d^x,H_d^y,H_d^z)$ in the form as two-dimensional Fourier integrals in the spatial harmonics:

$$H_d^\alpha=\frac{1}{2\pi}\int_{-\infty}^{+\infty}dq\ \exp\{iqx\}\int_{-\infty}^{+\infty}du\ \exp\{iuy\}\times C_d^\alpha(q,u)\exp\{z\sqrt{1+q^2+u^2}\}, \tag{8}$$

with $\alpha=x,y,z$. The field $\mathbf{H}_s$ generated by a vortex outside the superconductor ($z>0$) is a potential field, and the potential $U_s$ corresponding to this field,

$$\mathbf{H}_s=-\nabla U_s, \tag{9}$$

satisfies the Laplace equation

$$\Delta U_s=0. \tag{10}$$

We look for a solution of Eq. (10) in the form

$$U_s=\frac{1}{2\pi}\int_{-\infty}^{+\infty}dq\ \exp\{iqx\}\int_{-\infty}^{+\infty}du\ \exp\{iuy\}\times C_s(q,u)\exp\{-z\sqrt{q^2+u^2}\}. \tag{11}$$

Substituting the Fourier expansions (8) and (11) into Eq. (7) and the boundary conditions (4), we arrive at a system of linear algebraic equations for the unknown coefficients $C_d^\alpha$ and $C_s$. Solving it, we find (for $q^2+u^2\neq 0$) that

$$\begin{aligned} C_d^x&=-\frac{q\sqrt{1+q^2+u^2}\,S_v(q,u)}{\pi\sqrt{q^2+u^2}(\sqrt{1+q^2+u^2}+\sqrt{q^2+u^2})},\\ C_d^y&=-\frac{u\sqrt{1+q^2+u^2}\,S_v(q,u)}{\pi\sqrt{q^2+u^2}(\sqrt{1+q^2+u^2}+\sqrt{q^2+u^2})},\\ C_d^z&=\frac{i\sqrt{q^2+u^2}\,S_v(q,u)}{\pi(\sqrt{1+q^2+u^2}+\sqrt{q^2+u^2})},\\ C_s&=-\frac{i\sqrt{1+q^2+u^2}\,S_v(q,u)}{\pi\sqrt{q^2+u^2}(\sqrt{1+q^2+u^2}+\sqrt{q^2+u^2})}. \end{aligned} \tag{12}$$

When $q=u=0$ holds, all the coefficients $C_d^\alpha$ and $C_s$ in the expansions (8) and (11) are zero. Here the function $S_v(q,u)$ is determined by the spatial Fourier spectrum of the distribution of the magnetic field $H_v$ (Eq. (6)) in the $z=0$ plane:

$$S_v(q,u)=\frac{i}{2}\int_{-\infty}^{+\infty}dx\ \exp\{-iqx\}\times\int_{-\infty}^{+\infty}dy\ \exp\{-iuy\}H_v(y,x). \tag{13}$$

Substituting into the expansion (13) the distribution of the magnetic field $H_v$ given by (Eq. (6), we can easily arrive at the following integral representation for the spectral function $S_v(q,u)$:

$$S_v(q,u)=\rho_v\int_0^{+\infty}dy\ \sin(uy)\times\begin{cases}I_1(y\sqrt{1+q^2})K_1(\rho_v\sqrt{1+q^2}) & \text{if } y\leq\rho_v,\\ I_1(\rho_v\sqrt{1+q^2})K_1(y\sqrt{1+q^2}) & \text{if } y\geq\rho_v.\end{cases} \tag{14}$$

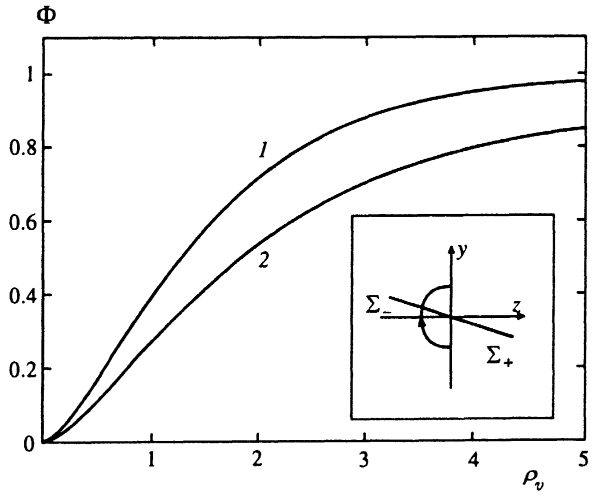


FIG. 2. The magnetic flux $\Phi_v$ in a toroidal Abrikosov vortex (curve *1*) and the magnetic flux $\Phi_{\Sigma_-}$ (curve *2*) as functions of the radius $\rho_v$ of the vortex line. The inset depicts the section plane $\Sigma=\Sigma_-+\Sigma_+$.

The details of the numerical calculation of the spectral function $S_v(q,u)$ are given in the Appendix. Employing the asymptotic behavior of the modified Bessel functions $I_1(z)$ and $K_1(z)$ with $z\gg 1$, we can show that for large values of the spectral variable $u$ the following approximate expression for $S_v(q,u)$ is valid:

$$S_v(q,u)\sim \frac{\sin(u\rho_v)}{u^2}, \quad u\gg 1, \quad u\gg q. \tag{15}$$

The above solutions completely determine the distribution of the magnetic field in a vortex half-ring that is near the surface of a type-II superconductor. The field $\mathbf{H}_d$ describes the perturbation of the structure of the toroidal Abrikosov vortex (6) introduced by the surface $S$ in the region where the vortex line is situated, $\mathbf{H}_d\cdot\mathbf{H}_v<0$. Thus, the effect of the superconducting surface is such that the magnetic field on the vortex axis diminishes. The characteristic scale of variation of $\mathbf{H}_d$ along the $z$ axis is determined by the vortex radius $\rho_v$ and the depth of penetration $\lambda$ by the magnetic field. Since the vortex line emerges from the superconductor's surface, a fringe field $\mathbf{H}_s$ appears outside the superconductor. This field resembles the magnetic field of a dipole and noticeably diminishes over distances of order $\rho_v$ from $S$.

To calculate the magnitude of the magnetic flux $\Phi$ in the vortex half-ring, we select an intersecting plane $\Sigma$ that passes through the center of the vortex half-ring and the $x$ axis, as shown in the inset in Fig. 2. Since the magnetic field and the supercurrent in the vortex are localized near the vortex line and disappear at large distances from the line, the total magnetic flux $\Phi_\Sigma$ through $\Sigma$ is zero:

$$\Phi_\Sigma=\Phi_{\Sigma_-}+\Phi_{\Sigma_+}\equiv 0, \tag{16}$$

where $\Phi_{\Sigma_\pm}$ are the magnetic fluxes through the corresponding half-planes $\Sigma_\pm$. Using Gauss's divergence theorem, one can easily establish that the fluxes $\Phi_{\Sigma_\pm}$ are independent of the orientation of the intersecting plane $\Sigma$ in relation to the superconductor's surface. Hence selecting $y=0$ as $\Sigma$ for simplicity, we get

$$\Phi_{\Sigma_-}=\Phi_v+\Phi_d, \tag{17}$$

$$\Phi_v=\int_{-\infty}^{+\infty}dx\int_{-\infty}^{0}dz\, H_v(z,x),$$

$$\Phi_d=\int_{-\infty}^{+\infty}dx\int_{-\infty}^{0}dz\, H_d^y(x,0,z),$$

$$\Phi_{\Sigma_+}\equiv\Phi_s=\int_{-\infty}^{+\infty}dx\int_{0}^{+\infty}dz\, H_s^y(x,0,z). \tag{18}$$

The term $\Phi_v$ is the magnetic flux in an undistorted toroidal Abrikosov vortex and was calculated in Ref. 8,

$$\Phi_v(\rho_v)=1-\rho_v K_1(\rho_v), \tag{19}$$

and the magnetic fluxes $\Phi_d$ and $\Phi_s$ are completely determined by the form of the spectral function $S_v(q,u)$:

$$\Phi_d=-\frac{2}{\pi}\int_0^{+\infty}du\,\frac{S_v(0,u)}{u+\sqrt{1+u^2}}, \tag{20}$$

$$\Phi_s=-\frac{2}{\pi}\int_0^{+\infty}du\,\frac{\sqrt{1+u^2}\,S_v(0,u)}{u(u+\sqrt{1+u^2})}. \tag{21}$$

The results of calculating $\Phi_v$ and $\Phi_{\Sigma_-}$ are illustrated by Fig. 2. The magnetic flux $\Phi_{\Sigma_-}$ in a vortex half-ring strongly depends on the radius $\rho_v$ of the vortex line and asymptotically approaches the flux quantum as $\rho_v$ becomes large. As Fig. 2 shows, $\Phi_{\Sigma_-}$ differs noticeably from $\Phi_v$ in a toroidal Abrikosov vortex. This means that the surface of the superconductor has a strong effect on the distribution of the magnetic field in the vortex half-ring, effectively lowering the magnetic field strength in the region with the vortex line.[10]

Let us calculate the free energy $G_0$ of a vortex half-ring near the surface of the superconductor as shown in Fig. 1. In the region occupied by the superconductor ($z\leq 0$) the London equation (1) has the following expression for the free energy $G_-$ corresponding to it:[11]

$$G_-=\frac{1}{8\pi}\int_{z=0}dx\,dy\,[\mathbf{H}\times\mathrm{curl}\,\mathbf{H}]_z$$
$$+\frac{1}{8\pi}\int_{z\leq 0}dV\,\mathbf{H}\cdot(\mathbf{H}+\mathrm{curl}\,\mathrm{curl}\,\mathbf{H}). \tag{22}$$

The energy of the magnetic field $\mathbf{H}_s$ outside the superconductor ($z>0$) is defined in the ordinary manner:

$$G_+\equiv\frac{1}{8\pi}\int_{z>0}dV\,(\mathbf{H}_s)^2. \tag{23}$$

Substituting the above solutions describing the distribution of the magnetic field in a vortex half-ring near the flat surface of a superconductor into Eqs. (22) and (23), we see that the free energy of such a vortex,

$$G_0=G_-+G_+, \tag{24}$$

can be written in the form of the sum

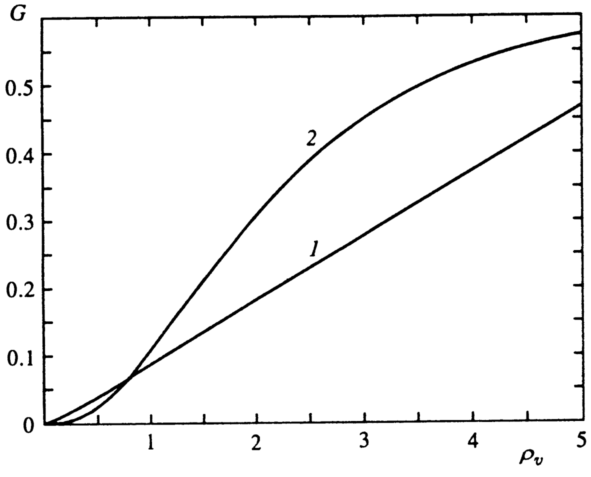


FIG. 3. The free energy $G_v$ of one-half of a toroidal Abrikosov vortex (curve 1) and the contribution $G_s$ of the superconductor's surface to the free energy of the vortex half-ring (curve 2) as functions of the radius $\rho_v$ of the vortex line ($\kappa=100$). To preserve the scale, we have plotted the quantity $100G_s$ along the vertical axis.

$$G_0=G_v+G_s. \tag{25}$$

Here $G_v$ is the energy of one-half of a toroidal Abrikosov vortex determined by the magnetic field strength $H_v$ at the center of the vortex line:[7,8]

$$G_v=\frac{\rho_v}{8}H_v(\rho_v,0), \tag{26}$$

and the term $G_s$ takes into account the overall contribution of the surface and is expressed in terms of the spectral function $S_v(q,v)$ in the following manner:

$$G_s=\frac{\rho_v}{2\pi^3}\int_0^{+\infty}dq\int_0^{+\infty}du \times\frac{u\sqrt{1+q^2+u^2}\,S_v(q,u)D(q,u)}{\sqrt{q^2+u^2}(\sqrt{1+q^2+u^2}+\sqrt{q^2+u^2})},$$

$$D(q,u)=\frac{\sin(u\rho_v)}{u\rho_v}-\int_0^{\pi/2}d\varphi\left[\cos\varphi\cos(u\rho_v\sin\varphi)+\frac{q^2+u^2}{u\sqrt{1+q^2+u^2}}\sin\varphi\sin(u\rho_v\sin\varphi)\right]\times\exp\{-\rho_v\cos\varphi\sqrt{1+q^2+u^2}\}. \tag{27}$$

Figure 3 depicts $G_v$ and $G_s$ as functions of the radius $\rho_v$ of the vortex half-ring. Since we have $G_s\ll G_v$, the superconductor surface has a negligible effect on the vortex free energy $G_0$, and with good accuracy we can ignore the contribution of the surface, assuming that

$$G_0(\rho_v)\simeq G_v(\rho_v). \tag{28}$$

Thus, the free energy $G_0(\rho_v)$ of an Abrikosov vortex shaped as a semicircle of radius $\rho_v$ near the flat surface of a type-II superconductor has increased only insignificantly in comparison to the free energy $G_v(\rho_v)$ of one-half of a toroidal vortex. At the same time, the distribution of the magnetic field in the vortex half-ring differs considerably from that in a toroidal Abrikosov vortex of the same radius $\rho_v$. The difference are the greatest in a surface layer of thickness $\lambda$. The effect of the surface is such that the magnetic field in the region of the kernel of the vortex half-ring proves to be weaker than in the toroidal vortex, and the magnetic flux in the half-ring, $\Phi_{\Sigma\_}$ (Eq. (18)), is lower than the magnetic flux $\Phi_v$ (Eq. (19)) even for vortices whose size is on the order of the depth of penetration by the magnetic field, $\lambda$ ($\rho_v\gtrsim 1$). Since the magnetic flux in the vortex half-ring closes through the space outside of the superconductor, forms a scattering field $\mathbf{H}_s$ forms near the surface, and the field noticeably diminishes over distances of the order of the vortex size. The energy of the field $\mathbf{H}_s$ contributes considerably to the free energy of the vortex and compensates for the decreases in the London energy $G_-$ (Eq. (22)) caused by the distortions of the structure of the toroidal vortex introduced by the superconductor surface.

## 3. A VORTEX HALF-RING IN AN EXTERNAL FIELD

As is well known,[11] the behavior of a superconductor in an external magnetic field is determined by the Gibbs free energy

$$F=G-\frac{1}{4\pi}\int_{z\leq 0}dV\,(\mathbf{H}\cdot\mathbf{H}_0). \tag{29}$$

If the external field $\mathbf{H}_0$ is parallel to the surface $S$, the distribution of the magnetic field in the superconducting half-space ($z\leq 0$; see Fig. 1) can be written as

$$\mathbf{H}=\mathbf{H}_V+\mathbf{H}_m. \tag{30}$$

The term $\mathbf{H}_V$ satisfies Eq. (1) with the boundary conditions (3) and describes the distribution of the magnetic field of an Abrikosov vortex with an arbitrary vortex line in a zero external field, and the term $\mathbf{H}_m=\mathbf{H}_0e^z$ describes the Meissner (vortex-free) distribution of the magnetic field near the surface of the superconductor. The free energy $G$ is determined by Eqs. (22)–(24), in which we must take (30) for the magnetic field $\mathbf{H}$ in the superconductor. Selecting the energy of the Meissner state as the reference point and allowing for the fact that the field $\mathbf{H}_m$ on the superconductor surface $S$ coincides with the external magnetic field $\mathbf{H}_0$, we can write the Gibbs free energy (29) as follows:

$$F=G_0+\frac{1}{4\pi}\int_{l_v}d\mathbf{l}_v\cdot\mathbf{H}_m-\frac{1}{4\pi}\int_{l_v}d\mathbf{l}_v\cdot\mathbf{H}_0. \tag{31}$$

Here $G_0$ is the free energy of an Abrikosov vortex, in a zero external magnetic field, with a vortex line specified in space by an arbitrary curve $l_v$, and $d\mathbf{l}_v=dl_v\mathbf{e}_v$ is an arc element $l_v$ directed along the tangent to the vortex line. Using Eq. (31), we can now easily write the Gibbs free energy of an Abrikosov vortex shaped as a half-circumference of an arbitrary radius $\rho_v$:

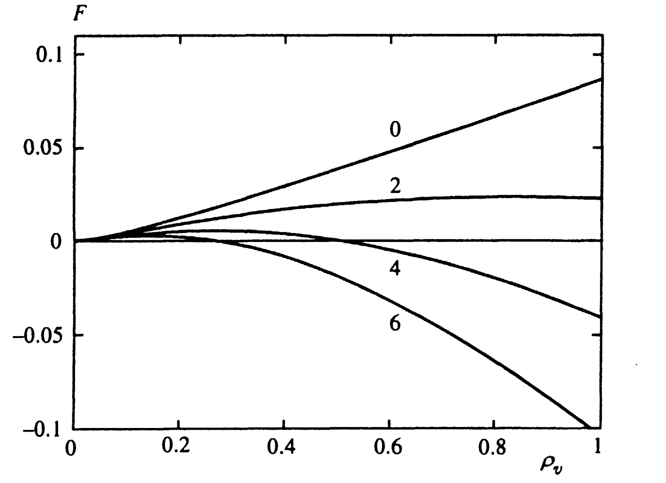


FIG. 4. The Gibbs free energy $F$ of a vortex half-ring near the flat surface of a type-II superconductor as a function of the radius $\rho_v$ of the vortex line for several values of the external magnetic field strength $H_0$ ($\kappa=100$). The number at each curve stands for the value of $H_0$ in units of the lower critical field $H_{c1}$.

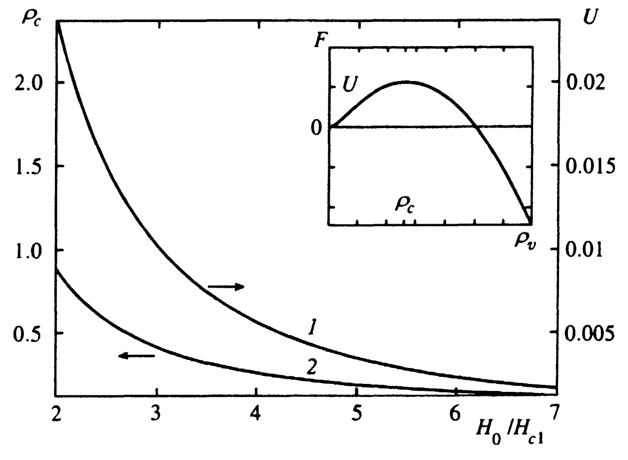


FIG. 5. The height $U$ of the surface barrier (curve *1*) for the vortex half-ring and the critical radius $\rho_c$ (curve *2*) as functions of the external magnetic field strength $H_0$ ($\kappa=100$). The region of the peak in the $F$ vs $\rho_v$ dependence is depicted in the inset on a larger scale. The Gibbs free energy reaches its maximum $U$ when the vortex radius is critical: $F(\rho_c)=U$ at $\rho_v=\rho_c$.

$$F(\rho_v)=G_0(\rho_v)-\frac{H_0\rho_v}{2\pi}\times\left(1-\int_0^{\pi/2}d\varphi\cos\varphi\exp\{-\rho_v\cos\varphi\}\right). \quad (32)$$

Figure 4 depicts the Gibbs free energy $F$ (Eq. (32)) as a function of the vortex radius $\rho_v$ for several values of the strength $H_0$ of the external magnetic field. The way in which $F$ depends on $\rho_v$ is greatly influenced by the magnitude of $H_0$. Bearing in mind the condition (28) and allowing for the fact that for a large vortex radius, $\rho_v\gg 1$, we have an approximate expression for the vortex free energy:

$$G_v(\rho_v)\simeq\frac{H_{c1}}{4}\rho_v,\quad \rho_v\gg 1, \quad (33)$$

where $H_{c1}=(1/4\pi)\ln\kappa$ is the lower critical field in the London model, we arrive at the following asymptotic expression for the Gibbs free energy $F(\rho_v)$ when $\rho_v\gg 1$:

$$F(\rho_v)\simeq\frac{\rho_v}{2\pi}\left(\frac{\pi}{2}H_{c1}-H_0\right),\quad \rho_v\gg 1. \quad (34)$$

We see that when $H_0<H_{c1}^*=\pi H_{c1}/2$ holds, the Gibbs free energy is positive for all $\rho_v\neq 0$ and that the lowest energy belongs to the Meissner state, which corresponds to vortex-free distributions of field and current in the superconductor. At $H_0=H_{c1}^*$ the energy of a state with a vortex in the form of a half-ring of a large radius ($\rho_v\gg 1$) is close to the energy of the Meissner state. Note that in this case an Abrikosov vortex proves to be asymptotically stable, since

$$\lim_{\rho_v\to\infty}\frac{dF}{d\rho_v}=0.$$

In other words, for a vortex half-ring the field $H_{c1}^*$ has the same meaning as the field $H_{c1}$ in the classical Bean–Livingston problem:[4] $H_{c1}^*$ is the lowest external field at which formation of a vortex half-ring in a type-II superconductor is energetically justified. For $H_0>H_{c1}^*$ the $F$ vs $\rho_v$ curve has a peak at a certain $\rho_v=\rho_c$, whose position ($\rho_c$) and height ($U$) decrease as $H_0$ grows (see the inset in Fig. 5). Figure 5 depicts the height $U$ of the potential barrier and the critical radius $\rho_c$ as functions of $H_0$. The potential barrier $U$ becomes completely flat when $\rho_c$ is on the order of the coherence length $\xi$. Using the approximate expression for the free energy $G_v$ of a vortex half-ring with a small vortex radius $\rho_v$ (see Refs. 13 and 14),

$$G_v(\rho_v)\simeq\frac{\rho_v}{4}\left(H_{c1}+\frac{\ln\rho_v}{4\pi}\right),\quad \rho_v<1, \quad (35)$$

we find that the potential barrier $U$ disappears,

$$\frac{dF}{d\rho_v}\simeq\frac{1}{16\pi}-\frac{\rho_v H_0}{4}\leqslant 0 \quad \text{at } \rho_v=\xi,$$

if the magnetic field $H_0$ is of the order of the thermodynamic critical field in the superconductor, $H_{cm}=\kappa/2\pi\sqrt{2}$. Thus, for a vortex half-ring near a superconductor surface the barrier $U$ disintegrates in approximately the same external fields as does the Bean–Livingston barrier for an Abrikosov vortex parallel to the surface.[4] Since at field strengths $H_0>H_{c1}^*$ the Gibbs free energy decreases with increasing vortex radius for $\rho_v>\rho_c$, the vortex half-ring that forms monotonically spreads to the point where its interaction with neighboring vortices becomes important.

## 4. DISCUSSION

Above we derived solutions of the London equation that describe the structure of an Abrikosov vortex shaped like a semicircle of arbitrary radius near the surface of a type-II superconductor and studied the interaction of such a vortex with an external magnetic field. The problem generalizes the well-known ideas about the Bean–Livingston barrier[4] to the

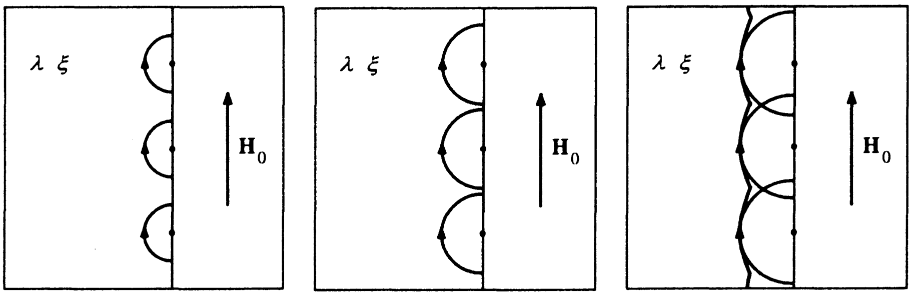


FIG. 6. Formation of an Abrikosov vortex with a vortex line having equilibrium shape near the flat surface of a type-II superconductor in a longitudinal magnetic field $\mathbf{H}_0$ as a result of intersection of the expanding vortex half-rings.

case of Abrikosov vortices with a curved vortex line. Such vortex loops serve as an example of compact magnetic structures[15] in which the magnetic field and the supercurrent are localized in all three directions. In view of the continuity of magnetic flux, the ends of the vortex half-ring are at the surface of the superconductor, and a corresponding scattering field $\mathbf{H}_s$ is generated outside the superconductor. A potential barrier hinders the vortex-loop formation near the superconductor's surface. As the strength $H_0$ of the external magnetic field grows, the barrier lowers. The barrier completely disappears in fields of the order of the thermodynamic critical field $H_{cm}$, when generation of vortex loops with dimensions of the order of the coherence length $\xi$ starts at the surface of the superconductor, without a threshold. The following Meissner current density corresponds to the magnetic field strength $H_0=H_{cm}$:

$$\mathbf{j}_m=\frac{1}{4\pi}\,\mathrm{curl}\,\mathbf{H}_m, \tag{36}$$

which close to the surface is comparable in magnitude to the unpairing current density $j_c=\kappa/12\sqrt{3}\,\pi^2$ (see Ref. 11). Since on a scale of order the coherence length $\xi$ the nonuniformity of the Meissner current is unimportant, the condition $j_m\sim j_c$ determines the threshold of thermodynamic stability of a homogeneous current state in superconductors[6,11] and agrees with the Landau criterion for roton excitation in a moving superfluid liquid.[13,16]

At a nonzero temperature $T$ and in fields $H_0<H_{cm}$, the magnetic flux can penetrate the superconductor because of thermally activated generation, subsequent expansion, and coalescence of vortex half-rings. Such an activation mechanism for surmounting the surface barrier plays an important role at temperatures $T$ close to the critical temperature $T_c$ (see Ref. 17), especially for superconductors with strong thermal fluctuations.[12,18]

Formation of an expanding vortex loop near the superconductor surface in an external field is similar to the formation of an expanding closed vortex ring in the presence of an external current.[12,13] The increase in the energy of the superconductor caused by formation of a vortex half-ring in the external magnetic field is related to the self-energy of an Abrikosov vortex: the larger the radius $\rho_v$ of the vortex line the greater the free energy $G_0$ of such a vortex. A decrease in the Gibbs free energy is determined by the work $\Delta W(\rho_v)$ done by the source of the external field in the expansion of the vortex to the size $\rho_v$ (see Refs. 3 and 11):

$$\Delta W(\rho_v)=\frac{H_0\rho_v}{2\pi}\left(1-\int_0^{\pi/2}d\varphi\,\cos\varphi\,\exp\{-\rho_v\cos\varphi\}\right). \tag{37}$$

The work $\Delta W(\rho_v)$ of the source is related to the displacement of the vortex line by the Lorentz force $f_L$ (see Ref. 19),

$$\mathbf{f}_L=\mathbf{j}_m\times\mathbf{e}_v, \tag{38}$$

from the Meissner current $\mathbf{j}_m$. Since this current is concentrated primarily in a layer $\sim\lambda$ at the superconductor surface, the interaction of the external magnetic field with the vortex half-ring strongly depends on the vortex size $\rho_v$. If $\rho_v\ll 1$ holds and the vortex line is entirely in the surface layer, where the current $\mathbf{j}_m$ may be assumed uniformly distributed, $\Delta W\sim\rho_v^2$ (Refs. 12–14). For large vortex radii ($\rho_v\gg 1$), the main part of the vortex is in the region where the Meissner current is practically zero. The work $\Delta W$ in this case is determined by the displacement of the sections of the vortex line that are in direct contact with the surface, so that $\Delta W\sim\rho_v$ for $\rho_v\gg 1$.

The equilibrium shape of an Abrikosov vortex is determined by the distribution of the external magnetic field and in the given case was found to be a straight vortex line parallel to the surface. An Abrikosov vortex shaped as a half-circumference of radius $\rho_v$ constitutes the proper structure for a vortex nucleation center if $\rho_v\ll 1$ and the inhomogeneities in the Meissner current density $\mathbf{j}_m$ are not large on the scale of the vortex dimensions. In this case the Lorentz force $f_L$ given by Eq. (38) is the same for all sections of the vortex line, and the vortex half-ring expands isotropically, i.e., without change of shape. When the size of the vortex half-ring becomes comparable to the depth of penetration $\lambda$ by the magnetic field ($\rho_v\gtrsim 1$), the Lorentz force $f_L$ for the sections of the vortex farther from the surface is found to be considerably weaker. This distorts the vortex shape so that the vortex loop finds itself stretched along the superconductor's surface, i.e., has a considerable section aligned with the external field.

The model of an isolated vortex half-ring expanding into the superconductor is valid as long as the vortex is considerably smaller than the characteristic distance $L$ between the vortices. If $L\sim 2\rho_v$ holds, the interaction of neighboring vortices becomes important and has a strong effect on the process of expansion of the vortex loops. Further convergence of the sections of neighboring vortices with oppositely directed magnetic fluxes may lead to an intersection of vortex lines.[3] Figure 6 depicts the sequence of formation of a straight Abrikosov vortex parallel to the surface of the superconductor as a result of intersection of vortex loops. Note that within this mechanism the surface barrier that hinders the formation of a straight Abrikosov vortex in fields $H_0<H_{cm}$ is determined by the potential barrier for a vortex half-ring, i.e., can be lower than the ordinary Bean–Livingston barrier.[4]

## 5. CONCLUSION

In this work we have calculated, in the London approximation, the distribution of the magnetic field, the magnetic flux, and the energy of an Abrikosov vortex shaped as a 3semicircle of arbitrary radius $\rho_v$ near the flat surface of a type-II superconductor in an external magnetic field $\mathbf{H}_0$ parallel to the surface. Formation of such a vortex is hindered by a surface barrier similar to the Bean–Livingston barrier for an Abrikosov vortex with a straight vortex line.[4] As the external magnetic field strength $H_0$ grows, the potential barrier flattens out and disappears completely in fields of the order of the thermodynamic critical field $H_{cm}$, when creation of vortex loops without a threshold with dimensions of the order of the coherence length $\xi$ becomes possible. At temperatures $T$ close to the critical temperature $T_c$ and in fields $H_0<H_{cm}$, the magnetic flux may penetrate the superconductor as a result of thermally activated creation of vortex half-rings at the surface of the superconductor and their subsequent merger. We have also discussed the mechanisms by which the equilibrium shape of an Abrikosov vortex forms via expansion and intersection of vortex loops.

## APPENDIX

Let us express the spectral function $S_v(q,u)$ in terms of an auxiliary function $w_{qu}(z)$ satisfying the following inhomogeneous differential equation with zero initial conditions:

$$w''_{qu}-\frac{1}{z}w'_{qu}-(1+q^2)w_{qu}=-\sin uz,$$
$$w_{qu}(0)=w'_{qu}(0)=0. \tag{A1}$$

Writing the Green's function for Eq. (A1), we obtain the following particular solution:

$$w_{qu}(z)=zK_1(z\sqrt{1+q^2})\int_0^z dy\ \sin(uy)\ I_1(y\sqrt{1+q^2})$$
$$-zI_1(z\sqrt{1+q^2})\int_0^z dy\ \sin(uy)\ K_1(y\sqrt{1+q^2}).$$

Using the tabulated value of the integral,[20]

$$\int_0^{+\infty} dy\ K_1(ay)\sin uy=\frac{\pi u}{2a\sqrt{a^2+u^2}},$$

we arrive at the following relationship linking the spectral function $S_v(q,u)$ and the auxiliary function $w_{qu}(z)$:

$$S_v(q,u)=w_{qu}(\rho_v)+\frac{\pi u\rho_v I_1(\rho_v\sqrt{1+q^2})}{2\sqrt{1+q^2}\ \sqrt{1+q^2+u^2}}. \tag{A2}$$

Thus, to calculate the value of $S_v(q,u)$ for arbitrary $q$ and $u$ we can solve the differential equation (A1) on the interval $z\in[0,\rho_v]$ and substitute the obtained value of $w_{qu}(\rho_v)$ into Eq. (A2). This method of calculating the spectral function $S_v(q,u)$ is especially effective for large $q$ and $u$ ($q$, $u\gg 1$) by making it possible to employ the well-known methods of solving the Cauchy problem for differential equations instead of calculating improper integrals of rapidly oscillating functions.


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